

INEQUALITIES FOR QUANTUM f -DIVERGENCE OF CONVEX FUNCTIONS AND MATRICES

SILVESTRU SEVER DRAGOMIR

ABSTRACT. Some inequalities for quantum f -divergence of matrices are obtained. It is shown that for normalised convex functions it is nonnegative. Some upper bounds for quantum f -divergence in terms of variational and χ^2 -distance are provided. Applications for some classes of divergence measures such as Umegaki and Tsallis relative entropies are also given.

1. Introduction

Let (X, \mathcal{A}) be a measurable space satisfying $|\mathcal{A}| > 2$ and μ be a σ -finite measure on (X, \mathcal{A}) . Let \mathcal{P} be the set of all probability measures on (X, \mathcal{A}) which are absolutely continuous with respect to μ . For $P, Q \in \mathcal{P}$, let $p = \frac{dP}{d\mu}$ and $q = \frac{dQ}{d\mu}$ denote the *Radon-Nikodym* derivatives of P and Q with respect to μ .

Two probability measures $P, Q \in \mathcal{P}$ are said to be *orthogonal* and we denote this by $Q \perp P$ if

$$P(\{q = 0\}) = Q(\{p = 0\}) = 1.$$

Let $f : [0, \infty) \rightarrow (-\infty, \infty]$ be a convex function that is continuous at 0, i.e., $f(0) = \lim_{u \downarrow 0} f(u)$.

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In 1963, I. Csiszár [3] introduced the concept of f -divergence as follows.

DEFINITION 1. Let $P, Q \in \mathcal{P}$. Then

$$(1.1) \quad I_f(Q, P) = \int_X p(x) f \left[\frac{q(x)}{p(x)} \right] d\mu(x),$$

is called the f -divergence of the probability distributions Q and P .

REMARK 1. Observe that, the integrand in the formula (1.1) is undefined when $p(x) = 0$. The way to overcome this problem is to postulate for f as above that

$$(1.2) \quad 0f \left[\frac{q(x)}{0} \right] = q(x) \lim_{u \downarrow 0} \left[uf \left(\frac{1}{u} \right) \right], \quad x \in X.$$

We now give some examples of f -divergences that are well-known and often used in the literature (see also [2]).

1.1. The Class of χ^α -Divergences. The f -divergences of this class, which is generated by the function χ^α , $\alpha \in [1, \infty)$, defined by

$$\chi^\alpha(u) = |u - 1|^\alpha, \quad u \in [0, \infty)$$

have the form

$$(1.3) \quad I_f(Q, P) = \int_X p \left| \frac{q}{p} - 1 \right|^\alpha d\mu = \int_X p^{1-\alpha} |q - p|^\alpha d\mu.$$

From this class only the parameter $\alpha = 1$ provides a distance in the topological sense, namely the *total variation distance* $V(Q, P) = \int_X |q - p| d\mu$. The most prominent special case of this class is, however, *Karl Pearson's χ^2 -divergence*

$$\chi^2(Q, P) = \int_X \frac{q^2}{p} d\mu - 1$$

that is obtained for $\alpha = 2$.

1.2. Dichotomy Class. From this class, generated by the function $f_\alpha : [0, \infty) \rightarrow \mathbb{R}$

$$f_\alpha(u) = \begin{cases} u - 1 - \ln u & \text{for } \alpha = 0; \\ \frac{1}{\alpha(1-\alpha)} [\alpha u + 1 - \alpha - u^\alpha] & \text{for } \alpha \in \mathbb{R} \setminus \{0, 1\}; \\ 1 - u + u \ln u & \text{for } \alpha = 1; \end{cases}$$

only the parameter $\alpha = \frac{1}{2}$ ($f_{\frac{1}{2}}(u) = 2(\sqrt{u} - 1)^2$) provides a distance, namely, the *Hellinger distance*

$$H(Q, P) = \left[\int_X (\sqrt{q} - \sqrt{p})^2 d\mu \right]^{\frac{1}{2}}.$$

Another important divergence is the *Kullback-Leibler divergence* obtained for $\alpha = 1$,

$$KL(Q, P) = \int_X q \ln \left(\frac{q}{p} \right) d\mu.$$

1.3. Matsushita's Divergences. The elements of this class, which is generated by the function φ_α , $\alpha \in (0, 1]$ given by

$$\varphi_\alpha(u) := |1 - u^\alpha|^{\frac{1}{\alpha}}, \quad u \in [0, \infty),$$

are prototypes of metric divergences, providing the distances $[I_{\varphi_\alpha}(Q, P)]^\alpha$.

1.4. Puri-Vincze Divergences. This class is generated by the functions Φ_α , $\alpha \in [1, \infty)$ given by

$$\Phi_\alpha(u) := \frac{|1 - u|^\alpha}{(u + 1)^{\alpha-1}}, \quad u \in [0, \infty).$$

It has been shown in [19] that this class provides the distances $[I_{\Phi_\alpha}(Q, P)]^{\frac{1}{\alpha}}$.

1.5. Divergences of Arimoto-type. This class is generated by the functions

$$\Psi_\alpha(u) := \begin{cases} \frac{\alpha}{\alpha-1} \left[(1 + u^\alpha)^{\frac{1}{\alpha}} - 2^{\frac{1}{\alpha}-1} (1 + u) \right] & \text{for } \alpha \in (0, \infty) \setminus \{1\}; \\ (1 + u) \ln 2 + u \ln u - (1 + u) \ln (1 + u) & \text{for } \alpha = 1; \\ \frac{1}{2} |1 - u| & \text{for } \alpha = \infty. \end{cases}$$

It has been shown in [21] that this class provides the distances $[I_{\Psi_\alpha}(Q, P)]^{\min(\alpha, \frac{1}{\alpha})}$ for $\alpha \in (0, \infty)$ and $\frac{1}{2}V(Q, P)$ for $\alpha = \infty$.

For f continuous convex on $[0, \infty)$ we obtain the **-conjugate* function of f by

$$f^*(u) = uf \left(\frac{1}{u} \right), \quad u \in (0, \infty)$$

and

$$f^*(0) = \lim_{u \downarrow 0} f^*(u).$$

It is also known that if f is continuous convex on $[0, \infty)$ then so is f^* .

The following two theorems contain the most basic properties of f -divergences. For their proofs we refer the reader to Chapter 1 of [20] (see also [2]).

THEOREM 1 (Uniqueness and Symmetry Theorem). *Let f, f_1 be continuous convex on $[0, \infty)$. We have*

$$I_{f_1}(Q, P) = I_f(Q, P),$$

for all $P, Q \in \mathcal{P}$ if and only if there exists a constant $c \in \mathbb{R}$ such that

$$f_1(u) = f(u) + c(u - 1),$$

for any $u \in [0, \infty)$.

