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MAPS PRESERVING JORDAN TRIPLE PRODUCT $A^*B + BA^*$ ON $*$-ALGEBRAS

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Abstract. Let $\mathcal{A}$ and $\mathcal{B}$ be two prime $*$-algebras. Let $\Phi : \mathcal{A} \to \mathcal{B}$ be a bijective and satisfies

$$\Phi(A \bullet B \bullet A) = \Phi(A) \bullet \Phi(B) \bullet \Phi(A),$$

for all $A, B \in \mathcal{A}$ where $A \bullet B = A^*B + BA^*$. Then, Φ is additive. Moreover, if $\Phi(I)$ is idempotent then we show that Φ is $\mathbb{R}$-linear $*$-isomorphism.

1. Introduction

Let $\mathcal{R}$ and $\mathcal{R}'$ be rings. We say the map $\Phi : \mathcal{R} \to \mathcal{R}'$ preserves product or is multiplicative if $\Phi(AB) = \Phi(A)\Phi(B)$ for all $A, B \in \mathcal{R}$, see [9]. Motivated by this, many authors pay more attention to the map on rings (and algebras) preserving different kinds of products to establish characteristics of Φ on rings. A natural problem is to study whether the map Φ preserving the new product on ring or algebra $\mathcal{R}$ is a ring or algebraic isomorphism. (for example [1–4, 6–8, 10–12]).

Recently, Liu and Ji [5] proved that a bijective map Φ on factor von Neumann algebras preserves, $A^*B + BA^*$ if and only if Φ is a $*$-isomorphism. Also, the authors in [14] considered such a bijective map $\Phi : \mathcal{A} \longrightarrow \mathcal{B}$ on prime C^*-algebras which preserves $A^*B + \eta BA^*$, where

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η is a non-zero scalar such that $\eta \neq \pm 1$. They proved that Φ is additive. Moreover, if $\Phi(I)$ is projection then Φ is $*$-isomorphism.

The authors of [13], proved that if the map Φ from a prime $*$-ring $\mathcal{A}$ onto a $*$-ring $\mathcal{B}$ is bijective and preserves Jordan triple product

$$\Phi(ABA) = \Phi(A)\Phi(B)\Phi(A)$$

or $*$-Jordan triple product

$$\Phi(AB^*A) = \Phi(A)\Phi(B)^*\Phi(A)$$

then it is additive. Also, we show that if $\Phi : \mathcal{A} \to \mathcal{B}$, where $\mathcal{A}$ and $\mathcal{B}$ are two prime rings, preserves Jordan triple product then it is multiplicative or anti-multiplicative. Also, we show that $\Psi(A) = \Phi(A)\Phi(I)^*$, for $A \in \mathcal{A}$, is a $\mathbb{C}$-linear or conjugate $\mathbb{C}$-linear $*$-isomorphism.

In this paper, motivated by the above results, we consider a map Φ on two prime $*$-algebras $\mathcal{A}$ and $\mathcal{B}$ with a nontrivial projection such that Φ is bijective and holds in the following condition

$$\Phi(A \bullet B \bullet A) = \Phi(A) \bullet \Phi(B) \bullet \Phi(A),$$

for all $A, B \in \mathcal{A}$ where $A \bullet B = A^*B + BA^*$. We show that Φ described in the above is additive. Also, if $\Phi(I)$ is idempotent then Φ is $\mathbb{R}$-linear $*$-isomorphism.

It is well known that C^*-algebra $\mathcal{A}$ is prime, in the sense that $A\mathcal{A}B = 0$ for $A, B \in \mathcal{A}$ implies either $A = 0$ or $B = 0$.

2. Main Results

We need the following lemma for proving our theorems.

LEMMA 2.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be two $*$-algebras and $\Phi : \mathcal{A} \to \mathcal{B}$ be a map which satisfies in the following case:*

$$\Phi(A \bullet B \bullet A) = \Phi(A) \bullet \Phi(B) \bullet \Phi(A). \tag{2.1}$$

If $\Phi(T) = \Phi(A) + \Phi(B)$ for $T, A, B \in \mathcal{A}$ then we have

$$\Phi(X \bullet T \bullet X) = \Phi(X \bullet A \bullet X) + \Phi(X \bullet B \bullet X)$$

for all $X, Y \in \mathcal{A}$.

