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HAUSDORFF OPERATORS ON WEIGHTED LORENTZ SPACES

Qinxiu Sun, Dashan Fan, and Hongliang Li*

Abstract. This paper is dedicated to studying some Hausdorff operators on the Heisenberg group $\mathbb{H}^n$. The sharp bounds on the strong-type weighted Lorentz spaces $\Lambda_u^p(w)$ and the weak-type weighted Lorentz spaces $\Lambda_u^{p,\infty}(w)$ are investigated. Especially, the results cover the classical power weighted space $L_\alpha^{p,q}$. The results are also extended to the product spaces $\Lambda_{u_1}^{p_1}(w_1) \times \Lambda_{u_2}^{p_2}(w_2)$, especially for $L_{\alpha_1}^{p_1,q_1} \times L_{\alpha_2}^{p_2,q_2}$. Our proofs are quite different from those in previous documents since the duality principle, and some well-known inequalities concerning the weights are adopted. The results recover the existing results as well as we obtain new results in the new and old settings.

1. Introduction

This paper is aimed to study the boundedness of Hausdorff operators on weighted Lorentz spaces with the Heisenberg group $\mathbb{H}^n$ as the underlying space. The theory of Hausdorff operator, while dating in a sense back to Hurwitz and Silverman [19] in 1917 with summability of number series, now becomes a notable ingredient in modern harmonic analysis

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* Corresponding author.

and has received an extensive attention in recent years. In order to save the length of this article, we refer the reader to the survey articles [7] and [24] for its background and historical developments. The reader can also see [5,6,8,13,17,23,25,26,28–30], among numerous research papers.

The classical one-dimensional Hausdorff operator is defined, in the integral form, by

$$H_\Phi(f)(x) = \int_0^\infty \frac{\Phi(x/t)}{t} f(t)dt, \ x > 0, \tag{1.1}$$

where Φ is a generating function that is fixed and assumed to make the integral to have sense for any f in the class S of all Schwartz functions. An extension of the n-dimensional Hausdorff operator (see [5]), is defined by

$$T_\Phi(f)(x) = \int_{\mathbb{R}^n} \frac{\Phi(x/|y|)}{|y|^n} f(y)dy, \tag{1.2}$$

for all $f \in S(\mathbb{R}^n)$. The Hausdorff operators (both $H_\Phi(f)$ and $T_\Phi(f)$) recover many classical operators by choosing suitable functions Φ. For instance, if we take

$$\Phi(t) = \frac{1}{t}\chi_{(1,\infty)}(t)$$

and

$$\Phi(t) = \frac{1}{|B(0,|t|)|}\chi_{(1,\infty)}(|t|)$$

in (1.1) and (1.2) respectively, then we obtain the one-dimensional Hardy operator

$$H(f)(x) = \frac{1}{x}\int_0^x f(t)dt \tag{1.3}$$

and the n-dimensional Hardy operator

$$\mathcal{H}(f)(x) = \frac{1}{|B(0,|x|)|}\int_{B(0,|x|)} f(y)dy, \tag{1.4}$$

respectively. Here $B(0,|x|)$ is the open ball centered at the origin and with radius $|x|$, and $|B(0,|x|)|$ denotes the volume of $B(0,|x|)$. As average operators, $H(f)$ is closely related to the fundamental theorem of calculus and $\mathcal{H}(f)$ is connected to the Hardy-Littlewood maximal function. Hence, the Hardy operators and their various varieties on the Euclidean space have received extensive studies (see [2,14,15,19,22,27,31,34,38]). In this paper, we are interested in studying the Hausdorff operators, since they are not only upgrades of the Hardy operators, but they also include the Cesáro operator, the Hardy-Littlewood-Pólya operator, the Riemann-Liouville fractional derivatives, the weighted Hardy operator, among many others (see [7]).

On the other hand, a changing of variables yields

$$H_\Phi(f)(x) = \int_0^\infty \frac{\Phi(t)}{t} f(x/t)dt, \ x > 0.$$

Thus, another form of n-dimensional Hausdorff operators on $\mathbb{R}^n$ is defined in [21] by

$$H_{\Phi,A}(f)(x) = \int_{\mathbb{R}^n} \frac{\Phi(y)}{|y|^n} f(A(y)x)dy, \tag{1.5}$$

where $A(y)$ is an $n \times n$ matrix with $\det A(y) \neq 0$ almost everywhere in the support of Φ (see also [7], [23], [31]). In [21], Lerner and Lifyand studied sufficient conditions for $H_{\Phi,A}$ to ensure its boundedness on the real Hardy space and on the Lebesgue spaces $L^p(\mathbb{R}^n)$ for $p \geq 1$. When

$$A(y) = diag[1/|y|, 1/|y|, ..., 1/|y|],$$

we denote it as $\widetilde{H}_\Phi(f)(x)$, namely

$$\widetilde{H}_\Phi(f)(x) = \int_{\mathbb{R}^n} \frac{\Phi(y)}{|y|^n} f(\frac{x}{|y|})dy.$$

It is notable that two extensions $T_\Phi(f)$ and $\widetilde{H}_\Phi(f)(x)$ are different when $n \geq 2$, although they coincide when $n = 1$. In [36], among other things, Wu and Chen [36] showed that the operator $\widetilde{H}_\Phi$ is bounded on the Lebesgue spaces L^p $(1 \leq p \leq +\infty)$ provided $C_0 = \int_{\mathbb{R}^n} |\Phi(x)| |x|^{-n(1-1/p)} dx < \infty$. Moreover, they proved that the constant C_0 is sharp if $\Phi(x) \geq 0$.

In this paper we are interested in studying Hausdorff operators on the Lorentz spaces. It is known that the Lorentz spaces $L^{p,q}(\mathbb{R}^n)$ play a pivot role in the theory of interpolation so that one can obtain the L^p boundedness of many important operators by establishing their weak boundedness with the help of the Lorentz spaces. As more general function spaces, we know $L^{p,p}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$. But this is not the main motivation we study Hausdorff operators on the Lorentz spaces. What makes us most interested is that we need to invoke a quite different method to establish the $L^{p,q}(\mathbb{R}^n)$ boundedness from those used on $L^p(\mathbb{R}^n)$. When we study the boundedness of Hausdorff operators on the L^p spaces, usually some inequalities (for instance, Hölder's inequality and the Minkowski inequality) can be employed. However the definition of $L^{p,q}$ is based on measure, rearrangement and distribution theory. Therefore, methods used on L^p is not adapted when we study similar problems on the Lorentz spaces and new method must be introduced. Since we always hope to establish theorems in a wide scope, in this paper we will study Hausdorff operators on the weighted Lorentz spaces $\Lambda_u^{p,q}(w)$ in a more general underlying space $\mathbb{H}^n$, where $\mathbb{H}^n$ denotes the Heisenberg group. It is known that the Heisenberg group is a non-commutative nilpotent Lie group, but it inherits some of basic structures of $\mathbb{R}^n$ and plays notable roles in many branches of mathematics, such as representation theory, harmonic analysis, several complex variables, partial differential equations, and quantum mechanics; see [34] for more details. Additionally, the Heisenberg group is widely applied in signal theory and many related topics. We notice that some research on $\mathbb{H}^n$ related to us already exists in recent publications. For instance, Fu and Wu [37] studied the L^p boundedness of Hardy operator (1.4); Guo, Sun and Zhao in [16] studied the L^p boundedness of Hausdorff operator; Ruan, Fan and Wu [31] studied the boundedness of Hausdorff operator on the Herz type space.

The Heisenberg group $\mathbb{H}^n$ [35] is the product space $\mathbb{R}^{2n} \times \mathbb{R}^+$ endowed with the following group law and dilation structure: for $x = (x_1, ..., x_{2n}, x_{2n+1})$, $y = (y_1, ..., y_{2n}, y_{2n+1}) \in \mathbb{R}^{2n} \times \mathbb{R}^+$,

$$x \cdot y = \left(x_1 + y_1, x_2 + y_2, ..., x_{2n} + y_{2n}, x_{2n+1} + y_{2n+1} + 2\sum_1^n (y_j x_{n+j} - x_j y_{n+j}) \right)$$

and

$$\delta_r x = (rx_1, rx_2, ..., rx_{2n}, r^2 x_{2n+1}), \ r > 0.$$

For any measurable set E in $\mathbb{H}^n$, with the notation $|E|$ for the measure of E coinciding with the usual Lebesgue measure on $\mathbb{R}^{2n+1} \times \mathbb{R}$, we have that

$$|\delta_r E| = r^Q |E|$$

and

$$d(\delta_r x) = r^Q dx.$$

where $Q = 2n + 2$ is the homogeneous dimension of $\mathbb{H}^n$. The norm of $x \in \mathbb{R}^{2n+1} \times \mathbb{R}$ is defined by

$$|x|_h = \left[\left(\sum_{i=1}^{2n} x_i^2 \right)^2 + x_{2n+1}^2 \right]^{\frac{1}{4}}.$$

