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THE FEKETE-SZEGÖ INEQUALITY FOR CERTAIN CLASS OF ANALYTIC FUNCTIONS DEFINED BY CONVOLUTION BETWEEN GENERALIZED AL-OBOUDI DIFFERENTIAL OPERATOR AND SRIVASTAVA-ATTIYA INTEGRAL OPERATOR

K. A. Challab, M. Darus*, and F. Ghanim

Abstract. The aim of this paper is to investigate the Fekete Szegö inequality for subclass of analytic functions defined by convolution between generalized Al-Oboudi differential operator and Srivastava-Attiya integral operator. Further, application to fractional derivatives are also given.

1. Introduction

Normally, we are considering the class of analytic function f of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

in the open unit disc $U = \{z : |z| < 1\}$ and denoted by A and normalized by $f(0) = f'(0) - 1 = 0$. Let the function which is defined by

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* Corresponding author.

$$(1-\lambda)f(z)+\lambda zf'(z) \tag{1.2}$$

be $F_\lambda(z)$. Then, we can write $F_\lambda(z)$ as

$$F_\lambda(z)=z+\sum_{n=2}^{\infty}[1+\lambda(n-1)]a_nz^n,$$

using (1.1) and (1.2).

In 2004, Al-Oboudi [2] investigated the operator

$$F_\lambda^k(z)=z+\sum_{n=2}^{\infty}[1+\lambda(n-1)]^k a_nz^n.$$

Later, Al-Oboudi differential operator was extended by Ramadan and Darus [19] given as follows:

$$D_{\alpha,\beta,\lambda,\delta}f(z)=z+\sum_{n=2}^{\infty}[1+(\lambda-\delta)(\beta-\alpha)(n-1)]a_nz^n$$

by having

$$D_{\alpha,\beta,\lambda,\delta}f(z)=[1-(\lambda-\delta)(\beta-\alpha)]f(z)+(\lambda-\delta)(\beta-\alpha)zf'(z),$$

thus

$$D^k_{\alpha,\beta,\lambda,\delta}f(z)=z+\sum_{n=2}^{\infty}[1+(\lambda-\delta)(\beta-\alpha)(n-1)]^k a_nz^n,$$

for

$(\alpha\geq 0,\beta\geq 0,\lambda\geq 0,\delta\geq 0,\lambda\geq\delta,\beta\geq\alpha)$ and $n\in\{0,1,2,...\}$.

The Hadamard product (or convolution) of f and g can be defined by

$$(f*g)=z+\sum_{n=2}^{\infty}a_nb_nz^n,\quad z\in U,$$

such that, $f\in A$ given by (1.1) and $g\in A$ $(g(z)=z+\sum_{n=2}^{\infty}b_nz^n)$.

An important operator given by [27] named as Srivastava-Attiya operator is as follows:

$$J_{s,a}:A\to A,$$

$$J_{s,a}f(z)=G_{s,a}*f(z),\quad (z\in U; a\in\mathbb{C}\setminus\{Z_0^-\}; s\in\mathbb{C}; f\in A), \tag{1.3}$$

where, for convenience,

$$G_{s,a}(z) := (1+a)^s[\Phi(z,s,a) - a^{-s}] \quad (z \in U), \tag{1.4}$$

$$J_a^s f(z) = z + \sum_{n=2}^{\infty} \left(\frac{1+a}{n+a}\right)^s a_n z^n,$$

and $\Phi(z,s,a) \equiv \sum_{k=0}^{\infty} \frac{z^k}{(a+k)^s}$, [see [1], Section 1.11, Eq.(1)] is obtained by the successful implementation of (1.1), (1.3) and (1.4).

Now, by the Hadamard product we define the following generalized operator

$$\Upsilon_{\alpha,\beta,\lambda,\delta}^{k,a,s} f(z) = z + \sum_{n=2}^{\infty} [1 + (\lambda-\delta)(\beta-\alpha)(n-1)]^k \left(\frac{1+a}{n+a}\right)^s a_n z^n, \tag{1.5}$$

where

$$\Upsilon_{\alpha,\beta,\lambda,\delta}^{k,a,s} f(z) = D_{\alpha,\beta,\lambda,\delta}^{k} f(z) * G_{s,a}(z)$$

or as

$$\Upsilon_{\alpha,\beta,\lambda,\delta}^{k,a,s} f(z) = z + \sum_{n=2}^{\infty} A_n z^n,$$

where

$$A_n = [1 + (\lambda-\delta)(\beta-\alpha)(n-1)]^k \left(\frac{1+a}{n+a}\right)^s a_n.$$

For suitable values of s, $\alpha, \beta, \delta, \lambda$, we can see that:

(i) when $s = 0$, $\Upsilon_{\alpha,\beta,\lambda,\delta}^{k,a,0} f(z)$ is the Ramadan and Darus differential operator as presented in [19],

(ii) when $s = 0$, $\alpha = 0$, $\delta = 0$, $\lambda = 1$, $\beta = 1$, $\Upsilon_{0,1,1,0}^{k,a,0} f(z)$ is the Salagean differential operator as presented in [22],

(iii) when $s = 0$, $\alpha = 0$, $\Upsilon_{0,\beta,\lambda,\delta}^{k,a,0} f(z)$ is the Darus and Ibrahim differential operator as presented in [10],

(iv) when $s = 0$, $\alpha = 0$, $\delta = 0$, $\beta = 1, \Upsilon_{0,1,\lambda,0}^{k,a,0} f(z)$ is the Al-Oboudi differential operator as presented in [2].

In the case where Σ is the class of functions $\Upsilon_{\alpha,\beta,\lambda,\delta}^{k,a,s} f(z)$, it is obvious that $\Upsilon_{\alpha,\beta,\lambda,\delta}^{k,a,s} f(z)$ is an analytical function in U [8].

If S is defined as the class of functions $\Upsilon^{k,a,s}_{\alpha,\beta,\lambda,\delta}f(z) \in \Sigma$, which are univalent in U.

Then we can recall that a Schwarzian function

$$w(z) = \sum_{n=1}^{\infty} c_n z^n$$

has the condition of $w(0) = 0$ and $|w(z)| < 1$ and is analytical in U.

If f and g are two analytical functions in U, then f is said to be subordinate to g (symbolically $f \prec g$) when a Schwarz function $w(z) \in U$ exists, such that $f(z) = g(w(z))$ ([9], [25]).

As studied by Goel and Mehrok [13], $S^*_\lambda(A,B)$ can be a subclass of the functions $\Upsilon^{k,a,s}_{\alpha,\beta,\lambda,\delta}f(z) \in \Sigma$ with the condition of

$$\frac{z\left(\Upsilon^{k,a,s}_{\alpha,\beta,\lambda,\delta}f(z)\right)'}{\Upsilon^{k,a,s}_{\alpha,\beta,\lambda,\delta}f(z)} \prec \frac{1+Az}{1+Bz}, \quad -1 \le B < A \le 1, z \in U,$$

where $S^*_\lambda(A,B)$ is a subclass of the starlike functions, and $S^*_0(A,B) \equiv S^*(A,B)$. In particular, $S^*_0(1,-1) \equiv S^*$, which is a class of the starlike functions [26].

$K_\lambda(A,B)$ is a subclass of the functions $\Upsilon^{k,a,s}_{\alpha,\beta,\lambda,\delta}f(z) \in \Sigma$, with the condition of

$$\frac{\left(z\left(\Upsilon^{k,a,s}_{\alpha,\beta,\lambda,\delta}f(z)\right)'\right)'}{\left(\Upsilon^{k,a,s}_{\alpha,\beta,\lambda,\delta}f(z)\right)'} \prec \frac{1+Az}{1+Bz}, -1 \le B < A \le 1, z \in U.$$

Note that, $K_0(A,B) \equiv K(A,B)$ and is a subclass of the convex functions. Most particularly $K_0(1,-1) \equiv K$, and is a class of the convex functions [26].

Now, we shall give a brief background on Fekete and Szegö problem which is our ultimate goal. Back in 1933, Fekete and Szegö [11] studied a special inequality which arises naturally from a combination of two coefficients a_2 and a_3 of a class of univalent analytic normalised function S. They obtained sharp upper bound of $|a_3 - \mu a_2^2|$, where μ is real. Later Keogh and Merkes [14], Koepf [15] and many others follow the same problems with different techniques for different classes.

The problems evolve and many new subclasses are focussed on Fekete-Szegö problem. For examples, Alhindi and Darus [5] studied the problems on a function belongs to a new subclass of Sakaguchi type which was determined through fractional derivatives, Ravichandran et al. [20] studied the problems for the class $z^{1-\alpha}f'(z)/f^{1-\alpha}(z)(\alpha \geq 0)$ which was located in a starlike region with respect to 1 and was symmetric with respect to the real axis, and Selvaraj and Kumar [23] studied for the class $\Re\left[1+\frac{1}{b}\left\{(1-\alpha)\frac{f(z)}{z}+\alpha f'(z)-1\right\}\right] \succ 0$. For other examples defined on various classes can be read in ([3], [4], [6], [7], [12], [16], [17], [19], [21], [24], [28]).

Motivated by all the work done above, we obtain sharp upper bound of $|a_3-\mu a_2^2|$ for classes of starlike $S^*_\lambda(A,B)$ and convex $K_\lambda(A,B)$, respectively.