Proof. By assumption we have

$$\Phi(T)^* = \Phi(A)^* + \Phi(B)^*. \tag{2.2}$$

Multiplying the left and right sides of (2.2) by $\Phi(X)$, we obtain

$$2\Phi(X)\Phi(T)^*\Phi(X) = 2\Phi(X)\Phi(A)^*\Phi(X) + 2\Phi(X)\Phi(B)^*\Phi(X). \tag{2.3}$$

Multiplying the left side of (2.2) by $\Phi(X)^2$, we obtain

$$\Phi(X)^2\Phi(T)^* = \Phi(X)^2\Phi(A)^* + \Phi(X)^2\Phi(B)^*. \tag{2.4}$$

Multiplying the right side of (2.2) by $\Phi(X)^2$, we obtain

$$\Phi(T)^*\Phi(X)^2 = \Phi(A)^*\Phi(X)^2 + \Phi(B)^*\Phi(X)^2. \tag{2.5}$$

Adding 2 times of (2.3), (2.4) and (2.5) together and making use of (2.1) we have

$$\Phi(X \bullet T \bullet X) = \Phi(X \bullet A \bullet X) + \Phi(X \bullet T \bullet X).$$

□

Our first theorem is as follows:

THEOREM 2.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be two prime $*$-algebras with unit $I_{\mathcal{A}}$ and $I_{\mathcal{B}}$ respectively, a nontrivial projection and $\Phi : \mathcal{A} \to \mathcal{B}$ be a bijective map which satisfies in the following condition*

$$\Phi(A \bullet B \bullet A) = \Phi(A) \bullet \Phi(B) \bullet \Phi(A) \tag{2.6}$$

for all $A, B \in \mathcal{A}$. Then Φ is additive.

Proof. Let P_1 be a nontrivial projection in $\mathcal{A}$ and $P_2 = I_{\mathcal{A}} - P_1$. Denote $\mathcal{A}_{ij} = P_i\mathcal{A}P_j,\ i,j = 1,2$, then $\mathcal{A} = \sum_{i,j=1}^{2} \mathcal{A}_{ij}$. For every $A \in \mathcal{A}$ we may write $A = A_{11} + A_{12} + A_{21} + A_{22}$. In all that follow, when we write A_{ij}, it indicates that $A_{ij} \in \mathcal{A}_{ij}$. For showing additivity of Φ on $\mathcal{A}$, we use above partition of $\mathcal{A}$ and give some claims that prove Φ is additive on each $\mathcal{A}_{ij},\ i,j = 1,2$.

We prove the above theorem by several claims.

CLAIM 1. *We show that $\Phi(0) = 0$.*

We know that for all $A, B \in \mathcal{A}$, the following holds

$$\Phi(A \bullet B \bullet A) = \Phi(A) \bullet \Phi(B) \bullet \Phi(A).$$

Let $B = 0$ then

$$\begin{aligned} \Phi(0) &= \Phi(A) \bullet \Phi(0) \bullet \Phi(A) \\ &= \Phi(A)\Phi(0)^*\Phi(A) + \Phi(0)^*\Phi(A)\Phi(A) \\ &\quad + \Phi(A)^2\Phi(0)^* + \Phi(A)\Phi(0)^*\Phi(A) \end{aligned}$$

for every $A \in \mathcal{A}$. Since Φ is surjective, we can find A such that $\Phi(A) = 0$, then we have $\Phi(0) = 0$.

Claim 2. *For each $A_{11} \in \mathcal{A}_{11}$ and $A_{22} \in \mathcal{A}_{22}$ we have*

$$\Phi(A_{11} + A_{22}) = \Phi(A_{11}) + \Phi(A_{22}).$$

Since Φ is surjective, there exists $T = T_{11} + T_{12} + T_{21} + T_{22} \in \mathcal{A}$ such that

$$\Phi(T) = \Phi(A_{11}) + \Phi(A_{22}). \tag{2.7}$$

By applying Lemma (2.1) to (2.7) for P_1 and P_2, we have

$$\Phi(P_1 \bullet T \bullet P_1) = \Phi(P_1 \bullet A_{11} \bullet P_1) + \Phi(P_1 \bullet A_{22} \bullet P_1) = \Phi(4A_{11}^*)$$

and

$$\Phi(P_2 \bullet T \bullet P_2) = \Phi(P_2 \bullet A_{11} \bullet P_2) + \Phi(P_2 \bullet A_{22} \bullet P_2) = \Phi(4A_{22}^*).$$