The distance on the Heisenberg group $\mathbb{H}^n$ is defined by

$$\rho(x,y) = |y^{-1}x|_h,$$

where $y^{-1} = -y$ is the inverse of y. The corresponding Lie algebra is generated by the left-invariant vector fields:

$$X_j = \frac{\partial}{\partial x_j} + 2x_{n+j}\frac{\partial}{\partial x_{2n+1}},\ j = 1, ..., n,$$

$$X_{n+j} = \frac{\partial}{\partial x_{n+j}} - 2x_j\frac{\partial}{\partial x_{2n+1}},\ j = 1, ..., n,$$

$$X_{2n+1} = \frac{\partial}{\partial x_{2n+1}}.$$

The only non-trivial commutator relations are

$$[X_j, X_{n+j}] = -4X_{2n+1},\ j = 1, ..., n.$$

For $x \in \mathbb{H}^n$, $r > 0$, the ball and sphere with center x and radius r on $\mathbb{H}^n$ is given respectively by

$$B(x,r) = \{y \in \mathbb{H}^n :\ \rho(x,y) < r\}$$

and

$$S(x,r) = \{y \in \mathbb{H}^n :\ \rho(x,y) = r\}.$$

It is known that

$$|B(x,r)| = |B(0,r)| = \nu_Q r^Q,$$

where ν_Q is the volume of $B(0,1)$ on $\mathbb{H}^n$, namely

$$\nu_Q = \frac{2\pi^{n+\frac{1}{2}}\gamma(\frac{n}{2})}{\Gamma(n+1)\Gamma(\frac{n+1}{2})}.$$

The unit sphere $S(0,1)$ is often simply denoted by S^{Q-1} whose area is $\omega_Q = Q\nu_Q$ (see [37] for more details). We define two types of Hausdorff operators on $\mathbb{H}^n$ by

$$T_\Phi(f)(x) = \int_{\mathbb{H}^n} \frac{\Phi(\delta_{|y|_h^{-1}}x)}{|y|_h^Q} f(y)dy \tag{1.6}$$

and

$$H_{\Phi,A}(f)(x) = \int_{\mathbb{H}^n} \frac{\Phi(y)}{|y|_h^Q} f(A(y)x)dy, \tag{1.7}$$

where $A(y)$ is a $(2n+1)\times(2n+1)$ matrix with $\det A(y)\neq 0$ almost everywhere in the support of Φ.

The paper is organized as follows. In the second section, we will introduce some preliminary knowledge of the weighted Lorentz spaces and some their basic properties that are necessary in our study. We will state our main results (Theorems 3.1-3.7) and their proofs in Section 3.

2. Preliminaries

Let (X,μ) be a σ-finite measure space and $\mathcal{M}(X,\mu)$ be the space of all μ-measurable real valued functions on X. The decreasing rearrangement f_μ^* of $f\in\mathcal{M}(X,\mu)$ is defined by the equality [3]

$$f_\mu^*(t)=\inf\{s:\lambda_f^\mu(s)\le t\},\ t\ge 0,$$

where

$$\lambda_f^\mu(s)=\mu\{x\in X:|f(x)|>s\},\ s\ge 0$$

is a distribution function of f. The function $w:\mathbb{H}^n\to\mathbb{R}_+$ or $w:\mathbb{R}_+\to\mathbb{R}_+$ is called a weight function, or simply a weight, whenever w is Lebesgue measurable, not identically equal to zero and integrable on sets of finite measure. If w is a weight on $\mathbb{R}_+$, then we denote $W(t)=\int_0^t w(s)\,ds$, and we always have that $W(t)<\infty$, where $t>0$. If $(X,\mu)=(\mathbb{H}^n,udx)$ or $(X,\mu)=(\mathbb{R}_+,udx)$, where u is a weight on $\mathbb{H}^n$ or $\mathbb{R}_+$, then we denote $\lambda_f^\mu=\lambda_f^u,\ f_\mu^*=f_u^*,\ \mu(E)=u(E)$ for every Lebesgue measurable subset E of $\mathbb{H}^n$ or $\mathbb{R}_+$. Let $0<p,\ q<\infty$. We say that $f\in\mathcal{M}(X,\mu)$ belongs to the Lorentz space $L^{p,q}(X)$ [3, 18] if

$$\|f\|_{p,q}=\left(\int_0^\infty(t^{1/p}f_\mu^*(t))^q\frac{dt}{t}\right)^{1/q}<\infty.$$

For $0<p\le\infty$, the space $L^{p,\infty}(X)$ is defined as the space of all $f\in\mathcal{M}(X,\mu)$ satisfying

$$\|f\|_{p,\infty}=\sup_{t>0}t^{1/p}f_\mu^*(t)<\infty,$$

where we agree, on convention, that $t^{1/p}=1$ for $p=\infty$. If $(X,\mu)=(\mathbb{H}^n,udx)$ or $(X,\mu)=(\mathbb{R}_+,udx)$, we use the notation $L^{p,q}(X)=L^{p,q}(u)$.

Let w be a weight on $\mathbb{R}_+$. Using the notation $\|g\|_{L^q(\frac{dy}{y})}=(\int_0^\infty|g(y)|^q\,\frac{dy}{y})^{1/q}$, following [12] or [10], define for $0<p,\ q<\infty$ the weighted Lorentz space

$\Lambda_X^{p,q}(w)$ as a class of all $f \in \mathcal{M}(X,\mu)$ such that

$$\|f\|_{\Lambda_X^{p,q}(w)} = \|f_\mu^*\|_{L^{p,q}(w)} = p^{\frac{1}{q}} \left\| y \left(\int_0^{\lambda_f^\mu(y)} w(t)dt \right)^{\frac{1}{p}} \right\|_{L^q(\frac{dy}{y})} < \infty,$$

and the weighted Lorentz space $\Lambda_X^{p,\infty}(w)$ consisting of $f \in \mathcal{M}(X,\mu)$ with

$$\|f\|_{\Lambda_X^{p,\infty}(w)} = \|f_\mu^*\|_{L^{p,\infty}(w)} = \sup_{y>0} y \left(\int_0^{\lambda_f^\mu(y)} w(t)dt \right)^{\frac{1}{p}} < \infty.$$

Denote $\Lambda_X^p(w) = \Lambda_X^{p,p}(w)$. Note that if $0 < p,\ q < \infty$, then $\Lambda_X^{p,q}(w) = \Lambda_X^q(\bar{w})$ where $\bar{w} = W^{\frac{q}{p}-1}w$.

Let $L_{dec}^p(w)$ be the cone of all decreasing functions in $L^p(w)$, $0 < p < \infty$. Ariño and Muckenhoupt [1] gave a characterization of the boundedness of the Hardy operator $H:\ L_{dec}^p(w) \to L^p(w)$ in terms of the inequality on $w \in B_p$, where $w \in B_p$ means that there exists $C > 0$ such that

$$r^p \int_r^\infty \frac{w(x)}{x^p} dx \le C \int_0^r w(x)dx,\ r > 0.$$

Carro and Soria [11] obtained similar characterization of boundedness of $H:\ L_{dec}^p(w) \to L^{p,\infty}(w)$ showing that H is bounded whenever $w \in B_{p,\infty}$. Here $w \in B_{p,\infty}$ means that there exists $C > 0$ such that if $p > 1$ then

$$\left(\int_0^r \left(\frac{1}{x} \int_0^x w(t)dt \right)^{-p'} w(x)dx \right)^{1/p'} \left(\int_0^r w(x)dx \right)^{1/p} \le Cr,\ r > 0,$$

and if $p \le 1$ then

$$\frac{1}{r^p} \int_0^r w(x)dx \le \frac{C}{s^p} \int_0^s w(x)dx,\ 0 < s < r.$$

It is worth indicating that $B_p = B_{p,\infty}$ if $p > 1$. In [33], Soria proved that $H:\ L_{dec}^{p,\infty}(w) \to L^{p,\infty}(w)$ if and only if $w \in B_p$. For other characterizations of $B_p,\ B_{p,\infty}$, we refer to [12, 20, 32, 33]. It is known [12, Theorem 2.2.5] that $\Lambda_X^p(w)$ is normable, namely there exists a norm in $\Lambda_X^p(w)$ equivalent to the expression $\|\cdot\|_{\Lambda_X^p(w)}$, if and only if $p \ge 1$ and $w \in B_{p,\infty}$, and $\Lambda_X^{p,\infty}(w)$ is normable if and only if $w \in B_p$. If $w \in B_p$, let

$$B(p,w) = \sup_{f\downarrow} \frac{\|Hf\|_{L^p(w)}}{\|f\|_{L^p(w)}},\ \overline{B(p,w)} = \sup_{f\downarrow} \frac{\|Hf\|_{L^{p,\infty}(w)}}{\|f\|_{L^{p,\infty}(w)}},$$

where the symbol $f\downarrow$ indicates that f is a nonnegative decreasing function in $\mathbb{R}_+$. The estimate of $B(p,w)$ and $\overline{B(p,w)}$ can be found in [4, 9, 10, 32, 33].

