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THE STUDY ON THE EINSTEIN'S CONNECTION IN 5-DIMENSIONAL ES-MANIFOLD FOR THE SECOND CLASS

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Abstract. The manifold $^*g-ESX_n$ is a generalized n-dimensional Riemannian manifold on which the differential geometric structure is imposed by the unified field tensor $^*g^{\lambda\nu}$ through the ES-connection which is both Einstein and semi-symmetric. The purpose of the present paper is to derive a new set of powerful recurrence relations and to prove a necessary and sufficient condition for a unique Einstein's connection to exist in 5-dimensional $^*g-ESX_5$ and to display a surveyable tnesorial representation of 5-dimensional Einstein's connection in terms of the unified field tensor, employing the powerful recurrence relations in the second class.

1. Preliminaries

This section is a brief collection of basic concepts, notations, and results needed in subsequent considerations. They are due to [1],[4] and [5].

(a) n-simensional $^*g$-unified field theory

Let X_n be an n-dimensional generalized Riemannian manifold referred to a real coordinate system x^ν, which obeys the coordinate transformations $x^\nu \to x^{\nu'}$ for which

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$$(1.1) \qquad det(\frac{\partial x'}{\partial x}) \neq 0$$

In $n-g-UFT$ the manifold X_n is endowed with a real nonsymmetric tensor $g_{\lambda\mu}$, which may be decomposed into its symmetric part $h_{\lambda\mu}$ and skew-symmetric part $k_{\lambda\mu}$:

$$(1.2) \qquad g_{\lambda\mu} = h_{\lambda\mu} + k_{\lambda\mu}$$

where

$$(1.3) \qquad \mathfrak{g} = det(g_{\lambda\mu}) \neq 0, \quad \mathfrak{h} = det(h_{\lambda\mu}) \neq 0, \quad \mathfrak{k} = det(k_{\lambda\mu})$$

In $n-{}^*g-UFT$ the algebraic structure on X_n is imposed by the basic real tensor ${}^*g^{\lambda\nu}$ defined by

$$(1.4) \qquad g_{\lambda\mu}{}^*g^{\lambda\nu} = g_{\mu\lambda}{}^*g^{\nu\lambda} = \delta_\mu^\nu$$

It may be also decomposed into its symmetric part ${}^*h^{\lambda\nu}$ and skew-symmetric part ${}^*k^{\lambda\nu}$:

$$(1.5) \qquad {}^*g^{\lambda\nu} = {}^*h^{\lambda\nu} + {}^*k^{\lambda\nu}$$

Since $det({}^*h^{\lambda\nu}) \neq 0$, we may define a unique tensor ${}^*h_{\lambda\mu}$ by

$$(1.6) \qquad {}^*h_{\lambda\mu}{}^*h^{\lambda\nu} = \delta_\mu^\nu$$

In $n-{}^*g$-UFT we use both ${}^*h^{\lambda\nu}$ and ${}^*h_{\lambda\mu}$ as tensors for raising and/or lowering indices of all tensors in X_n in the usual manner. We then have

$$(1.7) \qquad {}^*k_{\lambda\mu} = {}^*k^{\rho\sigma*}h_{\lambda\rho}{}^*h_{\mu\sigma}, \qquad {}^*g_{\lambda\mu} = {}^*g^{\rho\sigma*}h_{\lambda\rho}{}^*h_{\mu\sigma}$$

so that

$$(1.8) \qquad {}^*g_{\lambda\mu} = {}^*h_{\lambda\mu} + {}^*k_{\lambda\mu}$$

The differential geometric structure on X_n is imposed by the tensor ${}^*g^{\lambda\nu}$ by means of a connection $\Gamma_\lambda{}^\nu{}_\mu$ defined by a system of equations

$$(1.9) \qquad D_\omega{}^*g^{\lambda\nu} = -2S_{\omega\alpha}{}^\nu{}^*g^{\lambda\alpha}$$

where D_ω denotes the symbol of the covariant derivative with respect to $\Gamma_\lambda{}^\nu{}_\mu$ and $S_{\lambda\mu}{}^\nu$ is the torsion tensor of $\Gamma_\lambda{}^\nu{}_\mu$. Under certain conditions the system (1.9) admits a unique solutions $\Gamma_\lambda{}^\nu{}_\mu$.