THEOREM 2 (Range of Values Theorem). *Let $f : [0, \infty) \rightarrow \mathbb{R}$ be a continuous convex function on $[0, \infty)$.*

For any $P, Q \in \mathcal{P}$, we have the double inequality

$$(1.4) \quad f(1) \leq I_f(Q, P) \leq f(0) + f^*(0).$$

(i) *If $P = Q$, then the equality holds in the first part of (1.4).*

If f is strictly convex at 1, then the equality holds in the first part of (1.4) if and only if $P = Q$;

(ii) *If $Q \perp P$, then the equality holds in the second part of (1.4).*

If $f(0) + f^(0) < \infty$, then equality holds in the second part of (1.4) if and only if $Q \perp P$.*

The following result is a refinement of the second inequality in Theorem 2 (see [2, Theorem 3]).

THEOREM 3. *Let f be a continuous convex function on $[0, \infty)$ with $f(1) = 0$ (f is normalised) and $f(0) + f^*(0) < \infty$. Then*

$$(1.5) \quad 0 \leq I_f(Q, P) \leq \frac{1}{2} [f(0) + f^*(0)] V(Q, P)$$

for any $Q, P \in \mathcal{P}$.

For other inequalities for f -divergence see [1], [5]- [15].

Motivated by the above results, in this paper we obtain some new inequalities for quantum f -divergence of matrices. It is shown that for normalised convex functions it is nonnegative. Some upper bounds for quantum f -divergence in terms of variational and χ^2 -distance are provided. Applications for some classes of divergence measures such as Umegaki and Tsallis relative entropies are also given.

2. Quantum f -Divergence

Quasi-entropy was introduced by Petz in 1985, [22] as the quantum generalization of Csiszár's f -divergence in the setting of matrices or von Neumann algebras. The important special case was the relative entropy of Umegaki and Araki.

In what follows some inequalities for the quantum f -divergence of convex functions in the finite dimensional setting are provided.

Let \mathcal{M} denotes the algebra of all $n \times n$ matrices with complex entries and \mathcal{M}^+ the subclass of all positive matrices.

On complex Hilbert space $(\mathcal{M}, \langle \cdot, \cdot \rangle_2)$, where the *Hilbert-Schmidt inner product* is defined by

$$\langle U, V \rangle_2 := \text{tr}(V^*U), \quad U, V \in \mathcal{M},$$

for $A, B \in \mathcal{M}^+$ consider the operators $\mathfrak{L}_A : \mathcal{M} \rightarrow \mathcal{M}$ and $\mathfrak{R}_B : \mathcal{M} \rightarrow \mathcal{M}$ defined by

$$\mathfrak{L}_A T := AT \text{ and } \mathfrak{R}_B T := TB.$$

We observe that they are well defined and since

$$\langle \mathfrak{L}_A T, T \rangle_2 = \langle AT, T \rangle_2 = \text{tr}(T^*AT) = \text{tr}(|T^*|^2 A) \geq 0$$

and

$$\langle \mathfrak{R}_B T, T \rangle_2 = \langle TB, T \rangle_2 = \text{tr}(T^*TB) = \text{tr}(|T|^2 B) \geq 0$$

for any $T \in \mathcal{M}$, they are also positive in the operator order of $\mathcal{B}(\mathcal{M})$, the Banach algebra of all bounded operators on \mathcal{M} with the norm $\|\cdot\|_2$ where $\|T\|_2 = \text{tr}(|T|^2)$, $T \in \mathcal{M}$.

Since $\operatorname{tr}(|X^*|^2) = \operatorname{tr}(|X|^2)$ for any $X \in \mathcal{M}$, then also

$$\begin{aligned} \operatorname{tr}(T^*AT) &= \operatorname{tr}(T^*A^{1/2}A^{1/2}T) = \operatorname{tr}\left((A^{1/2}T)^*A^{1/2}T\right) \\ &= \operatorname{tr}\left(|A^{1/2}T|^2\right) = \operatorname{tr}\left(\left|(A^{1/2}T)^*\right|^2\right) = \operatorname{tr}\left(|T^*A^{1/2}|^2\right) \end{aligned}$$

for $A \geq 0$ and $T \in \mathcal{M}$.

We observe that \mathfrak{L}_A and \mathfrak{R}_B are commutative, therefore the product $\mathfrak{L}_A\mathfrak{R}_B$ is a selfadjoint positive operator in $\mathcal{B}(\mathcal{M})$ for any positive matrices $A, B \in \mathcal{M}^+$.

For $A, B \in \mathcal{M}^+$ with B invertible, we define the *Araki transform* $\mathfrak{A}_{A,B} : \mathcal{M} \rightarrow \mathcal{M}$ by $\mathfrak{A}_{A,B} := \mathfrak{L}_A\mathfrak{R}_{B^{-1}}$. We observe that for $T \in \mathcal{M}$ we have $\mathfrak{A}_{A,B}T = ATB^{-1}$ and

$$\langle \mathfrak{A}_{A,B}T, T \rangle_2 = \langle ATB^{-1}, T \rangle_2 = \operatorname{tr}(T^*ATB^{-1}).$$

Observe also, by the properties of trace, that

$$\begin{aligned} \operatorname{tr}(T^*ATB^{-1}) &= \operatorname{tr}(B^{-1/2}T^*A^{1/2}A^{1/2}TB^{-1/2}) \\ &= \operatorname{tr}\left((A^{1/2}TB^{-1/2})^*(A^{1/2}TB^{-1/2})\right) = \operatorname{tr}\left(|A^{1/2}TB^{-1/2}|^2\right) \end{aligned}$$

giving that

$$(2.1) \quad \langle \mathfrak{A}_{A,B}T, T \rangle_2 = \operatorname{tr}\left(|A^{1/2}TB^{-1/2}|^2\right) \geq 0$$

for any $T \in \mathcal{M}$.

We observe that, by the definition of operator order and by (2.1) we have $r1_{\mathcal{M}} \leq \mathfrak{A}_{A,B} \leq R1_{\mathcal{M}}$ for some $R \geq r \geq 0$ if and only if

$$(2.2) \quad r \operatorname{tr}(|T|^2) \leq \operatorname{tr}\left(|A^{1/2}TB^{-1/2}|^2\right) \leq R \operatorname{tr}(|T|^2)$$

for any $T \in \mathcal{M}$.

We also notice that a sufficient condition for (2.2) to hold is that the following inequality in the operator order of \mathcal{M} is satisfied

$$(2.3) \quad r|T|^2 \leq |A^{1/2}TB^{-1/2}|^2 \leq R|T|^2$$

for any $T \in \mathcal{B}_2(H)$.