Since Φ is injective, we obtain

$$T^* P_1 + 2P_1 T^* P_1 + P_1 T^* = 4A_{11}^*$$

and

$$T^* P_2 + 2P_2 T^* P_2 + P_2 T^* = 4A_{22}^*.$$

Hence, we have $A_{11} = T_{11}$, $A_{22} = T_{22}$ and $T_{12} = T_{12} = 0$. So,

$$\Phi(A_{11} + A_{22}) = \Phi(A_{11}) + \Phi(A_{22}).$$

Claim 3. *For each $A_{12} \in \mathcal{A}_{12}$, $A_{21} \in \mathcal{A}_{21}$ we have*

$$\Phi(A_{12} + A_{21}) = \Phi(A_{12}) + \Phi(A_{21}).$$

Since Φ is surjective, we can find $T = T_{11} + T_{12} + T_{21} + T_{22} \in \mathcal{A}$ such that

$$\Phi(T) = \Phi(A_{12}) + \Phi(A_{21}). \tag{2.8}$$

By applying Lemma (2.1) to (2.8) for $P_1 - P_2$, we have

$$\begin{aligned} &\Phi\left((P_1 - P_2) \bullet T \bullet (P_1 - P_2)\right) \\ &= \Phi\left((P_1 - P_2) \bullet A_{12} \bullet (P_1 - P_2)\right) \\ &+\Phi\left((P_1 - P_2) \bullet A_{21} \bullet (P_1 - P_2)\right) = 0. \end{aligned}$$

Since Φ is injective, we have

$$(P_1 - P_2) \bullet T \bullet (P_1 - P_2) = 0.$$

So, we obtain

$$T_{11}^* + T_{22}^* = 0$$

it follows that $T_{11} = T_{22} = 0$.
On the other hand, by applying Lemma (2.1) to (2.8) for X_{12} and X_{21} we have

$$\begin{aligned}\Phi(X_{12} \bullet T \bullet X_{12}) &= \Phi(X_{12} \bullet A_{12} \bullet X_{12}) + \Phi(X_{12} \bullet A_{21} \bullet X_{12}) \\ &= \Phi(2X_{12}A_{12}^*X_{12})\end{aligned}$$

and

$$\begin{aligned}\Phi(X_{21} \bullet T \bullet X_{21}) &= \Phi(X_{21} \bullet A_{12} \bullet X_{21}) + \Phi(X_{21} \bullet A_{21} \bullet X_{21}) \\ &= \Phi(2X_{21}A_{21}^*X_{21}).\end{aligned}$$

By injection, we have

$$X_{12} \bullet T \bullet X_{12} = 2X_{12}A_{12}^*X_{12},$$

for all $X_{12} \in \mathcal{A}_{12}$ and

$$X_{21} \bullet T \bullet X_{21} = 2X_{21}A_{21}^*X_{21},$$

for all $X_{21} \in \mathcal{A}_{21}$. Therefore, by primeness we have $T_{12} = A_{12} =$ and $T_{21} = A_{21}$.

Claim 4. *For each $A_{11} \in \mathcal{A}_{11}$, $A_{12} \in \mathcal{A}_{12}$, $A_{21} \in \mathcal{A}_{21}$ we have*

$$\Phi(A_{11} + A_{12} + A_{21}) = \Phi(A_{11}) + \Phi(A_{12}) + \Phi(A_{21})$$

and

$$\Phi(A_{22} + A_{12} + A_{21}) = \Phi(A_{22}) + \Phi(A_{12}) + \Phi(A_{21}).$$

Since Φ is surjective, there exists $T = T_{11} + T_{12} + T_{21} + T_{22}$ such that

$$\Phi(T) = \Phi(A_{11}) + \Phi(A_{12}) + \Phi(A_{21}). \tag{2.9}$$

By applying Lemma (2.1) to (2.9) for P_2 and implying Claim 3, we have

$$\begin{aligned}\Phi(P_2 \bullet T \bullet P_2) &= \Phi(P_2 \bullet A_{11} \bullet P_2) + \Phi(P_2 \bullet A_{12} \bullet P_2) + \Phi(P_2 \bullet A_{21} \bullet P_2) \\ &= \Phi(A_{12}^* + A_{21}^*).\end{aligned}$$

So, we have

$$T^*P_2 + 2P_2T^*P_2 + P_2T^* = A_{12}^* + A_{21}^*$$

then

$$4T_{22}^* + T_{12}^* + T_{21}^* = A_{12}^* + A_{21}^*.$$

Therefore, we have $T_{12} = A_{12}$, $T_{21} = A_{21}$ and $T_{22} = 0$.
For showing that $T_{11} = A_{11}$ we use the following trick. It is easy to check that

$$\begin{aligned}\Phi(4T_{11}^*) &= \Phi((P_1 - P_2) \bullet T \bullet (P_1 - P_2)) \\ &= \Phi((P_1 - P_2) \bullet A_{11} \bullet (P_1 - P_2)) + \Phi((P_1 - P_2) \bullet A_{12} \bullet (P_1 - P_2)) \\ &\quad + \Phi((P_1 - P_2) \bullet A_{21} \bullet (P_1 - P_2)) \\ &= \Phi(4A_{11}^*).\end{aligned}$$