It has been shown in [4] that if the system (1.9) admits $\Gamma_\lambda{}^\nu{}_\mu$, it must be of the form

$$\Gamma_\lambda{}^\nu{}_\mu = {}^*\left\{ \begin{array}{c} \nu \\ \lambda\mu \end{array} \right\} + U^\nu{}_{\lambda\mu} + S_{\lambda\mu}{}^\nu, \tag{1.10}$$

where

$$U_{\nu\lambda\mu} = \overset{100}{S}{}_{(\lambda\mu)\nu} + 2\overset{(10)0}{S}{}_{\nu(\lambda\mu)}. \tag{1.11}$$

(b) Some notations and results

The following quantities are frequently used in our further considerations:

$$ {}^*g = det({}^*g_{\lambda\mu}), \quad {}^*h = det({}^*h_{\lambda\mu}), \quad {}^*k = det({}^*k_{\lambda\mu}) \tag{1.12}$$

$$ {}^*g = \frac{{}^*g}{{}^*h}, \quad {}^*k = \frac{{}^*k}{{}^*h}. \tag{1.13}$$

$$K_p = {}^*k_{[\alpha_1}{}^{\alpha_1}\,{}^*k_{\alpha_2}{}^{\alpha_2}\cdots{}^*k_{\alpha_p]}{}^{\alpha_p}, \quad (p=0,1,2,\cdots). \tag{1.14}$$

$$ {}^{(0)*}k_\lambda{}^\nu = \delta_\lambda^\nu, \; {}^{(p)*}k_\lambda{}^\nu = {}^*k_\lambda{}^\alpha \; {}^{(p-1)*}k_\alpha{}^\nu \quad (p=1,2,\cdots). \tag{1.15}$$

$$K_{\omega\mu\nu} = \nabla_\nu{}^*k_{\omega\mu} + \nabla_\omega{}^*k_{\nu\mu} + \nabla_\mu{}^*k_{\omega\nu} \tag{1.16}$$

where ∇_ω is the symbolic vector of the covariant derivative with respect to the christoffel symbols ${}^*\left\{ \begin{array}{c} \nu \\ \lambda\mu \end{array} \right\}$ defined by ${}^*h_{\lambda\mu}$ in the usual way.

In X_n it was proved in [4] that

$$K_0 = 1, \; K_n = {}^*k \text{ if } n \text{ is even, and } \mathrm{K_n = 0 \text{ if } n \text{ is odd}}. \tag{1.17}$$

$$ {}^*g = 1 + K_2 + \cdots + K_{n-\sigma}. \tag{1.18}$$

$$\sum_{s=0}^{n-\sigma} K_s \; {}^{(n-s)*}k_\lambda{}^\nu = 0 \quad (p=0,1,2,\cdots). \tag{1.19}$$

We also use the following useful abbreviations, denoting an arbitrary tensor $T_{\omega\mu\nu}$ skew-symmetric in the first two indices by T:

$$\overset{pqr}{T} = \overset{pqr}{T}{}_{\omega\mu\lambda} = T_{\alpha\beta\gamma}{}^{(p)*}k_\omega{}^{\alpha(q)*}k_\mu{}^{\beta(r)*}k_\lambda{}^\gamma \tag{1.20}$$

and for an arbitrary tensor $T^{\cdots}_{\cdots}$ for $p = 1,2,3,\cdots$:

$$ {}^{(p)}T^{\nu\cdots}_{\cdots} ={}^{(p-1)}\;{}^*k^\nu{}_\alpha T^{\alpha\cdots}_{\cdots}. \tag{1.21}$$

DEFINITION 1.1. The tensors ${}^*g_{\lambda\mu}$ is said to be

(1) of the first class, if $K_{n-\sigma} \neq 0$

(2) of the second class with jth category ($j >= 1$), if

$$K_{2j} \neq 0, \quad K_{2j+2} = K_{2j+4} = \cdots = K_{n-\sigma} = 0 \tag{1.22}$$

(3) of the third class, if $K_2 = K_4 = \cdots = K_{n-\sigma} = 0$

In $5 - {}^*g - UFT$, there are three classes: namely the first class when $K_4 \neq 0$, the second class when $K_4 = 0, K_2 \neq 0$ and the third class when $K_2 = K_4 = 0$.