Let U be a selfadjoint linear operator on a complex Hilbert space $(K; \langle \cdot, \cdot \rangle)$. The *Gelfand map* establishes a $*$ -isometrically isomorphism Φ between the set $C(\operatorname{Sp}(U))$ of all *continuous functions* defined on the *spectrum* of U , denoted $\operatorname{Sp}(U)$, and the C^* -algebra $C^*(U)$ generated by U and the identity operator 1_K on K as follows:

For any $f, g \in C(\text{Sp}(U))$ and any $\alpha, \beta \in \mathbb{C}$ we have

- (i) $\Phi(\alpha f + \beta g) = \alpha\Phi(f) + \beta\Phi(g)$;
- (ii) $\Phi(fg) = \Phi(f)\Phi(g)$ and $\Phi(\bar{f}) = \Phi(f)^*$;
- (iii) $\|\Phi(f)\| = \|f\| := \sup_{t \in \text{Sp}(U)} |f(t)|$;
- (iv) $\Phi(f_0) = 1_K$ and $\Phi(f_1) = U$, where $f_0(t) = 1$ and $f_1(t) = t$, for $t \in \text{Sp}(U)$.

With this notation we define

$$f(U) := \Phi(f) \quad \text{for all } f \in C(\text{Sp}(U))$$

and we call it the *continuous functional calculus* for a selfadjoint operator U .

If U is a selfadjoint operator and f is a real valued continuous function on $\text{Sp}(U)$, then $f(t) \geq 0$ for any $t \in \text{Sp}(U)$ implies that $f(U) \geq 0$, i.e. $f(U)$ is a positive operator on K . Moreover, if both f and g are real valued functions on $\text{Sp}(U)$ then the following important property holds:

(P) $f(t) \geq g(t)$ for any $t \in \text{Sp}(U)$ implies that $f(U) \geq g(U)$

in the operator order of $B(K)$.

Let $f : [0, \infty) \rightarrow \mathbb{R}$ be a continuous function. Utilising the continuous functional calculus for the Araki selfadjoint operator $\mathfrak{A}_{Q,P} \in \mathcal{B}(\mathcal{M})$ we can define the *quantum f -divergence* for $Q, P \in S_1(\mathcal{M}) := \{P \in \mathcal{M}, P \geq 0 \text{ with } \text{tr}(P) = 1\}$ and P invertible, by

$$S_f(Q, P) := \langle f(\mathfrak{A}_{Q,P}) P^{1/2}, P^{1/2} \rangle_2 = \text{tr}(P^{1/2} f(\mathfrak{A}_{Q,P}) P^{1/2}).$$

If we consider the continuous convex function $f : [0, \infty) \rightarrow \mathbb{R}$, with $f(0) := 0$ and $f(t) = t \ln t$ for $t > 0$ then for $Q, P \in S_1(\mathcal{M})$ and Q, P invertible we have

$$S_f(Q, P) = \text{tr}[Q(\ln Q - \ln P)] =: U(Q, P),$$

which is the *Umegaki relative entropy*.

If we take the continuous convex function $f : [0, \infty) \rightarrow \mathbb{R}$, $f(t) = |t - 1|$ for $t \geq 0$ then for $Q, P \in S_1(H)$ with P invertible we have

$$S_f(Q, P) = \text{tr}(|Q - P|) =: V(Q, P),$$

where $V(Q, P)$ is the *variational distance*.

If we take $f : [0, \infty) \rightarrow \mathbb{R}$, $f(t) = t^2 - 1$ for $t \geq 0$ then for $Q, P \in S_1(\mathcal{M})$ with P invertible we have

$$S_f(Q, P) = \text{tr}(Q^2 P^{-1}) - 1 =: \chi^2(Q, P),$$

which is called the χ^2 -distance

Let $q \in (0, 1)$ and define the convex function $f_q : [0, \infty) \rightarrow \mathbb{R}$ by $f_q(t) = \frac{1-t^q}{1-q}$. Then

$$S_{f_q}(Q, P) = \frac{1 - \operatorname{tr}(Q^q P^{1-q})}{1 - q},$$

which is *Tsallis relative entropy*.

If we consider the convex function $f : [0, \infty) \rightarrow \mathbb{R}$ by $f(t) = \frac{1}{2}(\sqrt{t} - 1)^2$, then

$$S_f(Q, P) = 1 - \operatorname{tr}(Q^{1/2} P^{1/2}) =: h^2(Q, P),$$

which is known as *Hellinger discrimination*.

If we take $f : (0, \infty) \rightarrow \mathbb{R}$, $f(t) = -\ln t$ then for $Q, P \in S_1(\mathcal{M})$ and Q, P invertible we have

$$S_f(Q, P) = \operatorname{tr}[P(\ln P - \ln Q)] = U(P, Q).$$

The reader can obtain other particular quantum f -divergence measures by utilizing the normalized convex functions from Introduction, namely the convex functions defining the dichotomy class, Matsushita's divergences, Puri-Vincze divergences or divergences of Arimoto-type. We omit the details.

In the important case of finite dimensional spaces and the generalized inverse P^{-1} , numerous properties of the quantum f -divergence, mostly in the case when f is *operator convex*, have been obtained in the recent papers [17], [18], [22]- [25] and the references therein.

In what follows we obtain several inequalities for the larger class of convex functions on an interval.

3. Inequalities for f Convex and Normalized

Suppose that I is an interval of real numbers with interior $\overset{\circ}{I}$ and $f : I \rightarrow \mathbb{R}$ is a convex function on I . Then f is continuous on $\overset{\circ}{I}$ and has finite left and right derivatives at each point of $\overset{\circ}{I}$. Moreover, if $x, y \in \overset{\circ}{I}$ and $x < y$, then $f'_-(x) \leq f'_+(x) \leq f'_-(y) \leq f'_+(y)$, which shows that both f'_- and f'_+ are nondecreasing function on $\overset{\circ}{I}$. It is also known that a convex function must be differentiable except for at most countably many points.

For a convex function $f : I \rightarrow \mathbb{R}$, the subdifferential of f denoted by ∂f is the set of all functions $\varphi : I \rightarrow [-\infty, \infty]$ such that $\varphi(\overset{\circ}{I}) \subset \mathbb{R}$ and

$$(G) \quad f(x) \geq f(a) + (x - a)\varphi(a) \text{ for any } x, a \in I.$$

It is also well known that if f is convex on I , then ∂f is nonempty, $f'_-, f'_+ \in \partial f$ and if $\varphi \in \partial f$, then

$$f'_-(x) \leq \varphi(x) \leq f'_+(x) \text{ for any } x \in \overset{\circ}{I}.$$

In particular, φ is a nondecreasing function.

If f is differentiable and convex on $\overset{\circ}{I}$, then $\partial f = \{f'\}$.

We are able now to state and prove the first result concerning the quantum f -divergence for the general case of convex functions.