By, injection we have $T_{11} = A_{11}$.
Similarly, one can prove

$$\Phi(A_{22} + A_{12} + A_{21}) = \Phi(A_{22}) + \Phi(A_{12}) + \Phi(A_{21}).$$

Claim 5. *For each $A_{11} \in \mathcal{A}_{11}$, $A_{12} \in \mathcal{A}_{12}$, $A_{21} \in \mathcal{A}_{21}$, $A_{22} \in \mathcal{A}_{22}$, we have*

$$\Phi(A_{11} + A_{12} + A_{21} + A_{22}) = \Phi(A_{11}) + \Phi(A_{12}) + \Phi(A_{21}) + \Phi(A_{22}).$$

We assume T is an element in $\mathcal{A}$ such that

$$\Phi(T) = \Phi(A_{11}) + \Phi(A_{12}) + \Phi(A_{21}) + \Phi(A_{22}). \tag{2.10}$$

By applying Lemma (2.1) to (2.10) for P_1 and Claim 4, we have

$$\begin{aligned}\Phi(P_1 \bullet T \bullet P_1) &= \Phi(P_1 \bullet A_{11} \bullet P_1) + \Phi(P_1 \bullet A_{12} \bullet P_1) + \Phi(P_1 \bullet A_{21} \bullet P_1) \\ &\quad + \Phi(P_1 \bullet A_{22} \bullet P_1) \\ &= \Phi(4A_{11}^*) + \Phi(A_{12}^*) + \Phi(A_{21}^*) \\ &= \Phi(4A_{11}^* + A_{12}^* + A_{21}^*).\end{aligned}$$

Then, we have

$$4T_{11}^* + T_{12}^* + T_{21}^* = 4A_{11}^* + A_{12}^* + A_{21}^*$$

so, $T_{11} = A_{11}$, $T_{12} = A_{12}$ and $T_{21} = A_{21}$.
Similarly, by Lemma (2.1) to (2.10) for P_2 and Claim 4, we have

$$\Phi(P_2 \bullet T \bullet P_2) = \Phi(4A_{22}^* + A_{12}^* + A_{21}^*).$$

So, we obtain $T_{22} = A_{22}$. Hence, we obtain

$$\Phi(A_{11} + A_{12} + A_{21} + A_{22}) = \Phi(A_{11}) + \Phi(A_{12}) + \Phi(A_{21}) + \Phi(A_{22}).$$

Claim 6. *For each $A_{ij}, B_{ij} \in \mathcal{A}_{ij}$ such that $1 \leq i, j \leq 2$ and $i \neq j$, we have*

$$\Phi(A_{ij} + B_{ij}) = \Phi(A_{ij}) + \Phi(B_{ij}).$$

By a simple computation, we can show the following

$$(P_i + A_{ij}) \bullet (P_j + B_{ij}^*) \bullet (P_i + A_{ij}) = A_{ij} + B_{ij}.$$

By using Claim 5, we have

$$\begin{aligned}
\Phi(A_{ij} + B_{ij}) &= \Phi((P_i + A_{ij}) \bullet (P_j + B_{ij}^*) \bullet (P_i + A_{ij})) \\
&= \Phi(P_i + A_{ij}) \bullet \Phi(P_j + B_{ij}) \bullet \Phi(P_i + A_{ij}) \\
&= (\Phi(P_i) + \Phi(A_{ij})) \bullet (\Phi(P_j) + \Phi(B_{ij}^*)) \bullet (\Phi(P_i) + \Phi(A_{ij})) \\
&= \Phi(P_i) \bullet \Phi(P_j) \bullet \Phi(P_i) + \Phi(P_i) \bullet \Phi(B_{ij}^*) \bullet \Phi(P_i) \\
&\quad + \Phi(P_i) \bullet \Phi(P_j) \bullet \Phi(A_{ij}) + \Phi(P_i) \bullet \Phi(B_{ij}^*) \bullet \Phi(A_{ij}) \\
&\quad + \Phi(A_{ij}) \bullet \Phi(P_j) \bullet \Phi(P_i) + \Phi(A_{ij}) \bullet \Phi(P_j) \bullet \Phi(A_{ij}) \\
&\quad + \Phi(A_{ij}) \bullet \Phi(B_{ij}^*) \bullet \Phi(P_i) + \Phi(A_{ij}) \bullet \Phi(B_{ij}^*) \bullet \Phi(A_{ij}) \\
&= \Phi(P_i \bullet P_j \bullet P_i) + \Phi(P_i \bullet B_{ij}^* \bullet P_i) \\
&\quad + \Phi(P_i \bullet P_j \bullet A_{ij}) + \Phi(P_i \bullet B_{ij}^* \bullet A_{ij}) \\
&\quad + \Phi(A_{ij} \bullet P_j \bullet P_i) + \Phi(A_{ij} \bullet P_j \bullet A_{ij}) \\
&\quad + \Phi(A_{ij} \bullet B_{ij}^* \bullet P_i) + \Phi(A_{ij} \bullet B_{ij}^* \bullet A_{ij}) \\
&= \Phi(A_{ij}) + \Phi(B_{ij}).
\end{aligned}$$