Letting $0<p<\infty$, $f_\mu^{**}(t)=\frac{1}{t}\int_0^t f_\mu^*(s)ds$, $t>0$, for $f\in\mathcal{M}(X,\mu)$, define the space [12, Section 2.2.4]

$$\Gamma_X^p(w)=\{f\in M(X,\mu):\|f\|_{\Gamma_X^p(w)}=\int_0^\infty f_\mu^{**p}(t)w(t)dt<\infty\},$$

and if Φ is a nonnegative function on $\mathbb{R}_+$, define

$$\Gamma_X^{p,\infty}(d\Phi)=\{f\in M(X,\mu):\|f\|_{\Gamma_X^{p,\infty}(d\Phi)}=\sup_{t>0}f_\mu^{**}(t)\Phi^{1/p}(t)<\infty\}.$$

If $(X,\mu)=(\mathbb{H}^n,udx)$ or $(X,\mu)=(\mathbb{R}_+,udx)$, denote $\Gamma_X^p(w)=\Gamma_u^p(w)$ and $\Gamma_X^{p,\infty}(d\Phi)=\Gamma_u^{p,\infty}(d\Phi)$.

Throughout the paper, given $1\le p<\infty$ denote by p' its conjugate index that is $\frac{1}{p}+\frac{1}{p'}=1$. If g is a nonnegative function on $[0,\infty)$, $0<s<\infty$, let

$$\bar{g}(s)=\sup_{x\in[0,\infty)}\frac{g(sx)}{g(x)}.$$

3. Main results

Now we can state and prove our main results of the paper.

THEOREM 3.1. *Let $p\ge 1$, Φ be nonnegative, Φ and u be radial.*
(1) If w is decreasing, then

$$\|T_\Phi f\|_{\Lambda_u^p(w)\to\Lambda_u^p(w)}\le C_{\Phi,p,w},$$

where

$$C_{\Phi,p,w}=\omega_Q\int_0^\infty\frac{\Phi(s)}{s}(\overline{W}(s^Q\overline{u}(s)))^{1/p}ds.$$

(2) If w is increasing and $w\in B_p$, then for $f\in\Lambda_u^p(w)$, we have

$$\|T_\Phi f\|_{\Lambda_u^p(w)\to\Lambda_u^p(w)}\le B(p,w)^2C_{\Phi_1,p',w_1},$$

where $\Phi_1(t)=\frac{\Phi(\frac{1}{t})}{t^Q}\overline{u}(\frac{1}{t})$ and $w_1(t)=w^{1-p'}(t)$.

Proof. (1) Let

$$g_f(y) = \frac{1}{\omega_Q} \int_{|y'|_h=1} f(|y|_h y') dy'.$$

It follows that

$$\begin{aligned} T_\Phi(g_f)(x) &= \int_{\mathbb{H}^n} \frac{\Phi(\frac{x}{|y_1|_h})}{|y_1|_h^Q} g_f(y_1) dy_1 \\ &= \int_0^\infty r^{Q-1} dr \int_{|s'|=1} \frac{\Phi(\frac{x}{r})}{r^Q} \left(\frac{1}{\omega_Q} \int_{|y'|_h=1} f(ry') dy' \right) ds' \\ &= \int_0^\infty \frac{\Phi(\frac{x}{r})}{r} dr \int_{|y'|_h=1} f(ry') dy' \\ &= \int_{\mathbb{H}^n} \frac{\Phi(\frac{x}{|y|_h})}{|y|_h^Q} f(y) dy = T_\Phi f(x). \end{aligned}$$

Next we will verify that

$$\|g_f\|_{\Lambda_u^p(w)} \leq \|f\|_{\Lambda_u^p(w)}.$$

Indeed, since w is decreasing and $p \geq 1$, we get $\|\cdot\|_{\Lambda_u^p(w)}$ is a norm [12, Theorem 2.5.1] and thus by the Minkowski inequality,

$$\|g_f\|_{\Lambda_u^p(w)} \leq \frac{1}{\omega_Q} \int_{|y'|_h=1} \|f(|y|_h y')\|_{\Lambda_u^p(w)} dy' := I.$$

By the computation formula of the norm of the weighted Lorentz spaces and the rotation transformation,

$$I = p^{1/p} \frac{1}{\omega_Q} \int_{|y'|_h=1} \left(\int_0^\infty t^{p-1} W \left(\omega_Q \int_0^\infty r^{Q-1} \chi_{|f(ry')|>t} u(r) dr \right) dt \right)^{1/p} dy'.$$

By the Hölder inequality with the exponents p and p',

$$I \leq p^{1/p} \frac{1}{\omega_Q^{1/p}} \left(\int_{|y'|_h=1} \int_0^\infty t^{p-1} W \left(\omega_Q \int_0^\infty r^{Q-1} \chi_{|f(ry')|>t} u(r) dr \right) dt dy' \right)^{1/p}.$$

By the Fubini Theorem,

$$I \leq p^{1/p} \left(\int_0^\infty \frac{t^{p-1}}{\omega_Q} \int_{|y'|_h=1} W \left(\omega_Q \int_0^\infty r^{Q-1} \chi_{|f(ry')|>t} u(r) dr \right) dy' dt \right)^{1/p}.$$

Since w is decreasing which implies that W is concave, by the Jensen inequality it follows that

$$\begin{aligned} I &\leq p^{1/p}\left(\int_0^\infty t^{p-1}W\left(\int_{|y'|_h=1}\int_0^\infty r^{Q-1}\chi_{|f(ry')|>t}u(r)drdy'\right)dt\right)^{1/p} \\ &= p^{1/p}\left(\int_0^\infty t^{p-1}W(\lambda_f^u(t))dt\right)^{1/p} \\ &= \|f\|_{\Lambda_u^p(w)}. \end{aligned}$$

Hence

$$\frac{\|T_\Phi(f)\|_{\Lambda_u^p(w)}}{\|f\|_{\Lambda_u^p(w)}} \leq \frac{\|T_\Phi(g_f)\|_{\Lambda_u^p(w)}}{\|g_f\|_{\Lambda_u^p(w)}},$$

which implies that we only need to prove the theorem on a radial class. We check if f is radial, then

$$\|T_\Phi f\|_{\Lambda_u^p(w)} \leq C_{\Phi,p,w}\|f\|_{\Lambda_u^p(w)}.$$

Indeed, since $\Phi,\ f$ are radial,

$$T_\Phi f(x) = \omega_Q\int_0^\infty \frac{\Phi(\frac{|x|_h}{r})}{r}f(r)dr = \omega_Q\int_0^\infty \frac{\Phi(s)}{s}f\left(\frac{|x|_h}{s}\right)ds.$$

Hence by the Minkowski inequality,

$$\|T_\Phi(f)\|_{\Lambda_u^p(w)} \leq \omega_Q\int_0^\infty \frac{\Phi(s)}{s}\left\|f\left(\frac{|x|_h}{s}\right)\right\|_{\Lambda_u^p(w)}ds.$$

But for all $s>0$ and each measurable function f on $\mathbb{H}^n$, we have

$$\|f(\delta_s x)\|_{\Lambda_u^p(w)} = p^{1/p}\left(\int_0^\infty t^{p-1}W(\lambda^u_{f(\delta_r x)}(t))dt\right)^{1/p}.$$

And by change of coordinate,

$$\begin{aligned} \lambda^u_{f(\delta_r x)}(t) &= \int_{\mathbb{H}^n}\chi_{|f(\delta_r x)|>t}u(x)dx \\ &\leq \int_{\mathbb{H}^n}\chi_{|f(y)|>t}\overline{u}\left(\frac{1}{r}\right)u(y)r^{-Q}dy \\ &= r^{-Q}\overline{u}\left(\frac{1}{r}\right)\lambda_f^u(t), \end{aligned}$$

which yields that

$$\left\| f\left(\frac{|x|_h}{s}\right)\right\|_{\Lambda_u^p(w)} \leq (\overline{W}(s^Q\overline{u}(s)))^{1/p}\|f\|_{\Lambda_u^p(w)}. \tag{3.1}$$

Hence

$$\|T_\Phi f\|_{\Lambda_u^p(w)} \leq \omega_Q \int_0^\infty \frac{\Phi(s)}{s}(\overline{W}(s^Q\overline{u}(s)))^{1/p} ds \|f\|_{\Lambda_u^p(w)}.$$