THEOREM 4. *Let $f : [0, \infty) \rightarrow \mathbb{R}$ be a continuous convex function that is normalized, i.e. $f(1) = 0$. Then for any $Q, P \in S_1(\mathcal{M})$, with P invertible, we have*

$$(3.1) \quad 0 \leq S_f(Q, P).$$

Moreover, if f is continuously differentiable, then also

$$(3.2) \quad S_f(Q, P) \leq S_{\ell f'}(Q, P) - S_{f'}(Q, P),$$

where the function ℓ is defined as $\ell(t) = t, t \in \mathbb{R}$.

Proof. Since f is convex and normalized, then by the gradient inequality (G) we have

$$f(t) \geq (t - 1)f'_+(1)$$

for $t > 0$.

Applying the property (P) for the operator $\mathfrak{A}_{Q,P}$, then we have for any $T \in \mathcal{M}$

$$\begin{aligned} \langle f(\mathfrak{A}_{Q,P})T, T \rangle_2 &\geq f'_+(1) \langle (\mathfrak{A}_{Q,P} - 1_{B_2(H)})T, T \rangle_2 \\ &= f'_+(1) [\langle \mathfrak{A}_{Q,P}T, T \rangle_2 - \|T\|_2], \end{aligned}$$

which, in terms of trace, can be written as

$$(3.3) \quad \text{tr}(T^* f(\mathfrak{A}_{Q,P})T) \geq f'_+(1) \left[\text{tr} \left(|Q^{1/2}TP^{-1/2}|^2 \right) - \text{tr}(|T|^2) \right]$$

for any $T \in \mathcal{M}$.

Now, if we take in (3.3) $T = P^{1/2}$ where $P \in S_1(\mathcal{M})$, with P invertible, then we get

$$S_f(Q, P) \geq f'_+(1) [\text{tr}(Q) - \text{tr}(P)] = 0$$

and the inequality (3.1) is proved.

Further, if f is continuously differentiable, then by the gradient inequality we also have

$$(t - 1) f'(t) \geq f(t)$$

for $t > 0$.

Applying the property (P) for the operator $\mathfrak{A}_{Q,P}$, then we have for any $T \in \mathcal{M}$

$$\langle (\mathfrak{A}_{Q,P} - 1_{\mathcal{B}_2(H)}) f'(\mathfrak{A}_{Q,P}) T, T \rangle_2 \geq \langle f(\mathfrak{A}_{Q,P}) T, T \rangle_2,$$

namely

$$\langle \mathfrak{A}_{Q,P} f'(\mathfrak{A}_{Q,P}) T, T \rangle_2 - \langle f'(\mathfrak{A}_{Q,P}) T, T \rangle_2 \geq \langle f(\mathfrak{A}_{Q,P}) T, T \rangle_2,$$

for any $T \in \mathcal{M}$, or in terms of trace

$$(3.4) \quad \text{tr}(T^* \mathfrak{A}_{Q,P} f'(\mathfrak{A}_{Q,P}) T) - \text{tr}(T^* f'(\mathfrak{A}_{Q,P}) T) \geq \text{tr}(T^* f(\mathfrak{A}_{Q,P}) T),$$

for any $T \in \mathcal{M}$.

If in (3.4) we take $T = P^{1/2}$, where $P \in S_1(\mathcal{M})$, with P invertible, then we get the desired result (3.2). \square

REMARK 2. If we take in (3.2) $f : (0, \infty) \rightarrow \mathbb{R}$, $f(t) = -\ln t$ then for $Q, P \in S_1(\mathcal{M})$ and Q, P invertible we have

$$(3.5) \quad 0 \leq U(P, Q) \leq \chi^2(P, Q).$$

We need the following lemma.

LEMMA 1. Let S be a selfadjoint operator on the Hilbert space $(H, \langle \cdot, \cdot \rangle)$ and with spectrum $\text{Sp}(S) \subseteq [\gamma, \Gamma]$ for some real numbers γ, Γ . If $g : [\gamma, \Gamma] \rightarrow \mathbb{C}$ is a continuous function such that

$$(3.6) \quad |g(t) - \lambda| \leq \rho \text{ for any } t \in [\gamma, \Gamma]$$

for some complex number $\lambda \in \mathbb{C}$ and positive number ρ , then

$$(3.7) \quad |\langle Sg(S)x, x \rangle - \langle Sx, x \rangle \langle g(S)x, x \rangle| \leq \rho \langle |S - \langle Sx, x \rangle 1_H| x, x \rangle \\ \leq \rho [\langle S^2 x, x \rangle - \langle Sx, x \rangle^2]^{1/2}$$

for any $x \in H$, $\|x\| = 1$.

Proof. We observe that

$$(3.8) \quad \langle Sg(S)x, x \rangle - \langle Sx, x \rangle \langle g(S)x, x \rangle = \langle (S - \langle Sx, x \rangle 1_H)(g(S) - \lambda 1_H)x, x \rangle$$

for any $x \in H$, $\|x\| = 1$.

For any selfadjoint operator B we have the modulus inequality

$$(3.9) \quad |\langle Bx, x \rangle| \leq \langle |B| x, x \rangle \text{ for any } x \in H, \|x\| = 1.$$

Also, utilizing the continuous functional calculus we have for each fixed $x \in H, \|x\| = 1$

$$\begin{aligned} |(S - \langle Sx, x \rangle 1_H)(g(S) - \lambda 1_H)| &= |S - \langle Sx, x \rangle 1_H| |g(S) - \lambda 1_H| \\ &\leq \rho |S - \langle Sx, x \rangle 1_H|, \end{aligned}$$

which implies that

$$(3.10) \quad \langle |(S - \langle Sx, x \rangle 1_H)(g(S) - \lambda 1_H)| x, x \rangle \leq \rho \langle |S - \langle Sx, x \rangle 1_H| x, x \rangle$$

for any $x \in H, \|x\| = 1$.

Therefore, by taking the modulus in (3.8) and utilizing (3.9) and (3.10) we get

$$(3.11) \quad \begin{aligned} |\langle Sg(S)x, x \rangle - \langle Sx, x \rangle \langle g(S)x, x \rangle| \\ = |\langle (S - \langle Sx, x \rangle 1_H)(g(S) - \lambda 1_H)x, x \rangle| \\ \leq \langle |(S - \langle Sx, x \rangle 1_H)(g(S) - \lambda 1_H)| x, x \rangle \\ \leq \rho \langle |S - \langle Sx, x \rangle 1_H| x, x \rangle \end{aligned}$$

for any $x \in H, \|x\| = 1$, which proves the first inequality in (3.7).