Claim 7. *For each $A_{ii}, B_{ii} \in \mathcal{A}_{ii}$ such that $1 \le i \le 2$, we have*

$$\Phi(A_{ii} + B_{ii}) = \Phi(A_{ii}) + \Phi(B_{ii}).$$

Since Φ is surjective, we can find $T = T_{ii} + T_{ij} + T_{ji} + T_{jj} \in \mathcal{A}$ such that

$$\Phi(T) = \Phi(A_{ii}) + \Phi(B_{ii}). \tag{2.11}$$

By applying Lemma (2.1) to (2.11) for P_j, we have

$$\Phi(P_j \bullet T \bullet P_j) = \Phi(P_j \bullet A_{ii} \bullet P_j) + \Phi(I \bullet P_j \bullet B_{ii} \bullet P_j) = 0.$$

Since Φ is injective, we obtain

$$P_j T^* + T^* P_j + 2P_j T^* P_j = 0.$$

So, $T_{ij} = T_{ji} = T_{jj} = 0$. Hence, we have $T = T_{ii}$.
On the other hand, for each $C_{ij} \in \mathcal{A}_{ij}$ from Claim 6 and Lemma (2.1) for $P_j + C_{ij}$ we have

$$\begin{aligned}
\Phi(T_{ii}^* C_{ij}) &= \Phi\left((P_j + C_{ij}) \bullet T \bullet (P_j + C_{ij})\right) \\
&= \Phi\left((P_j + C_{ij}) \bullet A_{ii} \bullet P_j + C_{ij}\right) + \Phi\left((P_j + C_{ij}) \bullet B_{ii} \bullet (P_j + C_{ij})\right) \\
&= \Phi(A_{ii}^* C_{ij}) + \Phi(B_{ii}^* C_{ij}) \\
&= \Phi(A_{ii}^* C_{ij} + B_{ii}^* C_{ij}).
\end{aligned}$$

So, we have

$$(T_{ii}^* - A_{ii}^* - B_{ii}^*)C_{ij} = 0$$

for all $C_{ij} \in \mathcal{A}_{ij}$. By primeness, we have $T_{ii} = A_{ii} + B_{ii}$.

Hence, the additivity of Φ comes from the above claims. □

In the rest of this paper, we show that Φ is $*$-isomorphism.

THEOREM 2.3. *Let $\mathcal{A}$ and $\mathcal{B}$ be two prime $*$-algebras with unit $I_{\mathcal{A}}$ and $I_{\mathcal{B}}$ respectively, a nontrivial projection and $\Phi : \mathcal{A} \to \mathcal{B}$ be a bijective map which satisfies in the following condition*

$$\Phi(A \bullet B \bullet A) = \Phi(A) \bullet \Phi(B) \bullet \Phi(A) \tag{2.12}$$

for all $A, B \in \mathcal{A}$. If $\Phi(I_{\mathcal{A}})$ is idempotent, then Φ is $\mathbb{R}$-linear $*$-isomorphism.

Proof. We prove the above theorem by some claims.

CLAIM 1. *Φ is a $\mathbb{Q}$-linear map.*

By additivity of Φ, it is easy to check that Φ is $\mathbb{Q}$-linear.

CLAIM 2. *We show that Φ is unital.*

For any $A \in \mathcal{A}$, by knowing that $\Phi(I)$ is idempotent, we have

$$\begin{aligned}\Phi\left(\frac{I}{2} \bullet A \bullet \frac{I}{2}\right) &= \Phi\left(\frac{I}{2}\right) \bullet \Phi(A) \bullet \Phi\left(\frac{I}{2}\right) \\ &= \Phi\left(\frac{I}{2}\right)^2 \Phi(A)^* + \Phi(A)^* \Phi\left(\frac{I}{2}\right)^2 + 2\Phi\left(\frac{I}{2}\right) \Phi(A)^* \Phi\left(\frac{I}{2}\right).\end{aligned}$$

Since Φ is surjective, we can find A such that $\Phi(A) = I_{\mathcal{B}}$, then

$$\Phi\left(\frac{I}{2} \bullet A \bullet \frac{I}{2}\right) = \Phi(I)$$

by injectivity of Φ we get

$$\frac{I}{2} \bullet A \bullet \frac{I}{2} = I.$$

So, $A = I$.