2. Preliminary Results

To begin our results, P is defined as the family of all the functions r analytical in U, for which $\Re(r(z))>0$ and

$$r(z)=1+r_1z+r_2z^2+...$$

for $z\in U$. The following lemma was necessary to obtain our results:

Lemma 2.1. *[19] If $q(z)=1+r_1z+r_2z^2+...$ is an analytic function with positive real part in U, then*

$$\left|r_2-\nu r_1^2\right| := \begin{cases} -4\nu+2 & \nu\leq 0; \\ 2 & 0\leq\nu\leq 1; \\ 4\nu-2 & \nu\geq 1, \end{cases}$$

when $\nu<0$ or $\nu>1$, $q(z)$ is $(1+z)/(1-z)$ or one of its rotations if and only if the equality holds. If $0<\nu<1$, then $q(z)$ is $(1+z^2)/(1-z^2)$ or one of its rotations if and only if the equality holds. If $\nu=0$, then

$$q(z)=\left(\frac{1}{2}+\frac{1}{2}\gamma\right)\frac{1+z}{1-z}+\left(\frac{1}{2}-\frac{1}{2}\gamma\right)\frac{1-z}{1+z} \quad (0\leq\gamma\leq 1)$$

or one of its rotations if and only if the equality holds. If $\nu=1$, $q(z)$ is the reciprocal of one of the functions if and only if the equality holds *such that the equality holds in the case of $\nu = 0$. Also the above upper bound can be improved as follows when $0 < \nu < 1$.*

$$\left|r_2 - \nu r_1^2\right| + \nu|r_1|^2 \leq 2 \quad \left(0 < \nu \leq \frac{1}{2}\right)$$

and

$$\left|r_2 - \nu r_1^2\right| + (1-\nu)|r_1|^2 \leq 2 \quad \left(\frac{1}{2} < \nu \leq 1\right),$$

and it is sharp.

3. Fekete-Szegö Problem For $S_\lambda^*(A, B)$

The main result is the following:

THEOREM 3.1. *Let $\phi(z) = 1 + B_1 z + B_2 z^2 + \dots$. If $f(z)$ given by (1.1) belongs to $S_\lambda^*(A, B)$, then*

$$\left|a_3 - \mu a_2^2\right| := \begin{cases} \dfrac{-\mu(A-B)^2}{[(\lambda-\delta)(\beta-\alpha)+1]^{2k}\left(\frac{1+a}{2+a}\right)^{2s}} + \\ \dfrac{A(A-B)}{2[2(\lambda-\delta)(\beta-\alpha)+1]^{k}\left(\frac{1+a}{3+a}\right)^{s}} - \\ \dfrac{B(A-B)}{[2(\lambda-\delta)(\beta-\alpha)+1]^{k}\left(\frac{1+a}{3+a}\right)^{s}} & \mu \leq \sigma_1; \\ \dfrac{A-B}{2[2(\lambda-\delta)(\beta-\alpha)+1]^{k}\left(\frac{1+a}{3+a}\right)^{s}} & \sigma_1 \leq \mu \leq \sigma_2; \\ \dfrac{\mu(A-B)^2}{[(\lambda-\delta)(\beta-\alpha)+1]^{2k}\left(\frac{1+a}{2+a}\right)^{2s}} - \\ \dfrac{A(A-B)}{2[2(\lambda-\delta)(\beta-\alpha)+1]^{k}\left(\frac{1+a}{3+a}\right)^{s}} + \\ \dfrac{B(A-B)}{[2(\lambda-\delta)(\beta-\alpha)+1]^{k}\left(\frac{1+a}{3+a}\right)^{s}} & \mu \geq \sigma_2, \end{cases} \tag{3.1}$$

where

$$\sigma_1 = \frac{(A-2B-1)[(\lambda-\delta)(\beta-\alpha)+1]^{2k}\left(\frac{1+a}{2+a}\right)^{2s}}{2(A-B)[2(\lambda-\delta)(\beta-\alpha)+1]^{k}\left(\frac{1+a}{3+a}\right)^{s}}$$

and

$$\sigma_2 = \frac{(A-2B+1)[(\lambda-\delta)(\beta-\alpha)+1]^{2k}\left(\frac{1+a}{2+a}\right)^{2s}}{2(A-B)[2(\lambda-\delta)(\beta-\alpha)+1]^{k}\left(\frac{1+a}{3+a}\right)^{s}}.$$

The result is sharp.

If we follow a method like the one used in several other articles for example [17], [19] and [20], the following proof of Theorem 3.1 is obtained.