(2) Due to $w \in B_p$, $p \geq 1$, by [12] we get $\Lambda^p(w) = \Gamma^p(w)$ and $\Lambda^p(w)$ can be normable with norm $\|\cdot\|_{\Gamma^p(w)}$. Thus $(\Lambda^p(w),\ \|\cdot\|_{\Gamma^p(w)})$ is a Banach function space and $(\Lambda^p(w),\ \|\cdot\|_{\Gamma^p(w)})'' = (\Lambda^p(w),\ \|\cdot\|_{\Gamma^p(w)})$ where $(\Lambda^p(w),\ \|\cdot\|_{\Gamma^p(w)})''$ is the double conjugate space of the space $(\Lambda^p(w),\ \|\cdot\|_{\Gamma^p(w)})$ (see [3]). Thus the principle of duality shows that

$$\|T_\Phi\|_{\Gamma^p(w)\to\Gamma^p(w)} = \|T'_\Phi\|_{(\Gamma^p(w))'\to(\Gamma^p(w))'}$$

where T'_Φ is the dual operator of T_Φ. Thus by the following relations

$$\begin{aligned}\frac{1}{B(p,w)}\|T_\Phi\|_{\Gamma^p(w)\to\Gamma^p(w)} &\leq \|T_\Phi\|_{\Lambda^p(w)\to\Lambda^p(w)} \\ &\leq B(p,w)\|T_\Phi\|_{\Gamma^p(w)\to\Gamma^p(w)},\end{aligned}$$

$$\begin{aligned}\frac{1}{B(p,w)}\|T'_\Phi\|_{(\Gamma^p(w))'\to(\Gamma^p(w))'} &\leq \|T'_\Phi\|_{(\Lambda^p(w))'\to(\Lambda^p(w))'} \\ &\leq B(p,w)\|T'_\Phi\|_{(\Gamma^p(w))'\to(\Gamma^p(w))'},\end{aligned}$$

we get

$$\begin{aligned}\frac{1}{B(p,w)^2}\|T'_\Phi\|_{(\Lambda^p(w))'\to(\Lambda^p(w))'} &\leq \|T_\Phi\|_{\Lambda^p(w)\to\Lambda^p(w)} \\ &\leq B(p,w)^2\|T'_\Phi\|_{(\Lambda^p(w))'\to(\Lambda^p(w))'}.\end{aligned} \tag{3.2}$$

Calculate the dual operator T'_Φ of T_Φ as follows:

$$\begin{aligned}\langle T_\Phi f, g\rangle &= \int_{\mathbb{H}^n}\left(\int_{\mathbb{H}^n}\frac{\Phi(\frac{|x|_h}{|y|_h})}{|y|_h^Q} f(y)dy\right) g(x)u(x)dx \\ &= \int_{\mathbb{H}^n}\left(\int_{\mathbb{H}^n}\frac{\Phi(\frac{|x|_h}{|y|_h})}{|y|_h^Q}\frac{u(x)}{u(y)} g(x)dx\right) f(y)u(y)dy = \langle f, T'_\Phi g\rangle\end{aligned}$$

which implies that

$$T'_\Phi g(y) = \int_{\mathbb{H}^n} \frac{\Phi(\frac{|x|_h}{|y|_h})}{|y|_h^Q} \frac{u(x)}{u(y)} g(x)dx.$$

Noting Φ is nonnegative, Φ and u are radial, we obtain

$$\frac{\Phi(\frac{|x|_h}{|y|_h})}{|y|_h^Q} \frac{u(x)}{u(y)} = \frac{\Phi(\frac{|x|_h}{|y|_h})}{|y|_h^Q} \frac{u(|x|_h)}{u(|y|_h)} \leq \frac{\Phi(\frac{|x|_h}{|y|_h})}{|y|_h^Q} \overline{u}(\frac{|x|_h}{|y|_h}) = \frac{\Phi_1(\frac{|y|_h}{|x|_h})}{|x|_h^Q} \tag{3.3}$$

where

$$\Phi_1(t) = \frac{\Phi(\frac{1}{t})}{t^Q} \overline{u}(\frac{1}{t}).$$

So by (3.3) for all nonnegative functions g,

$$T'_\Phi g(y) \leq T_{\Phi_1} g(y). \tag{3.4}$$

On the other hand, by [12, Theorem 2.4.12] we know $(\Lambda^p(w))' = \Lambda^{p'}(w^{1-p'})$ with equality of norms since w is increasing. So by (3.2), (3.4) and the result of (1) we obtain

$$\begin{aligned} \|T_\Phi\|_{\Lambda^p(w)\to\Lambda^p(w)} &\leq B(p,w)^2 \|T'_\Phi\|_{(\Lambda^p(w))'\to(\Lambda^p(w))'} \\ &= B(p,w)^2 \|T'_\Phi\|_{\Lambda^{p'}(w^{1-p'})\to\Lambda^{p'}(w^{1-p'})} \\ &\leq B(p,w)^2 C_{\Phi_1,p',w_1}, \end{aligned}$$

which completes the proof. □

If $u(x) = |x|_h^\alpha$, $w(t) = t^{\frac{q}{p}-1}$, $\alpha \in \mathbb{R}$, $1 \leq p < \infty$, $1 \leq q < \infty$, Then it is easy to check $\Lambda_u^q(w) = L^{p,q}(\mathbb{H}^n, |x|^\alpha dx)$. In this case, we are able to obtain the operator norm $\|T_\Phi\|_{L_\alpha^{p,q}\to L_\alpha^{p,q}}$ in the following theorem. Here, for simplicity we will use the notation $L^{p,q}(\mathbb{H}^n, |x|^\alpha dx) = L_\alpha^{p,q}$.

Theorem 3.2. *Let $p = q = 1$ or $1 < p < \infty$ and $1 \leq q < \infty$, $\alpha \in \mathbb{R}$, Φ be radial, and*

$$C_{\Phi,p,\alpha} = \omega_Q \int_0^\infty \Phi(s) s^{\frac{Q+\alpha}{p}-1} ds.$$

(1) If $p \geq q$, then

$$\|T_\Phi\|_{L_\alpha^{p,q}\to L_\alpha^{p,q}} = C_{\Phi,p,\alpha}.$$

(2) If $p < q$, then

$$C_{\Phi,p,\alpha} \leq \|T_\Phi\|_{L_\alpha^{p,q}\to L_\alpha^{p,q}} \leq p'^2 C_{\Phi,p,\alpha}.$$

Proof. (1) First let $p \geq q$. Since $w(t) = t^{\frac{q}{p}-1}$ is decreasing, $\overline{u}(s) = s^{\alpha}$, and $\overline{W}(t) = t^{\frac{q}{p}}$, it follows by Theorem 3.1 (1) that

$$\|T_{\Phi} f\|_{L^{p,q}_{\alpha}} \leq C_{\Phi,p,\alpha} \|f\|_{L^{p,q}_{\alpha}}. \tag{3.5}$$

(2) Consider another case $p < q$. Since $w(t) = t^{q/p-1}$ is increasing, $w \in B_p$ and $\overline{u}(s) = s^{\alpha}$, $w_1(t) = w^{1-p'}(t) = t^{\frac{q'}{p'}-1}$ and $\overline{W_1}(t) = t^{\frac{q'}{p'}}$, By Theorem 3.1 (2) we get

$$\|T_{\Phi} f\|_{L^{p,q}_{\alpha}} \leq B^2 C_{\Phi,p,\alpha} \|f\|_{L^{p,q}_{\alpha}}. \tag{3.6}$$

But $B(p, w) \leq p'$. So by (3.6)

$$\|T_{\Phi} f\|_{L^{p,q}_{\alpha}} \leq p'^2 C_{\Phi,p,\alpha} \|f\|_{L^{p,q}_{\alpha}}. \tag{3.7}$$

Now we verify the best constant. Let

$$f_{\epsilon}(x) = |x|_h^{-\frac{Q+\alpha}{p}+\epsilon} \chi_{\{|x|_h \leq 1\}}.$$

Then by polar coordinate transformation,

$$\begin{aligned} T_{\Phi}(f_{\epsilon})(x) &= \int_{\mathbb{H}^n} \frac{\Phi(\delta_{|y|_h^{-1}} x)}{|y|_h^Q} |y|_h^{-\frac{Q+\alpha}{p}+\epsilon} \chi_{\{|y|_h \leq 1\}}(y) dy \\ &= \int_0^{\infty} r^{Q-1} dr \int_{|y'|_h=1} \frac{\Phi(\delta_{r^{-1}} x)}{r^Q} r^{-\frac{Q+\alpha}{p}+\epsilon} \chi_{\{r \leq 1\}} dy' \\ &= \omega_Q \int_1^{\infty} r^{\frac{Q+\alpha}{p}-\epsilon-1} \Phi(\delta_r x) dr \\ &= \omega_Q |x|_h^{-\frac{Q+\alpha}{p}+\epsilon} \int_{|x|_h}^{\infty} r^{\frac{Q+\alpha}{p}-\epsilon-1} \Phi(r) dr. \end{aligned}$$