It is well known that the basic scalars M are solutions of the characteristic equation

$$M^\sigma (M^{n-\sigma} + K_2 M^{n-2-\sigma} + \cdots + K_{n-2-\sigma} M^2 + K_{n-\sigma}) = 0 \tag{1.23}$$

On the other hand, it has shown in [5] that the tensor $S_{\lambda\mu}{}^\nu$ satisfies

$$S = B - 3 \overset{(110)}{S} \tag{1.24}$$

where

$$2B_{\omega\mu\nu} = K_{\omega\mu\nu} + 3K_{\alpha[\mu\beta}{}^*k_{\omega]}{}^{\alpha *}k_\nu{}^\beta \tag{1.25}$$

In our subsequent chapter, we start with the relation (1.24) to solve the system (1.9). Furthermore, for the second class, the nonholonomic solution of (1.24) is given by

$$\underset{xyz}{M} S_{xyz} = B_{xyz} \tag{1.26}$$

or equivalently

$$\begin{aligned} 4\underset{xyz}{M} S_{xyz} = (2 + \underset{z}{M}\underset{x}{M} + \underset{z}{M}\underset{y}{M}) K_{xyz} + \underset{z}{M}(\underset{x}{M} + \underset{z}{M}) K_{zxy} \\ + \underset{z}{M}(\underset{y}{M} + \underset{z}{M}) K_{yzx} \end{aligned} \tag{1.27}$$

where

$$\underset{xyz}{M} = 1 + \underset{x}{M}\underset{y}{M} + \underset{y}{M}\underset{z}{M} + \underset{z}{M}\underset{x}{M} \tag{1.28}$$

Therefore, in virtue of (1.26), we see that a necessary and sufficient condition for the system (1.9) to have a unique solution in the second class is

$$\underset{xyz}{M} \neq 0 \;\; for\, all\, x, y, z. \tag{1.29}$$

DEFINITION 1.2. A connection $\Gamma_{\lambda}{}^{\nu}{}_{\mu}$ is said to be *semi-symmetric* if its torsion tensor $S_{\lambda\mu}{}^{\nu}$ is of the form

$$S_{\lambda\mu}{}^{\nu} = 2\delta^{\nu}_{[\lambda}X_{\mu]}. \tag{1.30}$$

for an arbitrary non-null vector X_{μ}.

A connection which is both semi-symmetric and Einstein is called an ES connection. An n-dimensional generalized Riemannian manifold X_n, on which the differential geometric structure is imposed by ${}^*g^{\lambda\nu}$ by means of an ES connection, is called an n-dimensional ${}^*g-ES$ manifold. We denote this manifold by ${}^*g - ESX_n$ in our further considerations.

THEOREM 1.3. *In the second class, the basic scalars in* ${}^*g - ESX_5$ *are given by*

$$\underset{1}{M} = -\underset{2}{M} = \sqrt{-K_2} \neq 0, \qquad \underset{3}{M} = \underset{4}{M} = \underset{5}{M} = 0 \tag{1.31}$$

Proof. For the second class of $5-{}^*g-UFT$, the characteristic equation (1.23) is reduced to

$$M(M^4 + K_2M^2) = 0 \tag{1.32}$$

from which our assertion follows. □

THEOREM 1.4. *The main recurrence relation in the second class is*

$$^{(p+3)*}k_{\lambda}{}^{\nu} = -K_2{}^{(p+1)*}k_{\lambda}{}^{\nu} \quad (p = 0, 1, 2, \cdots) \tag{1.33}$$

Proof. When ${}^*g_{\lambda\mu}$ belongs to the second class, the characteristic equation (1.23) is reduced to

$$\sum_{f=0}^{2} K_f M^{n-f} = M^{n-2}\sum_{f=0}^{2} K_f M^{2-f} \tag{1.34}$$

If $\underset{x}{M}$ is a root of (1.34), it satisfies

$$0 = \underset{x}{M}\sum_{f=0}^{2} K_f \underset{x}{M}^{2-f} = \sum_{f=0}^{2} K_f \underset{x}{M}^{2-f+1} \tag{1.35}$$

Multiplying δ^i_x to both sides of (1.35) and making use of (1.15), we have (1.33). □

The following theorem is a simple consequence of (1.31).