Proof. If $\Upsilon^{k,a,s}_{\alpha,\beta,\lambda,\delta}f(z) \in S^*_\lambda(A,B)$, then a Schwarz function $w(z)$ exists, such that

$$\frac{z\left(\Upsilon^{k,a,s}_{\alpha,\beta,\lambda,\delta}f(z)\right)'}{\Upsilon^{k,a,s}_{\alpha,\beta,\lambda,\delta}f(z)} = \phi(w(z)), \tag{3.2}$$

where

$$\phi(z) = \frac{1+Az}{1+Bz} = 1+(A-B)z - B(A-B)z^2 + B^2(A-B)z^3 + ...$$

$$= 1 + B_1 z + B_2 z^2 + B_3 z^3 + ... \ . \tag{3.3}$$

The function $p(z)$ is determined to be following:

$$p(z) = \frac{1+w(z)}{1-w(z)} = 1 + c_1 z + c_2 z^2 + c_3 z^3 + ... \ . \tag{3.4}$$

Because $w(z)$ is a Schwarz function. $Re(p(z)) > 0$ and $p(0) = 1$ is obvious. If we use

$$h(z) = \frac{z\left(\Upsilon^{k,a,s}_{\alpha,\beta,\lambda,\delta}f(z)\right)'}{\Upsilon^{k,a,s}_{\alpha,\beta,\lambda,\delta}f(z)} = 1 + b_1 z + b_2 z^2 + b_3 z^3 + ... \ , \tag{3.5}$$

to define the function of $h(z)$, we can find

$$h(z) = \phi\left(\frac{p(z)-1}{p(z)+1}\right) = \phi\left(\frac{c_1 z + c_2 z^2 + c_3 z^3 + ...}{2 + c_1 z + c_2 z^2 + c_3 z^3 + ...}\right)$$

$$= \phi\left(\frac{1}{2}c_1 z + \frac{1}{2}\left(c_2 - \frac{{c_1}^2}{2}\right)z^2 + \frac{1}{2}\left(c_3 - c_1 c_2 - \frac{{c_1}^3}{4}\right)z^3 + ...\right)$$

$$= 1 + \frac{B_1 c_1}{2}z + \left[\frac{B_1}{2}\left(c_2 - \frac{{c_1}^2}{2}\right) + \frac{B_2 {c_1}^2}{4}\right]z^2$$

$$+\left[\frac{B_1}{2}\left(c_3-c_1c_2+\frac{{c_1}^3}{4}\right)+\frac{B_2c_1}{2}\left(c_2-\frac{{c_1}^2}{2}\right)+\frac{B_3{c_1}^3}{8}\right]z^3+\ldots\ ,$$

from equations (3.2), (3.4) and (3.5). Thus,

$$b_1=\frac{B_1c_1}{2};b_2=\frac{B_1}{2}\left(c_2-\frac{{c_1}^2}{2}\right)+\frac{B_2{c_1}^2}{4}. \tag{3.6}$$

By applying (3.3) and (3.5) in (3.6)

$$a_2=\frac{(A-B)c_1}{2[(\lambda-\delta)(\beta-\alpha)+1]^k\left(\frac{1+a}{2+a}\right)^s},$$

and

$$a_3=\frac{A-B}{8[2(\lambda-\delta)(\beta-\alpha)+1]^k(\frac{1+a}{3+a})^s}\left[2c_2+(A-2B-1){c_1}^2\right],$$

are obtained.

As a result, we have

$$a_3-\mu{a_2}^2=\frac{A-B}{4[2(\lambda-\delta)(\beta-\alpha)+1]^k\left(\frac{1+a}{3+a}\right)^s}\Bigg[c_2-\left(\frac{\mu(A-B)[2(\lambda-\delta)(\beta-\alpha)+1]^k\left(\frac{1+a}{3+a}\right)^s}{[(\lambda-\delta)(\beta-\alpha)+1]^{2k}\left(\frac{1+a}{2+a}\right)^{2s}}-\frac{A-2B-1}{2}\right){c_1}^2\Bigg]$$

$$a_3-\mu{a_2}^2=\frac{A-B}{4[2(\lambda-\delta)(\beta-\alpha)+1]^k\left(\frac{1+a}{3+a}\right)^s}\left[c_2-\nu{c_1}^2\right],$$

where

$$\nu=\frac{1}{2}\left(\frac{2\mu(A-B)[2(\lambda-\delta)(\beta-\alpha)+1]^k\left(\frac{1+a}{3+a}\right)^s}{[(\lambda-\delta)(\beta-\alpha)+1]^{2k}\left(\frac{1+a}{2+a}\right)^{2s}}-(A-2B-1)\right).$$

If $\mu\le\sigma_1$, then if we apply Lemma 2.1

$$\left|a_3-\mu{a_2}^2\right|\le\frac{-\mu(A-B)^2}{[(\lambda-\delta)(\beta-\alpha)+1]^{2k}\left(\frac{1+a}{2+a}\right)^{2s}}+\frac{A(A-B)}{2[2(\lambda-\delta)(\beta-\alpha)+1]^k\left(\frac{1+a}{3+a}\right)^s}-\frac{B(A-B)}{[2(\lambda-\delta)(\beta-\alpha)+1]^k\left(\frac{1+a}{3+a}\right)^s}$$

is obtained.