Therefore

$$\begin{aligned} \|T_{\Phi}(f_{\epsilon})\|_{L^{p,q}_{\alpha}} &\geq \left\| T_{\Phi}(f_{\epsilon}) \chi_{|x|_h < \epsilon} \right\|_{L^{p,q}_{\alpha}} \\ &\geq \left\| \omega_Q |x|_h^{-\frac{Q+\alpha}{p}+\epsilon} \chi_{|x|_h < \epsilon} \int_{\epsilon}^{\infty} r^{\frac{Q+\alpha}{p}-\epsilon-1} \Phi(r) dr \right\|_{L^{p,q}_{\alpha}} \\ &= \omega_Q \int_{\epsilon}^{\infty} r^{\frac{Q+\alpha}{p}-\epsilon-1} \Phi(r) dr \left\| |x|_h^{-\frac{Q+\alpha}{p}+\epsilon} \chi_{|x|_h < \epsilon} \right\|_{L^{p,q}_{\alpha}}. \end{aligned}$$

But in view of (3.1) (The inequality of (3.1) turns to equality if $\Lambda^p_u(w) = L^{p,q}_{\alpha}$)

$$\left\| |x|_h^{-\frac{Q+\alpha}{p}+\epsilon} \chi_{|x|_h < \epsilon} \right\|_{L^{p,q}_{\alpha}}$$

$$
\begin{aligned}
&= \epsilon^{-\frac{Q+\alpha}{p}+\epsilon} \left\| |x/\epsilon|_h^{-\frac{Q+\alpha}{p}+\epsilon} \chi_{|x/\epsilon|_h<1} \right\|_{L_\alpha^{p,q}} \\
&= \epsilon^{-\frac{Q+\alpha}{p}+\epsilon} \epsilon^{\frac{Q+\alpha}{p}} \|f_\epsilon\|_{L_\alpha^{p,q}} \\
&= \epsilon^{\epsilon} \|f_\epsilon\|_{L_\alpha^{p,q}}.
\end{aligned}
$$

Thus

$$\|T_\Phi(f_\epsilon)\|_{L_\alpha^{p,q}} \geq \omega_Q \epsilon^\epsilon \int_\epsilon^\infty r^{\frac{Q+\alpha}{p}-\epsilon-1} \Phi(r) dr \|f_\epsilon\|_{L_\alpha^{p,q}}.$$

Letting $\epsilon \to 0$ we obtain

$$\|T_\Phi\|_{L_\alpha^{p,q} \to L_\alpha^{p,q}} \geq \omega_Q \int_0^\infty r^{\frac{Q+\alpha}{p}-1} \Phi(r) dr.$$

Combining with (3.5) and (3.7), we complete the proof. □

THEOREM 3.3. *Let* $W_1(t) = \int_0^t W^{-1/p}(s) ds$. *Then*

$$\|T_\Phi f\|_{\Lambda^{p,\infty}_{|\cdot|^\alpha}(w) \to \Lambda^{p,\infty}_{|\cdot|^\alpha}(w)} \leq D_1,$$

where

$$D_1 = \left\| \frac{\overline{W_1}(\omega_Q |x|_h^{Q+\alpha})}{|x|_h^Q} \right\|_{\Lambda^{p,\infty}_{|\cdot|^\alpha}(w)} \left\| \int_0^\infty W_1 \left(\int_0^\infty \frac{1}{\zeta^{Q+1-\alpha}} \chi_{\{|\Phi(\delta_\zeta x'^Q > s\}} d\zeta \right) ds \right\|_{L^\infty(S^{Q-1})}.$$

Proof. Using the Hölder inequality of weighted Lorentz spaces and noting that $(\Lambda_u^{p,\infty})' = \Lambda_u^1(W^{-1/p})$, we get

(3.8) $|T_\Phi(f)(x)| \leq \|g_x\|_{(\Lambda^{p,\infty}_{|\cdot|^\alpha}(w))'} \|f\|_{\Lambda^{p,\infty}_{|\cdot|^\alpha}(w)} = \|g_x\|_{\Lambda^1_{|\cdot|^\alpha}(W^{-1/p})} \|f\|_{\Lambda^{p,\infty}_{|\cdot|^\alpha}(w)}$

where $g_x(y) = \frac{\Phi(\delta_{|y|_h^{-1}} x)}{|y|_h^Q}$. But

$$\|g_x\|_{\Lambda^1_{|\cdot|^\alpha}(W^{-1/p})} = \int_0^\infty W_1(\lambda^u_{g_x}(t)) dt.$$

Note that

$$
\begin{aligned}
\lambda^u_{g_x}(t) &= \omega_Q |x|_h^Q \int_0^\infty \frac{1}{\zeta^{Q+1}} \chi_{\{|\Phi(\delta_\zeta x')|\zeta^Q > t|x|_h^Q\}} \left(\frac{|x|_h}{\zeta} \right)^\alpha d\zeta \\
&= \omega_Q |x|_h^{Q+\alpha} \int_0^\infty \frac{1}{\zeta^{Q+1+\alpha}} \chi_{\{|\Phi(\delta_\zeta x')|\zeta^Q > t|x|_h^Q\}} d\zeta.
\end{aligned}
$$

Therefore

$$\|g_x\|_{\Lambda^1_{|\cdot|^\alpha}(W^{-1/p})} = \int_0^\infty W_1 \left(\omega_Q |x|_h^{Q+\alpha} \int_0^\infty \frac{1}{\zeta^{Q+1+\alpha}} \chi_{\{|\Phi(\delta_\zeta x')|\zeta^Q > t|x|_h^Q\}} d\zeta \right) dt$$

$$\leq \quad \frac{\overline{W_1}(\omega_Q|x|_h^{Q+\alpha})}{|x|_h^Q}\int_0^\infty W_1\left(\int_0^\infty \frac{1}{\zeta^{Q+1+\alpha}}\chi_{\{|\Phi(\delta_\zeta x')|\zeta^Q>s\}}d\zeta\right)ds.$$

Thus by (3.8) the theorem is proved. □

If $w=1$, $\alpha=0$, then by some computations we obtain the following

Corollary 3.1. *Let $p>1$. Then*

$$\|T_\Phi f\|_{L^{p,\infty}\to L^{p,\infty}} \leq \frac{p'^2\omega_Q}{Q^{1/p}}D_2,$$

where

$$D_2=\left\|\int_0^\infty\left(\int_0^\infty \frac{1}{\zeta^{Q+1}}\chi_{\{|\Phi(\delta_\zeta x'^Q>s\}}d\zeta\right)^{1/p'}ds\right\|_{L^\infty(S^{Q-1})}.$$

Proof. Notice that $W_1(t)=\overline{W}_1(t)=p't^{1/p'}$ and

$$\left\|\frac{\overline{W_1}(\omega_Q|x|_h^Q)}{|x|_h^Q}\right\|_{L^{p,\infty}}=\frac{p'\omega_Q}{Q^{1/p}}.$$

□

Next we consider the Hausdorff operator $H_{\Phi,A}$ defined by (1.7). Let the norm of a matrix B be defined by

$$\|B\|=\sup_{x\in\mathbb{H}^n,x\neq 0}\frac{|Bx|_h}{|x|_h}.$$

Then

$$\|B\|^{-Q}\leq|\det(B^{-1})|\leq\|B^{-1}\|^Q. \tag{3.9}$$

Indeed, since $|Bx|_h\leq\|B\||x|_h$ which implies that $\|B\|^{-1}|x|_h\leq|B^{-1}x|_h\leq\|B^{-1}\||x|_h$, it follows that

$$\{x\in\mathbb{H}^n:\|B^{-1}\||x|_h\leq 1\}\leq\{x\in\mathbb{H}^n:|B^{-1}x|_h\leq 1\}$$
$$\leq\{x\in\mathbb{H}^n:\|B\|^{-1}|x|_h\leq 1\}.$$

Thus

$$\|B^{-1}\|^{-Q}\nu_Q\leq|\det(B)|\nu_Q\leq\|B\|^Q\nu_Q,$$

which yields (3.9).

Denote $F_{w,p,u,A,n}(y)=\overline{W}^{1/p}(\overline{u}(\|A(y)^{-1}\|)\|A(y)^{-1}\|^Q)$.

