BIFURCATION PROBLEM FOR THE SUPERLINEAR ELLIPTIC OPERATOR

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ABSTRACT. We investigate the number of solutions for the superlinear elliptic bifurcation problem with Dirichlet boundary condition. We get a theorem which shows the existence of at least k weak solutions for the superlinear elliptic bifurcation problem with boundary value condition. We obtain this result by using the critical point theory induced from invariant linear subspace and the invariant functional.

1. Introduction

Let Ω be a bounded domain in R^n, $(n \geq 3)$, with smooth boundary $\partial\Omega$. Let $a : \overline{\Omega} \to R$ be a continuous function and $g : R \to R$ be a C^1 function. In this paper we consider the number of the weak solutions of the following superlinear elliptic bifurcation problem with Dirichlet boundary condition

$$(1.1) \qquad \begin{aligned} -\Delta u &= \lambda a(x)u + g(u) \qquad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega. \end{aligned}$$

We assume that $a(x) > 0$ in $\overline{\Omega}$ and g satisfies the following:

(g1) $g \in C^1(R, R)$, $g(0) = 0$, $g(u) = o(\|u\|)$, $\|\cdot\|$ is the norm introduced in section 3.

(g2) There exist a constant $\beta \in]2, 2^*[$, $2^* = \frac{2n}{n-2}$, and $r_0 > 0$ such that

$$0 < \beta G(\xi) = \beta \int_0^\xi g(t)dt \leq \xi g(\xi) \qquad \text{for } |\xi| \geq r_0.$$

(g3) $g(u) \leq C_1 |u|^{\beta - 1}$ \qquad for $C_1 > 0$,

(g4) $g(-u) = -g(u)$.

We note that (g2) implies the existence of positive constants a_1, a_2, a_3 such that

(1.2) \qquad $\frac{1}{\beta}(\xi g(\xi) + a_1) \geq G(\xi) + a_2 \geq a_3 |\xi|^\beta$ \qquad for $\xi \in R$.

The eigenvalue problem

$$\Delta u + \lambda u = 0 \qquad \text{in } \Omega,$$

$$u = 0 \qquad \text{on } \partial\Omega$$

has infinitely many eigenvalues λ_j, $j \geq 1$ which is repeated as often as its multiplicity, and the corresponding eigenfunctions ϕ_j, $j \geq 1$ suitably normalized with respect to $L^2(\Omega)$ inner product. The eigenvalue problem

$$-\Delta u = \mu a(x) u \qquad \text{in } \Omega,$$

$$u = 0 \qquad \text{on } \partial\Omega,$$

has also infinitely many eigenvalues μ_j, $j \geq 1$ and corresponding eigenfunctions ψ_j, $j \geq 1$. We note that $\mu_1 < \mu_2 \leq \mu_3 \ldots$, $\mu_j \to +\infty$.

This type bifurcation problem was considered by some authors ([1],[3]). Rabinowitz [3] showed that if $g \in C^1(\overline{\Omega}, R)$, g satisfies $g(\xi) = o(|\xi|)$ as $\xi \to 0$, $a(x)$ is continuous and $a(x) > 0$ in $\overline{\Omega}$, then (1.1) has several kinds of solutions under some additional assumptions. He proved this result by the critical point theory and the variational method.

Chang [1] also proved that (1.1) has at least k solutions under the conditions $(g1) - g(3)$

$(g1)'$ There exists $\xi \geq 0$ such that $a(x)\xi + g(\xi) \leq 0 \; \forall x \in \overline{\Omega}$,

$(g2)'$ $g \in C^1(\overline{\Omega}, R)$, g satisfies $g(\xi) = o(|\xi|)$ as $\xi \to 0$, $a(x)$ is continuous on $\overline{\Omega}$ and $a(x) > 0$,

$(g3)'$ $g(-\xi) = -g(\xi)$.

He proved this result by the critical point theory.

Jung and Choi [2] investigated the multiple solutions for the following nonlinear elliptic equation with jumping nonlinearity and subcritical growth nonlinearity and Dirichlet boundary condition

$$(1.3) \qquad \begin{aligned} \Delta u + bu_+ - pu_-^{p-1} &= h(x) \qquad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega, \end{aligned}$$

where $2 < p < 2^*$, $2^* = \frac{2n}{n-2}$, $n \geq 3$, $u_+ = \max\{u, 0\}$, $u_- = -\min\{u, 0\}$, $u(x) \in W_0^{1,2}(\Omega)$ and $h(x) \in L^s(\Omega)$ for some $s > n$. They first showed the existence of a positive solution and next found the second nontrivial solution by applying the variational method and the mountain pass method in the critical point theory. By investigating that the functional I satisfies the mountain pass geometry they show the existence of at least two solutions for the equation.

Our main result is the following.