Using Schwarz inequality we also have

$$\begin{aligned} \langle |S - \langle Sx, x \rangle 1_H| x, x \rangle &\leq \langle (S - \langle Sx, x \rangle 1_H)^2 x, x \rangle^{1/2} \\ &= [\langle S^2 x, x \rangle - \langle Sx, x \rangle^2]^{1/2} \end{aligned}$$

for any $x \in H, \|x\| = 1$, and the lemma is proved. □

COROLLARY 1. *With the assumption of Lemma 1, we have*

$$(3.12) \quad \begin{aligned} 0 \leq \langle S^2 x, x \rangle - \langle Sx, x \rangle^2 &\leq \frac{1}{2} (\Gamma - \gamma) \langle |S - \langle Sx, x \rangle 1_H| x, x \rangle \\ &\leq \frac{1}{2} (\Gamma - \gamma) [\langle S^2 x, x \rangle - \langle Sx, x \rangle^2]^{1/2} \leq \frac{1}{4} (\Gamma - \gamma)^2, \end{aligned}$$

for any $x \in H, \|x\| = 1$.

Proof. If we take in Lemma 1 $g(t) = t$, $\lambda = \frac{1}{2}(\Gamma + \gamma)$ and $\rho = \frac{1}{2}(\Gamma - \gamma)$, then we get

$$(3.13) \quad 0 \leq \langle S^2x, x \rangle - \langle Sx, x \rangle^2 \leq \frac{1}{2}(\Gamma - \gamma) \langle |S - \langle Sx, x \rangle 1_H| x, x \rangle \\ \leq \frac{1}{2}(\Gamma - \gamma) [\langle S^2x, x \rangle - \langle Sx, x \rangle^2]^{1/2}$$

for any $x \in H$, $\|x\| = 1$.

From the first and last terms in (3.13) we have

$$[\langle S^2x, x \rangle - \langle Sx, x \rangle^2]^{1/2} \leq \frac{1}{2}(\Gamma - \gamma),$$

which proves the rest of (3.12). \square

We can prove the following result that provides simpler upper bounds for the quantum f -divergence when the operators P and Q satisfy the condition (2.2).

THEOREM 5. *Let $f : [0, \infty) \rightarrow \mathbb{R}$ be a continuous convex function that is normalized. If $Q, P \in S_1(\mathcal{M})$, with P invertible, and there exists $R \geq 1 \geq r \geq 0$ such that*

$$(3.14) \quad r \operatorname{tr}(|T|^2) \leq \operatorname{tr}(|Q^{1/2}TP^{-1/2}|^2) \leq R \operatorname{tr}(|T|^2)$$

for any $T \in \mathcal{M}$, then

$$(3.15) \quad 0 \leq S_f(Q, P) \leq \frac{1}{2} [f'_-(R) - f'_+(r)] V(Q, P) \\ \leq \frac{1}{2} [f'_-(R) - f'_+(r)] \chi(Q, P) \\ \leq \frac{1}{4} (R - r) [f'_-(R) - f'_+(r)].$$

Proof. Without losing the generality, we prove the inequality in the case that f is continuously differentiable on $(0, \infty)$.

Since f' is monotonic nondecreasing on $[r, R]$ we have that

$$f'(r) \leq f'(t) \leq f'(R) \text{ for any } t \in [r, R],$$

which implies that

$$\left| f'(t) - \frac{f'(R) + f'(r)}{2} \right| \leq \frac{1}{2} [f'(R) - f'(r)]$$

for any $t \in [r, R]$.

Applying Lemma 1 and Corollary 1 in the Hilbert space $(\mathcal{M}, \langle \cdot, \cdot \rangle_2)$ and for the selfadjoint operator $\mathfrak{A}_{Q,P}$ we have

$$\begin{aligned} & \left| \langle \mathfrak{A}_{Q,P} f'(\mathfrak{A}_{Q,P}) T, T \rangle_2 - \langle \mathfrak{A}_{Q,P} T, T \rangle_2 \langle f'(\mathfrak{A}_{Q,P}) T, T \rangle_2 \right| \\ & \leq \frac{1}{2} [f'(R) - f'(r)] \langle |\mathfrak{A}_{Q,P} - \langle \mathfrak{A}_{Q,P} T, T \rangle_2 1_{\mathcal{B}_2(H)}| T, T \rangle_2 \\ & \leq \frac{1}{2} [f'(R) - f'(r)] \left[\langle \mathfrak{A}_{Q,P}^2 T, T \rangle_2 - \langle \mathfrak{A}_{Q,P} T, T \rangle_2^2 \right]^{1/2} \\ & \leq \frac{1}{4} (R - r) [f'_-(R) - f'_+(r)] \end{aligned}$$

for any $T \in \mathcal{M}$, $\|T\|_2 = 1$.

If in this inequality we take $T = P^{1/2}$, $P \in S_1(\mathcal{M})$, with P invertible, then we get

$$\begin{aligned} & \left| \langle \mathfrak{A}_{Q,P} f'(\mathfrak{A}_{Q,P}) P^{1/2}, P^{1/2} \rangle_2 - \langle f'(\mathfrak{A}_{Q,P}) P^{1/2}, P^{1/2} \rangle_2 \right| \\ & \leq \frac{1}{2} [f'(R) - f'(r)] \langle |\mathfrak{A}_{Q,P} - \langle \mathfrak{A}_{Q,P} P^{1/2}, P^{1/2} \rangle_2 1_{\mathcal{B}_2(H)}| P^{1/2}, P^{1/2} \rangle_2 \\ & \leq \frac{1}{2} [f'(R) - f'(r)] \left[\langle \mathfrak{A}_{Q,P}^2 P^{1/2}, P^{1/2} \rangle_2 - \langle \mathfrak{A}_{Q,P} P^{1/2}, P^{1/2} \rangle_2^2 \right]^{1/2} \\ & \leq \frac{1}{4} (R - r) [f'_-(R) - f'_+(r)], \end{aligned}$$

which can be written as

$$\begin{aligned} |S_{\ell f'}(Q, P) - S_{f'}(Q, P)| & \leq \frac{1}{2} [f'_-(R) - f'_+(r)] V(Q, P) \\ & \leq \frac{1}{2} [f'_-(R) - f'_+(r)] \chi(Q, P) \\ & \leq \frac{1}{4} (R - r) [f'_-(R) - f'_+(r)]. \end{aligned}$$

Making use of Theorem 4 we deduce the desired result (3.15). □

REMARK 3. If we take in (3.15) $f(t) = t^2 - 1$, then we get

$$\begin{aligned} (3.16) \quad 0 \leq \chi^2(Q, P) & \leq \frac{1}{2} (R - r) V(Q, P) \leq \frac{1}{2} (R - r) \chi(Q, P) \\ & \leq \frac{1}{4} (R - r)^2 \end{aligned}$$

for $Q, P \in S_1(\mathcal{M})$, with P invertible and satisfying the condition (3.14).

If we take in (3.15) $f(t) = t \ln t$, then we get the inequality

$$(3.17) \quad 0 \leq U(Q, P) \leq \frac{1}{2} \ln \left(\frac{R}{r} \right) V(Q, P) \leq \frac{1}{2} \ln \left(\frac{R}{r} \right) \chi(Q, P) \\ \leq \frac{1}{4} (R - r) \ln \left(\frac{R}{r} \right)$$

provided that $Q, P \in S_1(H)$, with P, Q invertible and satisfying the condition (3.14).