This is the first part of assertion (3.1).

Later, if $\mu \geq \sigma_2$, we can apply Lemma 2.1 to get

$$\left|a_3 - \mu a_2{}^2\right| \leq \frac{\mu(A-B)^2}{[(\lambda-\delta)(\beta-\alpha)+1]^{2k}\left(\frac{1+a}{2+a}\right)^{2s}} - \frac{A(A-B)}{2[2(\lambda-\delta)(\beta-\alpha)+1]^{k}\left(\frac{1+a}{3+a}\right)^{s}} + \frac{B(A-B)}{[2(\lambda-\delta)(\beta-\alpha)+1]^{k}\left(\frac{1+a}{3+a}\right)^{s}}.$$

This is the third part of assertion (3.1).

If $\mu = \sigma_1$, then equality holds if and only if

$$p(z) = \left(\frac{1+\gamma}{2}\right)\frac{1+z}{1-z} + \left(\frac{1-\gamma}{2}\right)\frac{1-z}{1+z}, \quad (0 \leq \gamma \leq 1, z \in U)$$

or one of its rotations.

If $\mu = \sigma_2$, then we have

$$\nu = \frac{1}{2}\left(\frac{2\mu(A-B)[2(\lambda-\delta)(\beta-\alpha)+1]^{k}\left(\frac{1+a}{3+a}\right)^{s}}{[(\lambda-\delta)(\beta-\alpha)+1]^{2k}\left(\frac{1+a}{2+a}\right)^{2s}} - (A-2B-1)\right) = 0,$$

resulting in,

$$\frac{1}{p(z)} = \left(\frac{1+\gamma}{2}\right)\frac{1+z}{1-z} + \left(\frac{1-\gamma}{2}\right)\frac{1-z}{1+z}, \quad (0 \leq \gamma \leq 1, z \in U).$$

In the end, it is observed that

$$\left|a_3 - \mu a_2{}^2\right| = \frac{A-B}{4[2(\lambda-\delta)(\beta-\alpha)+1]^{k}\left(\frac{1+a}{3+a}\right)^{s}}\left|\left[c_2 - \left(\frac{\mu(A-B)[2(\lambda-\delta)(\beta-\alpha)+1]^{k}\left(\frac{1+a}{3+a}\right)^{s}}{[(\lambda-\delta)(\beta-\alpha)+1]^{2k}\left(\frac{1+a}{2+a}\right)^{2s}} - \frac{A-2B-1}{2}\right)c_1{}^2\right]\right|,$$

and

$$\max\left|\frac{1}{2}\left(\frac{2\mu(A-B)[2(\lambda-\delta)(\beta-\alpha)+1]^k\left(\frac{1+a}{3+a}\right)^s}{[(\lambda-\delta)(\beta-\alpha)+1]^{2k}\left(\frac{1+a}{2+a}\right)^{2s}}-(A-2B-1)\right)\right|\leq$$
$1, \quad (\sigma_1 \leq \mu \leq \sigma_2)$.

Thus, if Lemma 2.1 is used, the result is

$$\left|a_3-\mu a_2{}^2\right|=\frac{A-B}{4[2(\lambda-\delta)(\beta-\alpha)+1]^k\left(\frac{1+a}{3+a}\right)^s}\leq$$
$$\frac{A-B}{2[2(\lambda-\delta)(\beta-\alpha)+1]^k\left(\frac{1+a}{3+a}\right)^s}, \quad (\sigma_1 \leq \mu \leq \sigma_2)$$
.