THEOREM 3.4. *Let $p \geq 1$, Φ be nonnegative, u be radial and increasing.*

(1) If w is decreasing, then

$$\|H_{\Phi,A}f\|_{\Lambda_u^p(w)\to\Lambda_u^p(w)} \leq E_1,$$

and

$$\|H_{\Phi,A}f\|_{\Lambda_u^{p,\infty}(w)\to\Lambda_u^{p,\infty}(w)} \leq E_1,$$

where

$$E_1 = \int_{\mathbb{H}^n} \frac{\Phi(y)}{|y|_h^Q} F_{w,p,u,A,n}(y)dy.$$

(2) If $w \in B_p$, then

$$\|H_{\Phi,A}f\|_{\Lambda_u^p(w)\to\Lambda_u^p(w)} \leq B(p,w)E_1,$$

$$\|H_{\Phi,A}f\|_{\Lambda_u^{p,\infty}(w)\to\Lambda_u^{p,\infty}(w)} \leq \overline{B(p,w)}E_1.$$

Proof. (1) Since u is radial and increasing,

$$\begin{aligned}
\lambda^u_{|f(A(y)x)|}(t) &= \int_{|f(A(y)x)|>t} u(x)dx = \int_{|f(z)|>t} u(|A(y)^{-1}z|_h)|det(A(y)^{-1})|dz \\
&\leq \int_{|f(z)|>t} u(\|A(y)^{-1}\||z|_h)|det(A(y)^{-1})|dz \\
&\leq \int_{|f(z)|>t} \overline{u}(\|A(y)^{-1}\|)u(|z|_h)|det(A(y)^{-1})|dz \\
&= \overline{u}(\|A(y)^{-1}\|)|det(A(y)^{-1})|\lambda_f^u(t) \\
&= \overline{u}(\|A(y)^{-1}\|)\frac{1}{|det(A(y)|}\lambda_f^u(t). \qquad (3.10)
\end{aligned}$$

Note that by the Minkowski inequality since w is decreasing,

$$\|H_{\Phi,A}(f)\|_{\Lambda_u^p(w)} \leq \int_{\mathbb{H}^n} \frac{\Phi(y)}{|y|_h^Q}\|f(A(y)x)\|_{\Lambda_u^p(w)}dy.$$

But by (3.10)

$$\|f(A(y)x)\|^p_{\Lambda_u^p(w)} = p\int_0^\infty t^{p-1}W(\lambda^u_{|f(A(y)x)|})(t)dt$$

$$\leq \overline{W}\left(\overline{u}(\|A(y)^{-1}\|)\frac{1}{|det(A(y))|}\right)\|f\|^p_{\Lambda_u^p(w)} \qquad (3.11)$$

and

$$\|f(A(y)x)\|^p_{\Lambda^{p,\infty}_u(w)} = \sup_{t>0} t^p W(\lambda^u_{|f(A(y)x)|})(t)$$

$$\leq \overline{W}\left(\overline{u}(\|A(y)^{-1}\|)\frac{1}{|det(A(y))|}\right)\|f\|^p_{\Lambda^{p,\infty}_u(w)}. \tag{3.12}$$

Thus by (3.11) and (3.12)

$$\|H_{\Phi,A}(f)\|_{\Lambda^p_u(w)} \leq \int_{\mathbb{H}^n} \frac{\Phi(y)}{|y|_h^Q}\overline{W}^{1/p}\left(\overline{u}(\|A(y)^{-1}\|)\frac{1}{|det(A(y))|}\right)dy\|f\|_{\Lambda^p_u(w)},$$

and

$$\|H_{\Phi,A}(f)\|_{\Lambda^{p,\infty}_u(w)} \leq \int_{\mathbb{H}^n} \frac{\Phi(y)}{|y|_h^Q}\overline{W}^{1/p}\left(\overline{u}(\|A(y)^{-1}\|)\frac{1}{|det(A(y))|}\right)dy\|f\|_{\Lambda^{p,\infty}_u(w)}.$$

By (3.9), the result of (1) follows.

(2) Since $w \in B_p$, we get $\Lambda^p(w) = \Gamma^p(w)$, $\Lambda^{p,\infty}(w) = \Gamma^{p,\infty}(w)$, and $\Lambda^p(w)$, $\Lambda^{p,\infty}(w)$ can be normable with norm $\|\cdot\|_{\Gamma^p(w)}$ and $\|\cdot\|_{\Gamma^{p,\infty}(w)}$:

$$\|\cdot\|_{\Lambda^p(w)} \leq \|\cdot\|_{\Gamma^p(w)} \leq B(p,w)\|\cdot\|_{\Lambda^p(w)}$$

and

$$\|\cdot\|_{\Lambda^{p,\infty}(w)} \leq \|\cdot\|_{\Gamma^{p,\infty}(w)} \leq \overline{B(p,w)}\|\cdot\|_{\Lambda^{p,\infty}(w)}.$$

Thus by the same procedure of case (1), the result follows:

$$\begin{aligned}
\|H_{\Phi,A}(f)\|_{\Lambda^p_u(w)} &\leq \|H_{\Phi,A}(f)\|_{\Gamma^p_u(w)} \\
&\leq \int_{\mathbb{H}^n} \frac{\Phi(y)}{|y|_h^Q}\|f(A(y)x)\|_{\Gamma^p_u(w)}dy \\
&\leq B(p,w)\int_{\mathbb{H}^n} \frac{\Phi(y)}{|y|_h^Q}\|f(A(y)x)\|_{\Lambda^p_u(w)}dy \\
&\leq B(p,w)\int_{\mathbb{H}^n} \frac{\Phi(y)}{|y|_h^Q}F_{w,p,u,A,n}(y)dy\|f\|_{\Lambda^p_u(w)}
\end{aligned}$$

and

$$\|H_{\Phi,A}(f)\|_{\Lambda^{p,\infty}_u(w)} \leq \overline{B(p,w)}\int_{\mathbb{H}^n} \frac{\Phi(y)}{|y|_h^Q}F_{w,p,u,A,n}(y)dy\|f\|_{\Lambda^{p,\infty}_u(w)}.$$

□

THEOREM 3.5. *Let Φ be nonnegative, $p = q = 1$ or $1 < p < \infty,\ 1 \leq q \leq \infty$, $\alpha > 0$.*

(1) If $p \geq q$, then

$$\|H_{\Phi,A}\|_{L^{p,q}_\alpha \to L^{p,q}_\alpha} \leq E_2 \tag{3.13}$$

and if $p < q$, then

$$\|H_{\Phi,A}\|_{L^{p,q}_\alpha \to L^{p,q}_\alpha} \leq p' E_2, \tag{3.14}$$

where

$$E_2 = \int_{\mathbb{H}^n} \frac{\Phi(y)}{|y|_h^Q} \|A(y)^{-1}\|^{\frac{Q+\alpha}{p}} dy.$$

(2) On the contrary,

$$\|H_{\Phi,A}\|_{L^{p,q}_\alpha \to L^{p,q}_\alpha} \geq E_3 \tag{3.15}$$

where

$$E_3 = \int_{\mathbb{H}^n} \frac{\Phi(y)}{|y|_h^Q} \|A(y)\|^{\frac{-(Q+\alpha)}{p}} dy.$$

Especially, suppose that $\|A(y)^{-1}\| \leq C_1 \|A(y)\|^{-1}$ for all $y \in \mathbb{H}^n$. If $p \geq q$, then

$$C_1^{-\frac{Q+\beta}{p}} E_4 \leq \|H_{\Phi,A}\|_{L^{p,q}_\alpha \to L^{p,q}_\alpha} \leq C_1^{\frac{Q+\beta}{p}} E_4$$

and if $p < q$, then

$$C_1^{-\frac{Q+\beta}{p}} E_4 \leq \|H_{\Phi,A}\|_{L^{p,q}_\alpha \to L^{p,q}_\alpha} \leq p' C_1^{\frac{Q+\beta}{p}} E_4,$$

where

$$E_4 = \int_{\mathbb{H}^n} \frac{\Phi(y)}{|y|_h^Q} \left(\frac{1}{|det(A(y))|^{1+\frac{\alpha}{Q}}} \right)^{1/p} dy.$$

Proof. (1) Noticing $\overline{u}(s) = s^\alpha$, $\overline{W}(t) = t^{\frac{q}{p}}$, $B(p, w) = p'$ and the fact that $w(t) = t^{\frac{q}{p}-1}$ is decreasing if $1 \leq q \leq p < \infty$ and $w \in B_p$ if $1 < p < q < \infty$, we get (1) for the case $p = q = 1$ or $1 < p < \infty,\ 1 \leq q < \infty$ by Theorem 3.4. When $1 < p < \infty,\ q = \infty$, it follows that $1 \in B_p$ and $\overline{B(p, w)} = p'$ which also implies (1) by Theorem 3.4.