With the same conditions and if we take $f(t) = -\ln t$, then

$$(3.18) \quad 0 \leq U(P, Q) \leq \frac{R - r}{2rR} V(Q, P) \leq \frac{R - r}{2rR} \chi(Q, P) \leq \frac{(R - r)^2}{4rR}.$$

If we take in (3.15) $f(t) = f_q(t) = \frac{1-t^q}{1-q}$, then we get

$$(3.19) \quad 0 \leq S_{f_q}(Q, P) \leq \frac{q}{2(1-q)} \left(\frac{R^{1-q} - r^{1-q}}{R^{1-q}r^{1-q}} \right) V(Q, P) \\ \leq \frac{q}{2(1-q)} \left(\frac{R^{1-q} - r^{1-q}}{R^{1-q}r^{1-q}} \right) \chi(Q, P) \\ \leq \frac{q}{4(1-q)} \left(\frac{R^{1-q} - r^{1-q}}{R^{1-q}r^{1-q}} \right) (R - r)$$

provided that $Q, P \in S_1(\mathcal{M})$, with P, Q invertible and satisfying the condition (3.14).

4. Other Reverse Inequalities

Utilising different techniques we can obtain other upper bounds for the quantum f -divergence as follows. Applications for Umegaki relative entropy and χ^2 -divergence are also provided.

THEOREM 6. *Let $f : [0, \infty) \rightarrow \mathbb{R}$ be a continuous convex function that is normalized. If $Q, P \in S_1(\mathcal{M})$, with P invertible, and there exists $R \geq 1 \geq r \geq 0$ such that the condition (3.14) is satisfied, then*

$$(4.1) \quad 0 \leq S_f(Q, P) \leq \frac{(R - 1)f(r) + (1 - r)f(R)}{R - r}.$$

Proof. By the convexity of f we have

$$f(t) = f \left(\frac{(R - t)r + (t - r)R}{R - r} \right) \leq \frac{(R - t)f(r) + (t - r)f(R)}{R - r}$$

for any $t \in [r, R]$.

This inequality implies the following inequality in the operator order of $\mathcal{B}(\mathcal{M})$

$$f(\mathfrak{A}_{Q,P}) \leq \frac{(R1_{\mathcal{M}} - \mathfrak{A}_{Q,P})f(r) + (\mathfrak{A}_{Q,P} - r1_{\mathcal{M}})f(R)}{R - r},$$

which can be written as

$$(4.2) \quad \langle f(\mathfrak{A}_{Q,P})T, T \rangle_2 \leq \frac{f(r)}{R - r} \langle (R1_{\mathcal{M}} - \mathfrak{A}_{Q,P})T, T \rangle_2 + \frac{f(R)}{R - r} \langle (\mathfrak{A}_{Q,P} - r1_{\mathcal{M}})T, T \rangle_2$$

for any $T \in \mathcal{M}$.

Now, if we take in (4.2) $T = P^{1/2}$, $P \in S_1(\mathcal{M})$, then we get the desired result (4.2). \square

REMARK 4. If we take in (4.1) $f(t) = t^2 - 1$, then we get

$$(4.3) \quad 0 \leq \chi^2(Q, P) \leq (R - 1)(1 - r) \frac{R + r + 2}{R - r}$$

for $Q, P \in S_1(\mathcal{M})$, with P invertible and satisfying the condition (3.14).

If we take in (4.1) $f(t) = t \ln t$, then we get the inequality

$$(4.4) \quad 0 \leq U(Q, P) \leq \ln \left[r^{\frac{(R-1)r}{R-r}} R^{\frac{R(1-r)}{R-r}} \right]$$

provided that $Q, P \in S_1(\mathcal{M})$, with P, Q invertible and satisfying the condition (3.14).

If we take in (4.1) $f(t) = -\ln t$, then we get the inequality

$$(4.5) \quad 0 \leq U(P, Q) \leq \ln \left[r^{\frac{1-R}{R-r}} R^{\frac{r-1}{R-r}} \right]$$

for $Q, P \in S_1(\mathcal{M})$, with P, Q invertible and satisfying the condition (3.14).

We also have:

THEOREM 7. *Let $f : [0, \infty) \rightarrow \mathbb{R}$ be a continuous convex function that is normalized. If $Q, P \in S_1(\mathcal{M})$, with P invertible, and there*

exists $R > 1 > r \geq 0$ such that the condition (3.14) is satisfied, then

$$\begin{aligned}
 (4.6) \quad 0 \leq S_f(Q, P) &\leq \frac{(R-1)(1-r)}{R-r} \Psi_f(1; r, R) \\
 &\leq \frac{(R-1)(1-r)}{R-r} \sup_{t \in (r, R)} \Psi_f(t; r, R) \\
 &\leq (R-1)(1-r) \frac{f'_-(R) - f'_+(r)}{R-r} \\
 &\leq \frac{1}{4} (R-r) [f'_-(R) - f'_+(r)]
 \end{aligned}$$

where $\Psi_f(\cdot; r, R) : (r, R) \rightarrow \mathbb{R}$ is defined by

$$(4.7) \quad \Psi_f(t; r, R) = \frac{f(R) - f(t)}{R-t} - \frac{f(t) - f(r)}{t-r}.$$

We also have

$$\begin{aligned}
 (4.8) \quad 0 \leq S_f(Q, P) &\leq \frac{(R-1)(1-r)}{R-r} \Psi_f(1; r, R) \\
 &\leq \frac{1}{4} (R-r) \Psi_f(1; r, R) \\
 &\leq \frac{1}{4} (R-r) \sup_{t \in (r, R)} \Psi_f(t; r, R) \\
 &\leq \frac{1}{4} (R-r) [f'_-(R) - f'_+(r)].
 \end{aligned}$$

Proof. By denoting

$$\Delta_f(t; r, R) := \frac{(t-r)f(R) + (R-t)f(r)}{R-r} - f(t), \quad t \in [r, R]$$

we have

$$\begin{aligned}
 (4.9) \quad \Delta_f(t; r, R) &= \frac{(t-r)f(R) + (R-t)f(r) - (R-r)f(t)}{R-r} \\
 &= \frac{(t-r)f(R) + (R-t)f(r) - (T-t+t-r)f(t)}{R-r} \\
 &= \frac{(t-r)[f(R) - f(t)] - (R-t)[f(t) - f(r)]}{M-m} \\
 &= \frac{(R-t)(t-r)}{R-r} \Psi_f(t; r, R)
 \end{aligned}$$

for any $t \in (r, R)$.