If $\sigma_1 < \mu < \sigma_2$, then we obtain

$$p(z)=\frac{1+\gamma z^2}{1-\gamma z^2}, \quad (0\leq\gamma\leq 1).$$

Our result is now followed by the implementation of Lemma 2.1. The function Q_n^ϕ (n=2,3,...) can be defined by

$$h(z)=\frac{z\left(\Upsilon^{k,a,s}_{\alpha,\beta,\lambda,\delta}Q_n^\phi\right)'}{\Upsilon^{k,a,s}_{\alpha,\beta,\lambda,\delta}Q_n^\phi}=\phi(z^{n-1}), Q_n^\phi(0)=0=[Q_n^\phi(0)]'-1$$

and the function F_γ and E_γ, $(0\leq\gamma\leq 1)$ can be defined by

$$\frac{z\left(\Upsilon^{k,a,s}_{\alpha,\beta,\lambda,\delta}F_\gamma(z)\right)'}{\Upsilon^{k,a,s}_{\alpha,\beta,\lambda,\delta}F_\gamma(z)}=\phi\left(\frac{z(z+\gamma)}{1+\gamma z}\right), F_\gamma(0)=0=[F_\gamma(0)]'-1$$

to show that the bounds are sharp, and

$$\frac{z\left(\Upsilon^{k,a,s}_{\alpha,\beta,\lambda,\delta}E_\gamma(z)\right)'}{\Upsilon^{k,a,s}_{\alpha,\beta,\lambda,\delta}E_\gamma(z)}=\phi\left(-\frac{z(z+\gamma)}{1+\gamma z}\right), E_\gamma(0)=0=[E_\gamma(0)]'-1.$$

It is obvious that, the functions of $Q_n^\phi, F_\gamma, E_\gamma \in S^*_\lambda(A,B)$, $Q^\phi = Q_2^\phi$ can also be written.

The equality holds for $\mu<\sigma_1$ or $\mu>\sigma_2$, if and only if f is Q^ϕ or one of its rotations. The equality holds When $\sigma_1<\mu<\sigma_2$, if and only if f is Q_3^ϕ or one of its rotations. The equality holds for $\mu=\sigma_1$, if and only if f is F_γ or one of its rotations. The equality holds for $\mu = \sigma_2$, if and only if f is E_γ or one of its rotations. $\square$

Remark 3.2. Then, in view of Lemma 2.1, Theorem3.1 can be improved If $\sigma_1 \leq \mu \leq \sigma_2$.

Now, if

$$\frac{(A-2B)[(\lambda-\delta)(\beta-\alpha)+1]^{2k}\left(\frac{1+a}{2+a}\right)^{2s}}{2(A-B)[2(\lambda-\delta)(\beta-\alpha)+1]^{k}\left(\frac{1+a}{3+a}\right)^{s}}$$

gives σ_3, then for the values of $\sigma_1 \leq \mu \leq \sigma_3$

$$\left|a_3-\mu a_2{}^2\right| + \Big[\frac{2\mu(A-B)[2(\lambda-\delta)(\beta-\alpha)+1]^{k}\left(\frac{1+a}{3+a}\right)^{s}}{2(A-B)[2(\lambda-\delta)(\beta-\alpha)+1]^{k}\left(\frac{1+a}{3+a}\right)^{s}} - \frac{(A-2B-1)[(\lambda-\delta)(\beta-\alpha)+1]^{2k}\left(\frac{1+a}{2+a}\right)^{2s}}{2(A-B)[2(\lambda-\delta)(\beta-\alpha)+1]^{k}\left(\frac{1+a}{3+a}\right)^{s}}\Big] \times |a_2|^2 \leq \frac{(A-B)}{2[2(\lambda-\delta)(\beta-\alpha)+1]^{k}\left(\frac{1+a}{3+a}\right)^{s}}$$

is obtained.

Likewise,

$$\left|a_3-\mu a_2{}^2\right| + \Big[\frac{(A-2B+1)[(\lambda-\delta)(\beta-\alpha)+1]^{2k}\left(\frac{1+a}{2+a}\right)^{2s}}{2(A-B)[2(\lambda-\delta)(\beta-\alpha)+1]^{k}\left(\frac{1+a}{3+a}\right)^{s}} - \frac{2\mu(A-B)[2(\lambda-\delta)(\beta-\alpha)+1]^{k}\left(\frac{1+a}{3+a}\right)^{s}}{2(A-B)[2(\lambda-\delta)(\beta-\alpha)+1]^{k}\left(\frac{1+a}{3+a}\right)^{s}}\Big] \times |a_2|^2 \leq \frac{(A-B)}{2[2(\lambda-\delta)(\beta-\alpha)+1]^{k}\left(\frac{1+a}{3+a}\right)^{s}},$$

is obtained for the values $\sigma_3 \leq \mu \leq \sigma_2$.