(2) Consider necessity. Let

$$f_k(x) = |x|_h^\beta \chi_{\{|x|_h \leq 1\}},$$

where $\beta = -\frac{Q+\alpha}{p} + \frac{1}{k}$. Then

$$\begin{aligned}
&\lambda^u_{H_\Phi f_k}(t) = u\left\{x : \left|\int_{\mathbb{H}^n} \frac{\Phi(y)}{|y|_h^Q} f_k(A(y)x)dy\right| > t\right\} \\
&= u\left\{x : \int_{\mathbb{H}^n} \frac{\Phi(y)}{|y|_h^Q} |A(y)x|_h^\beta \chi_{|A(y)x|_h \le 1} dy > t\right\} \\
&\ge u\left\{x : \int_{\mathbb{H}^n} \frac{\Phi(y)}{|y|_h^Q} (\|A(y)\||x|_h)^\beta \chi_{\|A(y)\||x|_h \le 1} dy > t\right\} \\
&\ge u\left\{x : \int_{\|A(y)\| \le 1/|x|_h} \frac{\Phi(y)}{|y|_h^Q} (\|A(y)\||x|_h)^\beta \chi_{\|A(y)\||x|_h \le 1} dy \chi_{|x|_h < 1/k} > t\right\} \\
&\ge u\left\{x : \int_{\|A(y)\| \le k} \frac{\Phi(y)}{|y|_h^Q} \|A(y)\|^\beta dy |x|_h^\beta| \chi_{|x|_h < 1/k} > t\right\} \\
&= \lambda^u_{f_k(k\cdot)|}\left(\frac{k^\beta t}{A_k}\right) \\
&= k^{-(\alpha+Q)} \lambda^u_{f_k}\left(\frac{k^\beta t}{A_k}\right),
\end{aligned}$$

where $A_k = \int_{\|A(y)\| \le k} \frac{\Phi(y)}{|y|_h^Q} \|A(y)\|^\beta dy$. Thus if $q \ne \infty$,

$$\begin{aligned}
\|H_\Phi f_k\|_{L^{p,q}_\alpha} &= \left(p\int_0^\infty t^{q-1} (\lambda^u_{|H_\Phi f_k|})^{q/p}(t)dt\right)^{1/q} \\
&\ge \left(pk^{-\frac{q(\alpha+Q)}{p}} \int_0^\infty t^{q-1} \left(\lambda^u_{f_k}\left(\frac{k^\beta t}{A_k}\right)\right)^{q/p} dt\right)^{1/q} \\
&= A_k k^{-\beta - \frac{Q+\alpha}{p}} \|f_k\|_{L^{p,q}_\alpha} = A_k k^{-\frac{1}{k}} \|f_k\|_{L^{p,q}_\alpha},
\end{aligned}$$

and if $q = \infty$,

$$\|H_\Phi f_k\|_{L^{p,\infty}_\alpha} = \sup_{t>0} t(\lambda^u_{|H_\Phi f_k|})^{\frac{1}{p}}(t) \ge A_k k^{-\frac{1}{k}} \|f_k\|_{L^{p,\infty}_\alpha},$$

i.e.,

$$\|H_{\Phi,A}\|_{L^{p,q}_\alpha \to L^{p,q}_\alpha} \ge A_k \frac{1}{k^{1/k}}. \tag{3.16}$$

Letting $k \to \infty$ in (3.16), we get (3.15).

If $\|A(y)^{-1}\| \le C_1\|A(y)\|^{-1}$ for all $y \in \mathrm{supp}\Phi$, then by (3.9)
(3.17)
$$\frac{1}{C_1}\left(\frac{1}{|\det(A(y))|}\right)^{1/Q} \le \|A(y))\|^{-1} \le \|A(y)^{-1}\| \le C_1\left(\frac{1}{|\det(A(y))|}\right)^{1/Q}.$$
Combining with (3.13), (3.14), (3.15) and (3.17), we come to the conclusion. □

REMARK 3.1. Suppose that $A(y) = \mathrm{diag}[\lambda_1(y), ..., \lambda_{2n}(y), \lambda_{2n+1}(y)]$ with $\lambda_i(y) \ne 0$. Let
$$M(y) = \max\{|\lambda_1(y)|, ..., |\lambda_{2n}(y)|, |\lambda_{2n+1}(y)|^{1/2}\},$$
$$m(y) = \min\{|\lambda_1(y)|, ..., |\lambda_{2n}(y)|, |\lambda_{2n+1}(y)|^{1/2}\}.$$
Then $\|A(y)^{-1}\| = \frac{1}{m(y)}$ and $\|A(y)\| = M(y)$. If $M(y) \le Cm(y)$ for some $C \ge 1$ and all y, Then $A(y)$ satisfies the conditions of Theorem 3.5. Particularly, if we take $\lambda_i(y) = \frac{1}{\|y\|_h}$, $i = 1, ..., 2n$, and $\lambda_{2n+1}(y) = \frac{1}{\|y\|_h^2}$, then $M(y) = m(y)$, $A(y)x = \delta_{\|y\|_h^{-1}}x$ and the operator $H_{\Phi,A}$ reduces to

$$H_\Phi f(x) = \int_{\mathbb{H}^n} \frac{\Phi(y)}{|y|_h^Q} f(\delta_{\|y\|_h^{-1}}x)dy \tag{3.18}$$

whose $L^p(\mathbb{H}^n)$ boundedness was studied in [31]. In the light of Theorem 3.5, it follows that if $1 \le q \le p < \infty$, then
$$\|H_\Phi\|_{L^{p,q}_\alpha \to L^{p,q}_\alpha} = E_5$$
and if $1 < p < q \le \infty$, then
$$E_5 \le \|H_\Phi\|_{L^{p,q}_\alpha \to L^{p,q}_\alpha} \le p' E_5,$$
where
$$E_5 = \int_{\mathbb{H}^n} \Phi(y)|y|_h^{-\frac{Q}{p'}+\frac{\alpha}{p}} dy.$$

We also can study the product Hausdorff operator on the Heisenburg group, generalizing the results in Euclidean spaces (see [36]). Let $Q_n = 2n+2$, $n \in \mathbb{N}$. Let Φ be a locally integrable function on $\mathbb{H}^{n_1} \times \mathbb{H}^{n_2}$. The Hausdorff operator $\mathcal{H}_{\Phi,A_1,A_2}$ is defined by
$$\mathcal{H}_{\Phi,A_1,A_2} f(x,y) = \int_{\mathbb{H}^{n_1} \times H^{n_2}} \frac{\Phi(\xi,\eta)}{|\xi|^{Q_{n_1}}|\eta|^{Q_{n_2}}} f(A_1(\xi)x, A_2(\eta)y)d\xi d\eta.$$

THEOREM 3.6. *Let Φ be nonnegative, $p_i \geq 1$, u_i be radial and increasing, $i = 1,\ 2$.*

(1) If w_i, $i = 1,\ 2$, are decreasing, then we have

$$\|\mathcal{H}_{\Phi,A_1,A_2} f\|_{\Lambda_{u_1}^{p_1}(w)\times\Lambda_{u_2}^{p_2}(w)\to\Lambda_{u_1}^{p_1}(w)\times\Lambda_{u_2}^{p_2}(w)} \leq \widetilde{E_1} \tag{3.19}$$

where

$$\widetilde{E_1} = \int_{\mathbb{H}^{n_1}\times\mathbb{H}^{n_2}} \frac{\Phi(\xi,\eta)}{|\xi|^{Q_{n_1}}|\eta|^{Q_{n_2}}} F_{w_1,p_1,u_1,A_1,n_1}(\xi) F_{w_2,p_2,u_2,A_2,n_2}(\eta) d\xi d\eta.$$

(2) If $w_i \in B_{p_i}$, $i = 1,\ 2$, then we have

$$\|\mathcal{H}_{\Phi,A_1,A_2} f\|_{\Lambda_{u_1}^{p_1}(w)\times\Lambda_{u_2}^{p_2}(w)\to\Lambda_{u_1}^{p_1}(w)\times\Lambda_{u_2}^{p_2}(w)} \leq B(p_1,w_1)B(p_2,w_2)\widetilde{E_1},$$

and

$$\|\mathcal{H}_{\Phi,A_1,A_2} f\|_{\Lambda_{u_1}^{p_1,\infty}(w)\times\Lambda_{u_2}^{p_2,\infty}(w)\to\Lambda_{u_1}^{p_1,\infty}(w)\times\Lambda_{u_2}^{p_2,\infty}(w)} \leq \overline{B(p_1,w_1)B(p_2,w_2)}\widetilde{E_1}.$$

Proof. (1) Notice that if w_i, $i = 1,\ 2$, are decreasing, then by the Minkowski inequality

$$\|\mathcal{H}_{\Phi,A_1,A_2} f\|_{\Lambda_{u_1}^{p_1}(w_1)\times\Lambda_{u_2}^{p_2}(w_2)} = \|\|\mathcal{H}_{\Phi,A_1,A_2} f(x,y)\|_{\Lambda_{u_1}^{p_1}(w_1)}\|_{\Lambda_{u_2}^{p_2}(w_2)}$$
$$\leq \int_{\mathbb{H}^{n_1}\times\mathbb{H}^{n_2}} \frac{\Phi(\xi,\eta)}{|\xi|^{Q_{n_1}}|\eta|^{Q_{n_2}}} \|\|f(A(\xi)x, B(\eta)y)\|_{\Lambda_{u_1}^{p_1}(w_1)}\|_{\Lambda_{u_2}^{p_2}(w_2)} d\xi d\eta.$$

But by Theorem 3.4 (1),

$$\|\|f(A(\xi)x, B(\eta)y)\|_{\Lambda_{u_1}^{p_1}(w_1)}\|_{\Lambda_{u_2}^{p_2}(w_2)}$$
$$\leq \|\|f(x,y)\|_{\Lambda_{u_1}^{p_1}(w_1)}\|_{\Lambda_{u_2}^{p_2}(w_2)} F_{w_1,p_1,u_1,A_1,n_1}(\xi) F_{w_2,p_2,u_2,A_2,n_2}(\eta).$$

Thus (3.19) holds.