CLAIM 3. *Φ preserves projections on the both sides.*

Suppose that $P \in \mathcal{A}$ is a projection. From the Claim 1 we have

$$\begin{aligned} \Phi(P) &= \Phi\left(\frac{I}{2} \bullet P \bullet \frac{I}{2}\right) \\ &= \left(\Phi\left(\frac{I}{2}\right)^* \Phi(P) + \Phi(P)\Phi\left(\frac{I}{2}\right)^*\right) \bullet \Phi\left(\frac{I}{2}\right) \\ &= \Phi(P) \bullet \Phi\left(\frac{I}{2}\right) = \Phi(P)^*. \end{aligned}$$

Then

$$\Phi(P) = \Phi(P)^*.$$

Also,

$$\begin{aligned} \Phi(P) &= \Phi\left(P \bullet \frac{I}{4} \bullet P\right) \\ &= \left(\Phi(P)^* \Phi\left(\frac{I}{4}\right) + \Phi\left(\frac{I}{4}\right)\Phi(P)^*\right) \bullet \Phi(P) \\ &= \frac{1}{2}\Phi(P) \bullet \Phi(P) \\ &= \frac{\Phi(P)^*\Phi(P) + \Phi(P)\Phi(P)^*}{2} \\ &= \Phi(P)^2. \end{aligned}$$

So,

$$\Phi(P) = \Phi(P)^2.$$

Since Φ^{-1} has the same characteristics of Φ then Φ is the preserver of the projections on the both sides.

Remark 2.4. We note here that if P_i and $P_j = I - P_i$ are two Orthogonal projections then $\Phi(P_i)$ and $\Phi(P_j)$ are so.

$$\begin{aligned} \Phi(P_i)\Phi(P_j) &= \Phi(P_i)(\Phi(I) - P_i) \\ &= \Phi(P_i)(\Phi(I) - \Phi(P_i)) \\ &= 0. \end{aligned}$$

Remark 2.5. From

$$\Phi\left(\frac{I}{2} \bullet A \bullet \frac{I}{2}\right) = \Phi\left(\frac{I}{2}\right) \bullet \Phi(A) \bullet \Phi\left(\frac{I}{2}\right)$$

we have $\Phi(A^*) = \Phi(A)^*$. It means that Φ preserves star.

Claim 4. $\Phi(\mathcal{A}_{ii}) = \mathcal{B}_{ii}$.

Let $X \in \mathcal{A}_{ii}$ be an arbitrary element, then we obtain

$$\begin{aligned}\Phi(4X) &= \Phi(P_i \bullet X^* \bullet P_i) \\ &= (\Phi(P_i)^*\Phi(X^*) + \Phi(X^*)\Phi(P_i)^*) \bullet \Phi(P_i) \\ &= \Phi(P_i)\Phi(X) + \Phi(X)\Phi(P_i) + 2\Phi(P_i)\Phi(X)\Phi(P_i)\end{aligned}$$

since Φ is $\mathbb{Q}$-linear, so we show that

$$4\Phi(X) = \Phi(P_i)\Phi(X) + 2\Phi(P_i)\Phi(X)\Phi(P_i) + \Phi(X)\Phi(P_i).$$

From the above equation, we obtain the following relations

$$\Phi(P_i)\Phi(X)\Phi(P_j) = 0,$$

$$\Phi(P_j)\Phi(X)\Phi(P_i) = 0$$

and

$$\Phi(P_j)\Phi(X)\Phi(P_j) = 0.$$

So, we have

$$\Phi(X) = \sum_{i,j=1}^{2} \Phi(P_i)\Phi(X)\Phi(P_j) = \Phi(P_i)\Phi(X)\Phi(P_i)$$

it follows that

$$\Phi(A_{ii}) \subseteq B_{ii}.$$

Since Φ^{-1} has the properties as Φ then we have $\Phi(A_{ii}) = B_{ii}$.

Claim 5. *$\Phi(A_{ij})\Phi(P_j) = \Phi(P_i)\Phi(A_{ij})\Phi(P_j)$ for $A_{ij} \in \mathcal{A}_{ij}$ and $1 \leq i, j \leq 2$ such that $i \neq j$.*

Since Φ is star preserving we have

$$\begin{aligned}\Phi(A_{ij}) &= \Phi(P_i \bullet A_{ij}^* \bullet P_i) \\ &= \Phi(P_i) \bullet \Phi(A_{ij}^*) \bullet \Phi(P_i) \\ &= (\Phi(P_i)^*\Phi(A_{ij}^*) + \Phi(A_{ij}^*)\Phi(P_i)) \bullet \Phi(P_i) \\ &= \Phi(P_i)\Phi(A_{ij}) + \Phi(A_{ij})\Phi(P_i) + 2\Phi(P_i)\Phi(A_{ij})\Phi(P_i).\end{aligned}$$