THEOREM 1.1. *Assume that* $a : \overline{\Omega} \to R$ *is a continuous function,* $a(x) > 0$, g *satisfies the conditions* $(g1) - (g4)$ *and* $\mu_k < \lambda < \mu_{k+1}$, $k \geq 1$. *Then (1.1) has at least* k *weak solutions.*

We prove Theorem 1.1 by the critical point theory induced from the invariant subspace and invariant functional. The outline of the proof of Theorem 1.1 is as follows: In section 2, we introduce a Hilbert space H and closed invariant linear subspaces of H which are invariant under the operator $u \mapsto \int_\Omega |\nabla u|^2 dx$, and the invariant functional on H. We obtain some results on the norm $\|\cdot\|$ and the functional $I(u)$, and recall a critical point theory in terms of the invariant functional and invariant subspaces which plays a crucial role for the proof of the main result. In section 3, we prove Theorem 1.1.

2. Recall of the critical point theory

Let E be a real Hilbert space on which the action Z_2 acts orthogonally. For $u \in E$, we define Z_2- actions on E by

$$Tu = u \quad \text{or} \quad Tu = -u,$$

that is, the Z_2 action have the identity map and the antipodal map as an action. Thus Z_2-action acts freely on the subspace $\{u| \; Tu = -u\}$. Let Fix_{Z_2} be the set of fixed points of the action, i.e.,

$$\text{Fix}_{Z_2} = \{u \in E| \; Tu(x) = u(x), \text{ for all } \; x \in \Omega, \; u \in E, \; Z_2 - \text{action } T\}.$$

We note that $\text{Fix}_{Z_2} = \{0\}$. Let

$$X_1 = \text{Fix}_{Z_2} = \{0\} \qquad X_2 = X_1^{\perp}.$$

Thus Z_2- action has the representation $x \mapsto -x$, for $x \in X_2$ and $E = X_1 \oplus X_2$. We say a subset B of E an Z_2-invariant set if for all $u \in B$, $Tu \in B$. A function $I : E \to R^1$ is called Z_2-invariant if $I(Tu) = I(u)$, $\forall u \in E$. Let $C(B, E)$ be the set of continuous functions from B into E. If B is an invariant set we say $h \in C(B, E)$ is an equivariant map if $h(Tu) = Th(u)$ for all $u \in B$.

Now we recall the critical point theory in terms of the invariant subspace and invariant functional which is proved in Theorem 4.1 of [1] which plays a crucial role for the proof of Theorem 1.1: Let S_ρ be the sphere centered at the origin of radius ρ. Let $I : E \to R$ be a functional of the form

$$I(u) = \frac{1}{2}(Lu)u - F(u),$$

where $L : E \to E$ is linear, continuous, symmetric and equivariant, $F : E \to R$ is of class C^1 and invariant and $DF : E \to E$ is compact.

THEOREM 2.1. *Assume that $I \in C^1(E, R^1)$ is Z_2-invariant and there exist two closed invariant linear subspaces V, W of E and $\rho > 0$ with the following properties:*

(a) *$V + W$ is closed and of finite codimension in E;*

(b) *$\text{Fix}_{Z_2} \subseteq V + W$;*

(c) *$L(W) \subseteq W$;*

(d) $\sup_{S_\rho \cap V} I < +\infty$ and $\inf_W I > -\infty$;

(e) $u \notin Fix_{Z_2}$ whenever $DI(u) = 0$ and

$$\inf_W I \leq I(u) \leq \sup_{S_\rho \cap V} I.$$

(f) I satisfies $(P.S.)_c$ condition whenever $\inf_W I \leq c \leq \sup_{S_\rho \cap V} I$.
Then I possesses at least

$$dim(V \cap W) - codim_H(V + W)$$

distinct critical orbits in $I^{-1}([\inf_W I, \sup_{S_\rho \cap V} I])$.

3. Proof of theorem 1.1

Let $L^2(\Omega)$ be a square integrable function space defined on Ω. Any element u in $L^2(\Omega)$ can be written as

$$u = \sum h_k \phi_k \quad \text{with} \quad \sum h_k^2 < \infty.$$

We define a subspace H of $L^2(\Omega)$ as follows

(3.1) $$H = \{u \in L^2(\Omega) | \sum |\mu_k| h_k^2 < \infty\}.$$

Then this is a complete normed space with a norm

$$\|u\| = [\sum |\mu_k| h_k^2]^{\frac{1}{2}}.$$

Since $\mu_k \to +\infty$, we have

(i) $-\Delta u \in H$ implies $u \in H$,

(ii) $\|u\| \geq C\|u\|_{L^2(\Omega)}$ for some $C > 0$,

(iii) $\|u\|_{L^2(\Omega)} = 0$ if and only if $\|u\| = 0$,

which can be proved easily.