From the proof of Theorem 6 we have

$$\begin{aligned}
 (4.10) \quad & \langle f(\mathfrak{A}_{Q,P})T, T \rangle_2 \\
 & \leq \frac{f(r)}{R-r} \langle (R1_{\mathcal{M}} - \mathfrak{A}_{Q,P})T, T \rangle_2 + \frac{f(R)}{R-r} \langle (\mathfrak{A}_{Q,P} - r1_{\mathcal{M}})T, T \rangle_2 \\
 & = \frac{(\langle \mathfrak{A}_{Q,P}T, T \rangle_2 - r) f(R) + (R - \langle \mathfrak{A}_{Q,P}T, T \rangle_2) f(r)}{R-r}
 \end{aligned}$$

for any $T \in \mathcal{M}$, $\|T\|_2 = 1$.

This implies that

$$\begin{aligned}
 (4.11) \quad & 0 \leq \langle f(\mathfrak{A}_{Q,P})T, T \rangle_2 - f(\langle \mathfrak{A}_{Q,P}T, T \rangle_2) \\
 & \leq \frac{(\langle \mathfrak{A}_{Q,P}T, T \rangle_2 - r) f(R) + (R - \langle \mathfrak{A}_{Q,P}T, T \rangle_2) f(r)}{R-r} - f(\langle \mathfrak{A}_{Q,P}T, T \rangle_2) \\
 & = \Delta_f(\langle \mathfrak{A}_{Q,P}T, T \rangle_2; r, R) \\
 & = \frac{(R - \langle \mathfrak{A}_{Q,P}T, T \rangle_2)(\langle \mathfrak{A}_{Q,P}T, T \rangle_2 - r)}{R-r} \Psi_f(\langle \mathfrak{A}_{Q,P}T, T \rangle_2; r, R)
 \end{aligned}$$

for any $T \in \mathcal{M}$, $\|T\|_2 = 1$.

Since

$$\begin{aligned}
 (4.12) \quad & \Psi_f(\langle \mathfrak{A}_{Q,P}T, T \rangle_2; r, R) \leq \sup_{t \in (r, R)} \Psi_f(t; r, R) \\
 & = \sup_{t \in (r, R)} \left[\frac{f(R) - f(t)}{R-t} - \frac{f(t) - f(r)}{t-r} \right] \\
 & \leq \sup_{t \in (r, R)} \left[\frac{f(R) - f(t)}{R-t} \right] + \sup_{t \in (r, R)} \left[-\frac{f(t) - f(r)}{t-r} \right] \\
 & = \sup_{t \in (r, R)} \left[\frac{f(R) - f(t)}{R-t} \right] - \inf_{t \in (r, R)} \left[\frac{f(t) - f(r)}{t-r} \right] \\
 & = f'_-(R) - f'_+(r),
 \end{aligned}$$

and, obviously

$$(4.13) \quad \frac{1}{R-r} (R - \langle \mathfrak{A}_{Q,P}T, T \rangle_2) (\langle \mathfrak{A}_{Q,P}T, T \rangle_2 - r) \leq \frac{1}{4} (R-r),$$

then by (4.11)-(4.13) we have

(4.14)

$$\begin{aligned}
 0 &\leq \langle f(\mathfrak{A}_{Q,P}T, T) \rangle_2 - f(\langle \mathfrak{A}_{Q,P}T, T \rangle_2) \\
 &\leq \frac{(R - \langle \mathfrak{A}_{Q,P}T, T \rangle_2)(\langle \mathfrak{A}_{Q,P}T, T \rangle_2 - r)}{R - r} \Psi_f(\langle \mathfrak{A}_{Q,P}T, T \rangle_2; r, R) \\
 &\leq \frac{(R - \langle \mathfrak{A}_{Q,P}T, T \rangle_2)(\langle \mathfrak{A}_{Q,P}T, T \rangle_2 - r)}{R - r} \sup_{t \in (r, R)} \Psi_f(t; r, R) \\
 &\leq (R - \langle \mathfrak{A}_{Q,P}T, T \rangle_2)(\langle \mathfrak{A}_{Q,P}T, T \rangle_2 - r) \frac{f'_-(R) - f'_+(r)}{R - r} \\
 &\leq \frac{1}{4}(R - r)[f'_-(R) - f'_+(r)]
 \end{aligned}$$

for any $T \in \mathcal{M}$, $\|T\|_2 = 1$.

Now, if we take in (4.14) $T = P^{1/2}$, then we get the desired result (4.6).

The inequality (4.8) is obvious from (4.6). \square

REMARK 5. If we consider the convex normalized function $f(t) = t^2 - 1$, then

$$\Psi_f(t; r, R) = \frac{R^2 - t^2}{R - t} - \frac{t^2 - r^2}{t - r} = R - r, \quad t \in (r, R)$$

and we get from (4.6) the simple inequality

$$(4.15) \quad 0 \leq \chi^2(Q, P) \leq (R - 1)(1 - r)$$

for $Q, P \in S_1(\mathcal{M})$, with P invertible and satisfying the condition (3.14), which is better than (4.3).

If we take the convex normalized function $f(t) = t^{-1} - 1$, then we have

$$\Psi_f(t; r, R) = \frac{R^{-1} - t^{-1}}{R - t} - \frac{t^{-1} - r^{-1}}{t - r} = \frac{R - r}{rRt}, \quad t \in [r, R].$$

Also

$$S_f(Q, P) = \chi^2(P, Q).$$

Using (4.6) we get

$$(4.16) \quad 0 \leq \chi^2(P, Q) \leq \frac{(R - 1)(1 - r)}{Rr}$$

for $Q, P \in S_1(\mathcal{M})$, with Q invertible and satisfying the condition (3.14).

If we consider the convex function $f(t) = -\ln t$ defined on $[r, R] \subset (0, \infty)$, then

$$\begin{aligned} \Psi_f(t; r, R) &= \frac{-\ln R + \ln t}{R - t} - \frac{-\ln t + \ln r}{t - r} \\ &= \frac{(R - r) \ln t - (R - t) \ln r - (t - r) \ln R}{(M - t)(t - m)} \\ &= \ln \left(\frac{t^{R-r}}{r^{R-t} M^{t-r}} \right)^{\frac{1}{(R-t)(t-r)}}, \quad t \in (r, R). \end{aligned}$$

Then by (4.6) we have

$$(4.17) \quad 0 \leq U(P, Q) \leq \ln \left[r^{\frac{1-R}{R-r}} R^{\frac{r-1}{R-r}} \right] \leq \frac{(R - 1)(1 - r)}{rR}$$

for $Q, P \in S_1(\mathcal{M})$, with P, Q invertible and satisfying the condition (3.14).