So, we showed that

$$\Phi(A_{ij}) = \Phi(P_i)\Phi(A_{ij}) + \Phi(A_{ij})\Phi(P_i) + 2\Phi(P_i)\Phi(A_{ij})\Phi(P_i).$$

We multiply the right side of above equation by $\Phi(P_j)$, to obtain

$$\Phi(A_{ij})\Phi(P_j) = \Phi(P_i)\Phi(A_{ij})\Phi(P_j). \tag{2.13}$$

Similarly, one can show that

$$\Phi(P_j)\Phi(A_{ji}) = \Phi(P_j)\Phi(A_{ji})\Phi(P_i).$$

Claim 6. *Suppose that $A_{ii}, B_{ii} \in \mathcal{A}_{ii}$ for $1 \leq i \leq 2$. Then*

$$\Phi(A_{ii}B_{ii}) = \Phi(A_{ii})\Phi(B_{ii}).$$

Let $C_{ij} \in \mathcal{A}_{ij}$ be an arbitrary element. Therefore, we have

$$\begin{aligned}
\Phi(A_{ii}B_{ii}C_{ij}) &= \Phi((P_j + C_{ij}) \bullet (B_{ii}^*A_{ii}^*) \bullet (P_j + C_{ij})) \\
&= \Phi((P_j + C_{ij}) \bullet \Phi(B_{ii}^*A_{ii}^*) \bullet \Phi(P_j + C_{ij}) \\
&= ((\Phi(P_j + C_{ij})^*\Phi(B_{ii}^*A_{ii}^*) \\
&\quad +\Phi(B_{ii}^*A_{ii}^*)\Phi(P_j + C_{ij})^*) \bullet \Phi(P_j + C_{ij}) \\
&= (\Phi(C_{ij})^*\Phi(B_{ii}^*A_{ii}^*)) \bullet \Phi(P_j) \\
&= \Phi(A_{ii}B_{ii})\Phi(C_{ij}).
\end{aligned}$$

So, by the above equation, we obtain

$$\begin{aligned}
\Phi(A_{ii}B_{ii})\Phi(C_{ij}) &= \Phi(A_{ii}(B_{ii}C_{ij})) \\
&= \Phi(A_{ii})\Phi(B_{ii}C_{ij}) \\
&= \Phi(A_{ii})\Phi(B_{ii})\Phi(C_{ij}).
\end{aligned}$$

We have

$$(\Phi(A_{ii}B_{ii}) - \Phi(A_{ii})\Phi(B_{ii}))\Phi(C_{ij}) = 0.$$

We multiply the above equation by $\Phi(P_j)$ from the left side, then we have

$$(\Phi(A_{ii}B_{ii}) - \Phi(A_{ii})\Phi(B_{ii}))\Phi(C_{ij})\Phi(P_j) = 0.$$

By Claim 5, we have

$$(\Phi(A_{ii}B_{ii}) - \Phi(A_{ii})\Phi(B_{ii}))\Phi(P_i)\Phi(C_{ij})\Phi(P_j) = 0.$$

By primeness, we obtain

$$\Phi(A_{ii}B_{ii}) = \Phi(A_{ii})\Phi(B_{ii}).$$

Claim 7. *Suppose that $A_{ij} \in \mathcal{A}_{ii}$ and $B_{ji} \in \mathcal{B}_{ji}$. Then*

$$\Phi(A_{ij}B_{ji}) = \Phi(A_{ij})\Phi(B_{ji}).$$

Since Φ preserves star, we have

$$\begin{aligned}
& \Phi(A_{ij}B_{ji} + B_{ji}A_{ij}) \\
= & \Phi\left((A_{ij} + B_{ji}) \bullet \frac{I}{4} \bullet (A_{ij} + B_{ji})\right) \\
= & \Phi(A_{ij} + B_{ji}) \bullet \Phi\left(\frac{I}{4}\right) \bullet \Phi(A_{ij} + B_{ji}) \\
= & \left(\Phi(A_{ij} + B_{ij})^* \Phi\left(\frac{I}{4}\right) + \Phi\left(\frac{I}{4}\right) \Phi(A_{ij} + B_{ji})^*\right) \bullet \Phi(A_{ij} + B_{ij}) \\
= & \frac{1}{2}(\Phi(A_{ij})^* + \Phi(B_{ji})^*) \bullet \Phi(A_{ij} + B_{ji}) \\
= & \Phi(A_{ij})\Phi(B_{ji}) + \Phi(B_{ji})\Phi(A_{ij}).
\end{aligned}$$