Proof. For the values of $\sigma_1 \le \mu \le \sigma_3$, we have

$$\left|a_3 - \mu {a_2}^2\right| + (\mu - \sigma_1)|a_2|^2 = \frac{(A-B)}{4[2(\lambda-\delta)(\beta-\alpha)+1]^k \left(\frac{1+a}{3+a}\right)^s} \left|c_2 - \nu {c_1}^2\right| +$$

$$(\mu - \sigma_1)\frac{(A-B)^2|c_1|^2}{4[(\lambda-\delta)(\beta-\alpha)+1]^{2k} \left(\frac{1+a}{2+a}\right)^{2s}}$$

$$= \frac{(A-B)}{4[2(\lambda-\delta)(\beta-\alpha)+1]^k \left(\frac{1+a}{3+a}\right)^s} \left|c_2 - \nu {c_1}^2\right| + \Bigg(\mu -$$

$$\frac{(A-2B-1)[(\lambda-\delta)(\beta-\alpha)+1]^{2k}\left(\frac{1+a}{2+a}\right)^{2s}}{2(A-B)[2(\lambda-\delta)(\beta-\alpha)+1]^k \left(\frac{1+a}{3+a}\right)^s}\Bigg) \times \frac{(A-B)^2|c_1|^2}{4[(\lambda-\delta)(\beta-\alpha)+1]^{2k} \left(\frac{1+a}{2+a}\right)^{2s}}$$

$$= \frac{(A-B)}{2[2(\lambda-\delta)(\beta-\alpha)+1]^k \left(\frac{1+a}{3+a}\right)^s} \left(\frac{1}{2}\left[\left|c_2 - \nu {c_1}^2\right| + \nu|c_1|^2\right]\right) \le$$

$$\frac{(A-B)}{2[2(\lambda-\delta)(\beta-\alpha)+1]^k \left(\frac{1+a}{3+a}\right)^s}.$$

Similarly, for the value of $\sigma_3 \le \mu \le \sigma_2$, we get

$$\left|a_3 - \mu {a_2}^2\right| + (\sigma_2 - \mu)|a_2|^2 = \frac{(A-B)}{4[2(\lambda-\delta)(\beta-\alpha)+1]^k \left(\frac{1+a}{3+a}\right)^s} \left|c_2 - \nu {c_1}^2\right| +$$

$$(\sigma_2 - \mu)\frac{(A-B)^2|c_1|^2}{4[(\lambda-\delta)(\beta-\alpha)+1]^{2k} \left(\frac{1+a}{2+a}\right)^{2s}}$$

$$= \frac{(A-B)}{4[2(\lambda-\delta)(\beta-\alpha)+1]^k \left(\frac{1+a}{3+a}\right)^s} \left|c_2 - \nu {c_1}^2\right| +$$

$$\left(\frac{(A-2B+1)[(\lambda-\delta)(\beta-\alpha)+1]^{2k}\left(\frac{1+a}{2+a}\right)^{2s}}{2(A-B)[2(\lambda-\delta)(\beta-\alpha)+1]^k \left(\frac{1+a}{3+a}\right)^s} - \mu\right) \times \frac{(A-B)^2|c_1|^2}{4[(\lambda-\delta)(\beta-\alpha)+1]^{2k} \left(\frac{1+a}{2+a}\right)^{2s}}$$

$$= \frac{(A-B)}{2[2(\lambda-\delta)(\beta-\alpha)+1]^k \left(\frac{1+a}{3+a}\right)^s} \left(\frac{1}{2}\left[\left|c_2 - \nu {c_1}^2\right| + (1-\nu)|c_1|^2\right]\right) \le$$

$$\frac{(A-B)}{2[2(\lambda-\delta)(\beta-\alpha)+1]^k \left(\frac{1+a}{3+a}\right)^s}.$$

Therefore, Remark 3.2 remains true. □

4. Applications To Functions Defined By Fractional Derivatives

DEFINITION 4.1. [20] Let f be analytic function in a simply-connected region of the z-plane containing the region. The fractional derivative of f order η is defined by

$$C_z^\eta f(z)\frac{1}{\Gamma(1-\eta)}\frac{d}{dz}\int_0^z \frac{f(\zeta)}{(z-\zeta)^\eta}, \quad (0 \le \eta < 1),$$

where the multiplicity of $(z-\zeta)^\eta$ is removed by requiring that $\log(z-\zeta)$ is real for $z-\zeta > 0$. Using the above definition and its known extensions involving fractional derivatives and fractional integrals, Owa and Srivastava [18] introduced the operator $\Omega^\eta : A \to A$ defined by

$$(\Omega^\eta f)(z) = \Gamma(2-\eta)z^\eta C_z^\eta f(z), \quad (\eta \ne 2, 3, 4, ...).$$

The class $S_\lambda^*(A, B, \eta)$ consists the functions of $f \in A$ for which $\Omega^\eta f \in S_\lambda^*(A, B)$.

It can be noted that, when

$$g(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\eta)}{\Gamma(n+1-\eta)} z^n,$$

$S_\lambda^*(A, B, \eta)$ is the special case of the class $S_\lambda^*(A, B, g)$.