We note that H in (3.1) is a real Hilbert space on which the action Z_2 acts orthogonally. For $u \in H$, we define Z_2- actions on H by

$$Tu = u \quad \text{or} \quad Tu = -u,$$

that is, the Z_2-action have the identity map and the antipodal map as an action. Thus Z_2-action acts freely on the subspace $\{u| Tu = -u\}$.

Let Fix_{Z_2} be the set of fixed points of the action, i.e.,

$\mathrm{Fix}_{Z_2} = \{u \in H \mid Tu(x) = u(x), \text{ for all } x \in \Omega, u \in H, Z_2 - \text{action } T\}$.

We note that $\mathrm{Fix}_{Z_2} = \{0\}$. Let

$$X_1 = \mathrm{Fix}_{Z_2} = \{0\} \qquad X_2 = X_1^{\perp}.$$

Thus Z_2- action has the representation $x \mapsto -x$, for $x \in X_2$ and $H = X_1 \oplus X_2$. Let

$$(Lu)h = \int_{\Omega} [\Delta u \cdot \Delta h - c\nabla u \cdot \nabla h]dx.$$

We can check easily that $L(H) \subseteq H$, $L : H \to H$ is an isomorphism and $\nabla I(H) \subseteq H$. Therefore constrained critical points on H are in fact free critical points on H. Moreover, distinct critical orbits give rise to geometrically distinct solutions.

We are looking for the weak solutions of (1.1). By the following Proposition 3.1, the weak solutions of (1.1) coincide with the critical points of the associated functional

$$
\begin{aligned}
I(u) \quad &\in \quad C^1(H, R), \\
(3.2) \qquad I(u) \quad &= \quad \int_{\Omega} [\frac{1}{2}|\nabla u|^2 - \int_{\Omega} [\frac{1}{2}\lambda a(x)u^2 + G(u)]dx,
\end{aligned}
$$

where $G(\xi) = \int_0^{\xi} g(\tau)\tau$. By $(g1)$, I is well defined.

PROPOSITION 3.1. *Assume that* $a : \overline{\Omega} \to R$ *is a continuous function,* $a(x) > 0$ *and* g *satisfies* $(g1)-(g4)$. *Then* $I(u)$ *is continuous and* $Fr\acute{e}chet$ *differentiable in* H *with* $Fr\acute{e}chet$ *derivative*

$$\nabla I(u)h = \int_{\Omega} [\nabla u \cdot \nabla h - (\lambda a(x)u + g(u))h]dx.$$

If we set

$$F(u) = \int_{\Omega} [\frac{1}{2}\lambda a(x)u^2 + G(u)]dx,$$

then $F'(u)$ *is continuous with respect to weak convergence,* $F'(u)$ *is compact, and*

$$F'(u)h = \int_{\Omega} (\lambda a(x)u + g(u))hdx \qquad \text{for all } h \in H,$$

this implies that $I \in C^1(H, R)$ *and* $F(u)$ *is weakly continuous.*

The proof of Proposition 3.1 has the similar process to that of the proof in Appendix B in [2].

We have the following lemma which can be checked easily since $\text{Fix}_{Z_2} = \{0\}$:

LEMMA 3.1. *Assume that g satisfies the conditions $(g1) - (g4)$. Let $u \in \text{Fix}_{Z_2} = \{0\}$ and u be a critical point of the functional of I, i.e., $\nabla I(u) = 0$. Then $I(u) = 0$.*

To prove Theorem 1.1 we shall prove that the functional I satisfies the assumptions $(a) - (f)$ of Theorem 2.1.

We assume that g satisfies the conditions $(g1) - (g4)$. Let us set

$$
\begin{aligned}
H_k^+ &= \{u|\ u \in H,\ u \in \text{span}\{\psi_l,\ l \geq k\}\}, \\
H_k^- &= \{u|\ u \in H,\ u \in \text{span}\{\psi_l,\ 1 \leq l \leq k\}\}.
\end{aligned}
$$

We note that

(3.3)
$$
\forall u \in H_1^+ \ :\ (Lu)u \geq \mu_1 \int_\Omega a(x)u^2 dx,
$$

$$
\forall u \in H_k^- \ :\ (Lu)u \leq \mu_k \int_\Omega a(x)u^2 dx.
$$

LEMMA 3.2. *Assume that g satisfies the conditions $(g1)-(g4)$. Then there exist $\rho > 0$ and a sphere S_ρ centered at 0 in H such that the functional $I(u)$ is bounded from above on $S_\rho \cap H_k^-$ and from below on H_1^+. That is,*

$$
-\infty < \inf_{u \in H_1^+} I(u) \qquad and \qquad \sup_{u \in S_\rho \cap H_k^-} I(u) < 0.
$$