It follows that

$$\Phi(A_{ij}B_{ji}) + \Phi(B_{ji}A_{ij}) = \Phi(A_{ij})\Phi(B_{ji}) + \Phi(B_{ji})\Phi(A_{ij}).$$

Multiplying the left side of the above equation by $\Phi(P_i)$ and applying Claims 4 and 5 to obtain

$$\Phi(P_i)\Phi(A_{ij}B_{ji}) + \Phi(P_i)\Phi(B_{ji}A_{ij}) = \Phi(P_i)\Phi(A_{ij})\Phi(B_{ji}) + \Phi(P_i)\Phi(B_{ji})\Phi(A_{ij}).$$

So,

$$\Phi(A_{ij}B_{ji}) = \Phi(A_{ij})\Phi(B_{ji}).$$

Claim 8. *For $A_{ii} \in \mathcal{A}_{ii}$ and $B_{ij} \in \mathcal{A}_{ij}$ we have*

$$\Phi(A_{ii}B_{ij}) = \Phi(A_{ii})\Phi(B_{ij}).$$

Let T_{ji} in $\mathcal{A}_{ji}$ such that $i \neq j$, Claims 6 and 7 imply that

$$\begin{aligned}
\Phi(A_{ii}B_{ij})\Phi(T_{ji}) &= \Phi(A_{ii}B_{ij}T_{ji}) \\
&= \Phi(A_{ii})\Phi(B_{ij}T_{ji}) \\
&= \Phi(A_{ii})\Phi(B_{ij})\Phi(T_{ji}).
\end{aligned}$$

Since $\mathcal{B}$ is prime and by Claim 5, we have

$$\Phi(A_{ii}B_{ij}) = \Phi(A_{ii})\Phi(B_{ij}).$$

Claim 9. *For $A_{ij} \in \mathcal{A}_{ij}$ and $B_{jj} \in \mathcal{A}_{jj}$ we have*

$$\Phi(A_{ij}B_{jj}) = \Phi(A_{ij})\Phi(B_{jj}).$$

For each $T_{ji} \in \mathcal{A}_{ji}$ such that $i \neq j$, we have

$$\begin{aligned}\Phi(T_{ji})\Phi(A_{ij}B_{jj}) &= \Phi(T_{ji}A_{ij}B_{jj}) \\ &= \Phi(T_{ji}A_{ij})\Phi(B_{jj}) \\ &= \Phi(T_{ji})\Phi(A_{ij})\Phi(B_{jj}).\end{aligned}$$

So,

$$\Phi(A_{ij}B_{jj}) = \Phi(A_{ij})\Phi(B_{jj}).$$

It should be clear that

$$\begin{aligned}\Phi(AB) &= \Phi((A_{ii} + A_{ij} + A_{ji} + A_{jj})(B_{ii} + B_{ij} + B_{ji} + B_{jj})) \\ &= \Phi(A_{ii}B_{ii} + A_{ii}B_{ij} + A_{ij}B_{ji} + A_{ij}B_{jj} + A_{ji}B_{ii} \\ &\quad + A_{ji}B_{ij} + A_{jj}B_{ji} + A_{jj}B_{jj}) \\ &= \Phi(A_{ii})\Phi(B_{ii}) + \Phi(A_{ii})\Phi(B_{ij}) + \Phi(A_{ij})\Phi(B_{ji}) + \Phi(A_{ij})\Phi(B_{jj}) \\ &\quad + \Phi(A_{ji})\Phi(B_{ii}) + \Phi(A_{ji})\Phi(B_{ij}) + \Phi(A_{jj})\Phi(B_{ji}) + \Phi(A_{jj})\Phi(B_{jj}) \\ &= (\Phi(A_{ii}) + \Phi(A_{ij}) + \Phi(A_{ji}) + \Phi(A_{jj}))(\Phi(B_{ii}) + \Phi(B_{ij}) + \Phi(B_{ji}) \\ &\quad + \Phi(B_{jj})) \\ &= \Phi(A)\Phi(B).\end{aligned}$$

Claim 10. *Φ is $\mathbb{R}$-linear.*

For every $\lambda \in \mathbb{R}$, there exists two rational number sequences $\{r_n\}, \{s_n\}$ such that $r_n \leq \lambda \leq s_n$ and $\lim r_n = \lim s_n = \lambda$ when $n \to \infty$. It is clear that Ψ preserves positive elements, then Ψ preserves order. So, by the additivity of Φ we have

$$r_nI = \Phi(r_nI) \leq \Phi(\lambda I) \leq \Phi(s_nI) = s_nI.$$

Hence,

$$\Phi(\lambda I) = \lambda I$$

for $\lambda \in \mathbb{R}$. It means that Φ is $\mathbb{R}$-linear. □

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