We also have:

THEOREM 8. *Let $f : [0, \infty) \rightarrow \mathbb{R}$ be a continuous convex function that is normalized. If $Q, P \in S_1(\mathcal{M})$, with P invertible, and there exists $R > 1 > r \geq 0$ such that the condition (3.14) is satisfied, then*

$$(4.18) \quad 0 \leq S_f(Q, P) \leq 2 \left[\frac{f(r) + f(R)}{2} - f\left(\frac{r + R}{2}\right) \right].$$

Proof. We recall the following result (see for instance [4]) that provides a refinement and a reverse for the weighted Jensen's discrete inequality:

$$\begin{aligned} (4.19) \quad & n \min_{i \in \{1, \dots, n\}} \{p_i\} \left[\frac{1}{n} \sum_{i=1}^n f(x_i) - f\left(\frac{1}{n} \sum_{i=1}^n x_i\right) \right] \\ & \leq \frac{1}{P_n} \sum_{i=1}^n p_i f(x_i) - f\left(\frac{1}{P_n} \sum_{i=1}^n p_i x_i\right) \\ & \leq n \max_{i \in \{1, \dots, n\}} \{p_i\} \left[\frac{1}{n} \sum_{i=1}^n f(x_i) - f\left(\frac{1}{n} \sum_{i=1}^n x_i\right) \right], \end{aligned}$$

where $f : C \rightarrow \mathbb{R}$ is a convex function defined on the convex subset C of the linear space X , $\{x_i\}_{i \in \{1, \dots, n\}} \subset C$ are vectors and $\{p_i\}_{i \in \{1, \dots, n\}}$ are nonnegative numbers with $P_n := \sum_{i=1}^n p_i > 0$.

For $n = 2$ we deduce from (3.6) that

$$\begin{aligned}
 (4.20) \quad & 2 \min \{s, 1 - s\} \left[\frac{f(x) + f(y)}{2} - f\left(\frac{x+y}{2}\right) \right] \\
 & \leq sf(x) + (1-s)f(y) - f(sx + (1-s)y) \\
 & \leq 2 \max \{s, 1 - s\} \left[\frac{f(x) + f(y)}{2} - f\left(\frac{x+y}{2}\right) \right]
 \end{aligned}$$

for any $x, y \in C$ and $s \in [0, 1]$.

Now, if we use the second inequality in (4.20) for $x = r$, $y = R$, $s = \frac{R-t}{R-r}$ with $t \in [r, R]$, then we have

$$\begin{aligned}
 (4.21) \quad & \frac{(R-t)f(r) + (t-r)f(R)}{R-r} - f(t) \\
 & \leq 2 \max \left\{ \frac{R-t}{R-r}, \frac{t-r}{R-r} \right\} \left[\frac{f(r) + f(R)}{2} - f\left(\frac{r+R}{2}\right) \right] \\
 & = \left[1 + \frac{2}{R-r} \left| t - \frac{r+R}{2} \right| \right] \left[\frac{f(r) + f(R)}{2} - f\left(\frac{r+R}{2}\right) \right]
 \end{aligned}$$

for any $t \in [r, R]$.

This implies in the operator order of $\mathcal{B}(\mathcal{M})$

$$\begin{aligned}
 & \frac{(R1_{\mathcal{M}} - \mathfrak{A}_{Q,P})f(r) + (\mathfrak{A}_{Q,P} - r1_{\mathcal{M}})f(R)}{R-r} - f(\mathfrak{A}_{Q,P}) \\
 & \leq \left[\frac{f(r) + f(R)}{2} - f\left(\frac{r+R}{2}\right) \right] \\
 & \times \left[1_{\mathcal{M}} + \frac{2}{R-r} \left| \mathfrak{A}_{Q,P} - \frac{r+R}{2} 1_{\mathcal{M}} \right| \right]
 \end{aligned}$$

which implies that

(4.22)

$$\begin{aligned}
 0 &\leq \langle f(\mathfrak{A}_{Q,P}T, T) \rangle_2 - f(\langle \mathfrak{A}_{Q,P}T, T \rangle_2) \\
 &\leq \frac{(\langle \mathfrak{A}_{Q,P}T, T \rangle_2 - r) f(R) + (R - \langle \mathfrak{A}_{Q,P}T, T \rangle_2) f(r)}{R - r} - f(\langle \mathfrak{A}_{Q,P}T, T \rangle_2) \\
 &\leq \left[\frac{f(r) + f(R)}{2} - f\left(\frac{r + R}{2}\right) \right] \\
 &\times \left[1 + \frac{2}{R - r} \left\langle \left| \mathfrak{A}_{Q,P} - \frac{r + R}{2} 1_{\mathcal{M}} \right| T, T \right\rangle_2 \right] \\
 &\leq 2 \left[\frac{f(r) + f(R)}{2} - f\left(\frac{r + R}{2}\right) \right]
 \end{aligned}$$

for any $T \in \mathcal{M}$, $\|T\|_2 = 1$.

If we take in (4.22) $T = P^{1/2}$, $P \in S_1(\mathcal{M})$, then we get the desired result (4.18). \square

REMARK 6. If we take $f(t) = t^2 - 1$ in (4.18), then we get

$$0 \leq \chi^2(Q, P) \leq \frac{1}{2} (R - r)^2$$

for $Q, P \in S_1(\mathcal{M})$, with P invertible and satisfying the condition (3.14), which is not as good as (4.15).

If we take in (4.18) $f(t) = t^{-1} - 1$, then we have

$$(4.23) \quad 0 \leq \chi^2(P, Q) \leq \frac{(R - r)^2}{rR(r + R)}$$

for $Q, P \in S_1(\mathcal{M})$, with P invertible and satisfying the condition (3.14).

If we take in (4.18) $f(t) = -\ln t$, then we have

$$(4.24) \quad 0 \leq U(P, Q) \leq \ln \left(\frac{(R + r)^2}{4rR} \right)$$

for $Q, P \in S_1(\mathcal{M})$, with P invertible and satisfying the condition (3.14).

From (3.18) we have the following absolute upper bound

$$(4.25) \quad 0 \leq U(P, Q) \leq \frac{(R - r)^2}{4rR}$$

for $Q, P \in S_1(\mathcal{M})$, with P invertible and satisfying the condition (3.14).

Utilising the elementary inequality $\ln x \leq x - 1$, $x > 0$, we have that

$$\ln \left(\frac{(R+r)^2}{4rR} \right) \leq \frac{(R-r)^2}{4rR},$$

which shows that (4.24) is better than (4.25).

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Silvestru Sever Dragomir

Mathematics, School of Engineering & Science
Victoria University, PO Box 14428
Melbourne City, MC 8001, Australia.

E-mail: sever.dragomir@vu.edu.au

URL: <http://rgmia.org/dragomir>

DST-NRF Centre of Excellence
in the Mathematical and Statistical Sciences
School of Computer Science and Applied Mathematics
University of the Witwatersrand
Johannesburg, Private Bag 3, Wits 2050, South Africa