Proof. Let $u \in H_1^+$. Then we have

$$
\begin{aligned}
I(u) &= \frac{1}{2}(Lu)u - \int_\Omega [\frac{1}{2}\lambda a(x)u^2 + G(u)]dx \\
&\geq \frac{1}{2}(\mu_1 - \lambda)\int_\Omega a(x)u^2 dx - \int_\Omega G(u)dx \\
&\geq \frac{1}{2}(\mu_1 - \lambda)\int_\Omega a(x)u^2 dx - C_2 \int_\Omega |u|^\beta dx \\
&\geq \frac{1}{2}(\mu_1 - \lambda)\sup a(x)\|u\|_{L^2(\Omega)}^2 - C_2\|u\|_{L^2(\Omega)}^\beta > -\infty
\end{aligned}
$$

since $a(x) > 0$, $\mu_1 - \lambda < 0$ and $\beta > 2$. Thus we have $\inf_{u \in H_1^+} I(u) > -\infty$.

For $u \in H_k^-$,

$$
\begin{aligned}
I(u) &= \frac{1}{2}(Lu)u - \int_\Omega [\frac{1}{2}\lambda a(x)u^2 + G(u)]dx \\
&\leq \frac{1}{2}(\mu_k - \lambda)\int_\Omega a(x)u^2 dx - \int_\Omega G(u)dx.
\end{aligned}
$$

By (1.2),

$$
I(u) \leq \frac{1}{2}(\mu_k - \lambda)\int_\Omega a(x)u^2 dx - a_3 \int_\Omega |u|^\beta dx + a_2|\Omega|.
$$

for $C_3 > 0$ and $a_2 > 0$. Since $\beta > 2$ and $a(x) > 0$, we can choose a number $\rho > 0$ and a sphere S_ρ centered at 0 in H such that for any $u \in S_\rho \cap H_k^-$, $I(u) < 0$. Thus we have $\sup_{u \in S_\rho \cap H_k^-} I(u) < 0$.   □

LEMMA 3.3. *Assume that g satisfies the conditions $(g1)-(g4)$. Then the functional I satisfies $(P.S.)_c$ condition for every $c \in [\inf_{H_1^+} I(u), \sup_{S_\rho \cap H_k^-} I(u)]$.*

Proof. Let $u \in H$. Since $H = H_1^+$, the functional

$$
\begin{aligned}
I(u) &= \frac{1}{2}(Lu)u - \int_\Omega [\frac{1}{2}\lambda a(x)u^2 + G(u)]dx \\
&\geq \frac{1}{2}(\mu_1 - \lambda)\int_\Omega a(x)u^2 dx - \int_\Omega G(u)dx \\
&\geq \frac{1}{2}(\mu_1 - \lambda)\sup a(x)\|u\|^2_{L^2(\Omega)} - C_2\|u\|^\beta_{L^2(\Omega)} > -\infty.
\end{aligned}
$$

Thus $I(u)$ is bounded from below. Thus $I(u)$ satisfies the $(P.S.)_c$ condition.   □

PROOF OF THEOREM 1.1
If we set $V = H_k^-$ and $W = H_1^+ = H$, then V and W are invariant subspace of H with $V + W = H$ and $V + W$ has codimension 0 in H. We note that $\text{Fix}_{Z_2} = \{0\}$ and $\text{Fix}_{Z_2} = \{0\} \subseteq V + W = H$. We also note that $L(W) \subseteq W$. By Lemma 3.1,

$$
-\infty < \inf_W I \qquad \sup_{H_k^- \cap S_\rho} I < 0.
$$

Thus the condition (d) of Theorem 2.1 is satisfied. Suppose that u is a critical point of the functional of I and $\inf_W I \leq I(u) \leq \sup_{S_\rho \cap V} I$. We claim that $u \notin \text{Fix}_{Z_2}$. If not, then $u \in \text{Fix}_{Z_2} = \{0\}$ i.e., $u = 0$. Since $u = 0$ is a critcal point of $I(u)$ with $I(0) = 0$ and $0 \notin [\inf_W I, \sup_{S_\rho \cap V} I]$,

it leads to a contradiction to the fact that $\inf_W I \leq I(u) \leq \sup_{S_\rho \cap V} I$. Thus $u \notin \text{Fix}_{Z_2}$. Thus the condition (e) is satisfied. By Lemma 3.2, I satisfies $(P.S.)_c$ condition whenever $\inf_W I \leq c \leq \sup_{S_\rho \cap V} I$.

Thus the assumptions $(a) - (e)$ of Theorem 1.1 are satisfied. Thus by the Theorem 2.1, Then I possesses at least

$$dim(V \cap W) - codim_H(V + W) = k$$

distinct critical orbits in $I^{-1}([\inf_W I, \sup_{S_\rho \cap V} I])$.

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