Theorem 1.5. *In the second class, the basic scalars* $\underset{x}{M}$ *satisfy*

$$\underset{1}{M} + \underset{2}{M} = \underset{x}{M} + \underset{y}{M} = 0 \quad (x, y = 3, 4, 5) \tag{1.36}$$

$$\underset{1}{M}\underset{2}{M} = K_2, \quad \underset{1}{M}\underset{x}{M} = \underset{2}{M}\underset{x}{M} = \underset{x}{M}\underset{y}{M} = 0 \quad (x, y = 3, 4, 5) \tag{1.37}$$

In virtue of the above theorem, we have

Theorem 1.6. *In the second class, the following identities hold for all values of* x *and* y *when* $x \neq y$

$$\underset{x}{M}{}^2 \underset{y}{M}{}^2 = K_2 \underset{x}{M}\underset{y}{M} \tag{1.38}$$

$$\underset{x}{M}{}^{(2} \underset{y}{M}{}^{1)} = 0 \tag{1.39}$$

Theorem 1.7. *If* $T_{\omega\mu\nu}$ *is a tensor skew-symmetric in the first two indices, then the following recurrence relations hold in the second class of* $^*g - ESX_5$ *:*

$$\overset{(21)r}{T} = 0 \tag{1.40}$$

$$\overset{22r}{T} = K_2 \overset{11r}{T} \tag{1.41}$$

Proof. The relations (1.40) and (1.41) follow from (1.38) and (1.39). For example, the relation (1.41) is obtained as in the following way:

$$\begin{aligned}\overset{22r}{T} &= \textstyle\sum_{x,y,z} T_{xyz} \underset{x}{M}{}^2 \underset{y}{M}{}^2 \underset{z}{M}{}^r \overset{x}{A}_\omega \overset{y}{A}_\mu \overset{z}{A}_\nu \\ &= \textstyle\sum_{x,y,z} T_{xyz} (K_2 \underset{x}{M}\underset{y}{M}\underset{z}{M}{}^r) \overset{x}{A}_\omega \overset{y}{A}_\mu \overset{z}{A}_\nu \\ &= K_2 \overset{11r}{T}\end{aligned} \tag{1.42}$$

□

2. Einstein's connection $\Gamma_\lambda{}^\nu{}_\mu$ in the second class

In this section, we shall derive surveyable representations of $\Gamma_\lambda{}^\nu{}_\mu$ in terms of $^*g^{\lambda\nu}$, employing the recurrence relations.

In the following theorem, we shall prove two relations in X_n. These relations will be used in our subsequent theorem when we are concerned with the solution of (1.9).

THEOREM 2.1. *We have*

$$\overset{(pq)r}{B} = \overset{(pq)r}{S} + \overset{(p'q')r}{S} + \overset{(p'q)r'}{S} + \overset{(pq')r'}{S} \tag{2.1}$$

$$\begin{aligned} 2\overset{(pq)r}{B}{}_{\omega\mu\nu} &= \overset{(pq)r}{K}{}_{\omega\mu\nu} + \overset{r''(pq)}{K}{}_{\nu[\omega\mu]} + \\ &+ \tfrac{1}{2}\big(\overset{(pq')r'}{K}{}_{\omega\mu\nu} + \overset{(p'q)r'}{K}{}_{\omega\mu\nu} + \overset{r'p'q}{K}{}_{\nu[\omega\mu]} + \overset{r'q'p}{K}{}_{\nu[\omega\mu]}\big) \end{aligned} \tag{2.2}$$