(2) Using the following fact: if $w \in B_p$, then $\Lambda^p(w) = \Gamma^p(w)$ and $\Lambda^p(w)$ can be given a norm $\|\cdot\|_{\Gamma^p(w)}$ with the relation

$$\|\cdot\|_{\Lambda^p(w)} \leq \|\cdot\|_{\Gamma^p(w)} \leq B(p,w)\|\cdot\|_{\Lambda^p(w)}$$

and the result of (1), we get (2). The result on the space $\Lambda_{u_1}^{p_1,\infty}(w_1) \times \Lambda_{u_2}^{p_2,\infty}(w_2)$ can be similarly obtained.

□

THEOREM 3.7. *Let* $1 < p_i < \infty,\ 1 \le q_i \le \infty$, $i = 1,\ 2$, $\alpha > 0$.
(1) If $p_i \ge q_i$, *then*

$$\|\mathcal{H}_{\Phi,A_1,A_2}\|_{L^{p_1,q_1}_{\alpha_1}\times L^{p_2,q_2}_{\alpha_2}\to L^{p_1,q_1}_{\alpha_1}\times L^{p_2,q_2}_{\alpha_2}} \le \widetilde{E_2}; \tag{3.20}$$

and if $p_i < q_i$, *then*

$$\|\mathcal{H}_{\Phi,A_1,A_2}\|_{L^{p_1,q_1}_{\alpha_1}\times L^{p_2,q_2}_{\alpha_2}\to L^{p_1,q_1}_{\alpha_1}\times L^{p_2,q_2}_{\alpha_2}} \le p_1'p_2'\widetilde{E_2}, \tag{3.21}$$

where

$$\widetilde{E_2} = \int_{\mathbb{H}^{n_1}\times\mathbb{H}^{n_2}} \frac{\Phi(\xi,\eta)}{|\xi|^{Q_{n_1}}|\eta|^{Q_{n_2}}}\|A_1(\xi)^{-1}\|^{\frac{Q_{n_1}+\alpha_1}{p_1}}\|A_2(\eta)^{-1}\|^{\frac{Q_{n_2}+\alpha_2}{p_2}}d\xi d\eta.$$

(2) On the contrary,

$$\|\mathcal{H}_{\Phi,A_1,A_2}\|_{L^{p_1,q_1}_{\alpha_1}\times L^{p_2,q_2}_{\alpha_2}\to L^{p_1,q_1}_{\alpha_1}\times L^{p_2,q_2}_{\alpha_2}} \ge \widetilde{E_3} \tag{3.22}$$

where

$$\widetilde{E_3} = \int_{\mathbb{H}^{n_1}\times\mathbb{H}^{n_2}} \frac{\Phi(\xi,\eta)}{|\xi|^{Q_{n_1}}|\eta|^{Q_{n_2}}}\|A_1(\xi)\|^{-\frac{Q_{n_1}+\alpha_1}{p_1}}\|A_2(\eta)\|^{-\frac{Q_{n_2}+\alpha_2}{p_2}}d\xi d\eta.$$

Especially, assume that $\|A_i(y)^{-1}\| \le C_i\|A_i(y)\|^{-1}$, $i = 1,\ 2$, *for all* $y \in \mathbb{H}^n$. *then if* $p_i \ge q_i$, *we have that*

$$\Pi_{i=1}^2 C_i^{-\frac{Q_{n_i}+\beta_i}{p_i}}\widetilde{E_4} \le \|\mathcal{H}_{\Phi,A_1,A_2}\|_{L^{p_1,q_1}_{\alpha_1}\times L^{p_2,q_2}_{\alpha_2}\to L^{p_1,q_1}_{\alpha_1}\times L^{p_2,q_2}_{\alpha_2}} \le \Pi_{i=1}^2 C_i^{\frac{Q_{n_i}+\beta_i}{p_i}}\widetilde{E_4}; \tag{3.23}$$

and if $p_i < q_i$ *we have that*

$$\Pi_{i=1}^2 C_i^{-\frac{Q_{n_i}+\beta_i}{p_i}}\widetilde{E_4} \le \|\mathcal{H}_{\Phi,A_1,A_2}\|_{L^{p_1,q_1}_{\alpha_1}\times L^{p_2,q_2}_{\alpha_2}\to L^{p_1,q_1}_{\alpha_1}\times L^{p_2,q_2}_{\alpha_2}} \le \Pi_{i=1}^2 p_i' C_i^{\frac{Q_{n_i}+\beta_i}{p_i}}\widetilde{E_4}, \tag{3.24}$$

where

$$\widetilde{E_4} = \int_{\mathbb{H}^{n_1}\times\mathbb{H}^{n_2}} \frac{\Phi(\xi,\eta)}{|\xi|^{Q_{n_1}}|\eta|^{Q_{n_2}}}\left(\frac{1}{|det(A_1(\xi))|^{1+\frac{\alpha_1}{Q_{n_1}}}}\right)^{1/p_1}\left(\frac{1}{|det(A_2(\eta))|^{1+\frac{\alpha_2}{Q_{n_2}}}}\right)^{1/p_2}d\xi d\eta.$$

Proof. (1) Noticing $\overline{u_i}(s) = s^\alpha$, $\overline{W_i}(t) = t^{\frac{q_i}{p_i}}$, and the fact that $w_i(t) = t^{\frac{q_i}{p_i}-1}$ is decreasing if $p_i \ge q_i$; and $w_i \in B_{p_i}$, $B(p_i, w_i) = p_i'$ if $p_i < q_i < \infty$, we get (1) by Theorem 3.6. By the same token the weak-type bound follows.

(2) Let $f_k(\xi)=|\xi|^{\beta_1}\chi_{|\xi|\le 1}$, $g_k(\eta)=|\eta|^{\beta_2}\chi_{|\eta|\le 1}$ and $h_k(\xi,\eta)=f_k(\xi)g_k(\eta)$ where $\beta_i=-\frac{Q_{n_i}+\alpha_i}{p}+\frac{1}{k}$. Thus by the proof of Theorem 3.5 (2)

$$\begin{aligned}
&\|\mathcal{H}_{\Phi,A_1,A_2}h_k\|_{L^{p_1,q_1}_{\alpha_1}\times L^{p_2,q_2}_{\alpha_2}}=\|\|\mathcal{H}_{\Phi,A_1,A_2}h_k\|_{L^{p_2,q_2}_{\alpha_2}}\|_{L^{p_1,q_1}_{\alpha_1}}\\
\ge\ &\left\|\int_{\|A_1(\xi)\|\le k}\frac{\|A_1(\xi)\|^{\beta_1}}{|\xi|^{n_1}}\int_{\mathbb{H}^n}\frac{\Phi(\xi,\eta)}{|\eta|^{Q_{n_2}}}g_k(A_2(\eta)y)d\eta d\xi\right\|_{L^{p_2,q_2}_{\alpha_2}}k^{-\frac{1}{k}}\|f_k\|_{L^{p_1,q_1}_{\alpha_1}}\\
=\ &\left\|\int_{\mathbb{H}^n}\frac{\int_{\|A_1(\xi)\|\le k}\Phi(\xi,\eta)\frac{\|A_1(\xi)\|^{\beta_1}}{|\xi|^{Q_{n_1}}}d\xi}{|\eta|^{Q_{n_2}}}g_k(A_2(\eta)y)d\eta\right\|_{L^{p_2,q_2}_{\alpha_2}}k^{-\frac{1}{k}}\|f_k\|_{L^{p_1,q_1}_{\alpha_1}}\\
\ge\ &\int_{\|B(\eta)\|\le k}\frac{\int_{\|A_1(\xi)\|\le k}\Phi(\xi,\eta)\frac{\|A_1(\xi)\|^{\beta_1}}{|\xi|^{Q_{n_1}}}d\xi}{|\eta|^{Q_{n_2}}}\|A_2(\eta)\|^{\beta_2}d\eta k^{-\frac{1}{k}}k^{-\frac{1}{k}}\|g_k\|_{L^{p_2,q_2}_{\alpha_2}}\|f_k\|_{L^{p_1,q_1}_{\alpha_1}}.
\end{aligned}$$

Letting $k\to\infty$, we get (3.22). If $\|A_i(y)^{-1}\|\le C_i\|A_i(y)\|^{-1}$, $i=1,\ 2$, for all $y\in\mathbb{H}^n$, by (3.20), (3.21), (3.22) and (3.9), (3.23) and (3.24) follow. □

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Qinxiu Sun
Department of Mathematics
Zhejiang University of Science and Technology
Hangzhou, 310023, China
E-mail: `qxsun@126.com`

Dashan Fan
Department of Mathematics
University of Wisconsin-Milwaukee
Milwaukee, WI 53201, USA
E-mail: `fan@uwm.edu`

Hongliang Li
Department of Mathematics
Zhejiang International Studies University
Hangzhou, 310012, China
E-mail: `honglli@126.com`