where

$$p' = p+1, \quad q' = q+1, \quad r' = r+1, \quad r'' = r+2 \tag{2.3}$$

Proof. In virtue of (1.24) and (1.20), the first relation (2.1) is obtained as in the following way:

$$\begin{aligned} \overset{(pq)r}{B} &= \overset{(pq)r}{B}{}_{\omega\mu\nu} = \tfrac{1}{2}B_{\omega\beta\gamma}\big({}^{(p)*}k_\omega{}^{\alpha}\,{}^{(q)*}k_\mu{}^{\beta} + {}^{(q)*}k_\omega{}^{\alpha}\,{}^{(p)*}k_\mu{}^{\beta}\big){}^{(r)*}k_\nu{}^{\gamma} \\ &= \tfrac{1}{2}\big(S_{\alpha\beta\gamma} + S_{\epsilon\eta\gamma}{}^{*}k_\alpha{}^{\epsilon*}k_\beta{}^{\eta} + S_{\epsilon\beta\eta}{}^{*}k_\alpha{}^{\epsilon*}k_\gamma{}^{\eta} + S_{\alpha\epsilon\eta}{}^{*}k_\beta{}^{\epsilon*}k_\gamma{}^{\eta}\big)\times \\ &\qquad \times\big({}^{(p)*}k_\omega{}^{\alpha}\,{}^{(q)*}k_\mu{}^{\beta} + {}^{(q)*}k_\omega{}^{\alpha}\,{}^{(p)*}k_\mu{}^{\beta}\big){}^{(r)*}k_\nu{}^{\gamma} \end{aligned} \tag{2.4}$$

After a lengthy calculation, we note that the right-hand side of the above equation is equal to that of (2.1). Similarly, we verify (2.2) using (1.20) and (1.25).

□

THEOREM 2.2. *A necessary and sufficient condition for the system (1.9) to admit a unique solution $\Gamma_\lambda{}^\nu{}_\mu$ in the second class is that*

$$1 - (K_2)^2 \neq 0 \tag{2.5}$$

Proof. In virtue of (1.31), the symmetric scalars $\underset{xyz}{M}$ defined by (1.28) takes values as in the following 3 cases:

If two of the indices x, y, z are $1,2$ or $3,4$, then

$$\underset{xyz}{M} = 1 + K_2, \quad 1 \tag{2.6}$$

If at least one of x, y, z is 5 and no two take the values $1,2$ nor $3,4$, then

$$\underset{xyz}{M} = 1 \tag{2.7}$$

In the remaining cases,

$$\underset{xyz}{M} = \ 1 - K_2, \quad 1 \tag{2.8}$$

It is easily verified that the product of two factors in the right of (2.6) is $1+K_2$, that of one factor in the right of (2.7) is 1, and that of two factors in the right of (2.8) is $1-K_2$. Hence we have proved our assertion (2.5) in virtue of (1.29).

□

THEOREM 2.3. *The system of equations* (1.24) *in the second class is reduced to the following 5 equations:*

$$
\begin{cases}
B = S + 2\overset{(10)1}{S} + \overset{110}{S} \\
\overset{(10)1}{B} = \overset{(10)1}{S} + \overset{(20)2}{S} + \overset{112}{S} \\
\overset{(20)2}{B} = (K_2)^2\overset{(10)1}{S} + \overset{(20)2}{S} - K_2\overset{112}{S} \\
\overset{110}{B} = (1+K_2)\overset{110}{S} \\
\overset{112}{B} = (1+K_2)\overset{112}{S}
\end{cases}
\tag{2.9}
$$

Proof. This assertion follows from (2.1) using (1.40), (1.41) and (1.33) . □

THEOREM 2.4. *If the conditions* (2.5) *is satisfied, the unique solution of* (1.24) *is given by*

$$
(1-{K_2}^2)(S-B) = -2\overset{(10)1}{B} + (K_2-1)\overset{110}{B} + 2\overset{(20)2}{B} + 2\overset{112}{B}
\tag{2.10}
$$

Proof. (2.10) is a solution of (2.9). □

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