Let

$$g(z) = z + \sum_{n=2}^{\infty} g_n z^n, \quad (g_n > 0).$$

Since

$$\begin{aligned} &\Upsilon_{\alpha,\beta,\lambda,\delta}^{k,a,s} f(z) \\ = \quad & z + \sum_{n=2}^{\infty} [1 + (\lambda-\delta)(\beta-\alpha)(n-1)]^k \left(\frac{1+a}{n+a}\right)^s a_n z^n \in S_\lambda^*(A, B, g) \end{aligned}$$

if and only if

$$\Upsilon^{k,a,s}_{\alpha,\beta,\lambda,\delta}(f*g)(z) \\ = z + \sum_{n=2}^{\infty}[1+(\lambda-\delta)(\beta-\alpha)(n-1)]^k\left(\frac{1+a}{n+a}\right)^s a_n g_n z^n \in S^*_\lambda(A,B). \tag{4.1}$$

The estimation of the coefficient for the functions in the class $S^*_\lambda(A,B,g)$ is obtained from the estimation which corresponds to the functions of f in class $S^*_\lambda(A,B)$. If Theorem 3.1 is applied for operator (4.1), Theorem 4.1 is obtained after an obvious change of the parameter μ:

THEOREM 4.1. *Let $g(z) = z + \sum_{n=2}^{\infty} g_n z^n$, $(g_n > 0)$, and let the function $\phi(z)$ be given by $\phi(z) = 1+\sum_{n=1}^{\infty} B_n z^n$. If operator $\Upsilon^{k,a,s}_{\alpha,\beta,\lambda,\delta}f(z)$ given by (1.5) belongs to $S^*_\lambda(A,B,g)$, then*

$$\left|a_3 - \mu a_2^2\right| := \begin{cases} \frac{1}{g_3}\Big[\frac{-\mu(A-B)^2 g_3}{[(\lambda-\delta)(\beta-\alpha)+1]^{2k}\left(\frac{1+a}{2+a}\right)^{2s} g_2^2} + \\ \frac{A(A-B)}{2[2(\lambda-\delta)(\beta-\alpha)+1]^k\left(\frac{1+a}{3+a}\right)^s} - \\ \frac{B(A-B)}{[2(\lambda-\delta)(\beta-\alpha)+1]^k\left(\frac{1+a}{3+a}\right)^s}\Big] & \mu \le \sigma_1; \\ \frac{1}{g_3}\Big[\frac{A-B}{2[2(\lambda-\delta)(\beta-\alpha)+1]^k\left(\frac{1+a}{3+a}\right)^s}\Big] & \sigma_1 \le \mu \le \sigma_2; \\ \frac{1}{g_3}\Big[\frac{\mu(A-B)^2 g_3}{[(\lambda-\delta)(\beta-\alpha)+1]^{2k}\left(\frac{1+a}{2+a}\right)^{2s} g_2^2} - \\ \frac{A(A-B)}{2[2(\lambda-\delta)(\beta-\alpha)+1]^k\left(\frac{1+a}{3+a}\right)^s} + \\ \frac{B(A-B)}{[2(\lambda-\delta)(\beta-\alpha)+1]^k\left(\frac{1+a}{3+a}\right)^s}\Big] & \mu \ge \sigma_2, \end{cases}$$

where

$$\sigma_1 = \frac{g_2^2}{g_3}.\frac{(A-2B-1)[(\lambda-\delta)(\beta-\alpha)+1]^{2k}\left(\frac{1+a}{2+a}\right)^{2s}}{2(A-B)[2(\lambda-\delta)(\beta-\alpha)+1]^k\left(\frac{1+a}{3+a}\right)^s}$$

and

$$\sigma_2 = \frac{g_2^2}{g_3} . \frac{(A-2B+1)[(\lambda-\delta)(\beta-\alpha)+1]^{2k}\left(\frac{1+a}{2+a}\right)^{2s}}{2(A-B)[2(\lambda-\delta)(\beta-\alpha)+1]^{k}\left(\frac{1+a}{3+a}\right)^{s}}.$$

The result is sharp.

Since

$$\begin{aligned}&(\Omega^{\eta}\Upsilon^{k,a,s}_{\alpha,\beta,\lambda,\delta}f)(z)\\ = \quad & z+\sum_{n=2}^{\infty}\frac{\Gamma(n+1)\Gamma(2-\eta)}{\Gamma(n+1-\eta)}[1+(\lambda-\delta)(\beta-\alpha)(n-1)]^{k}\left(\frac{1+a}{n+a}\right)^{s}a_n z^n\end{aligned}$$

we have

$$g_2 := \frac{\Gamma(3)\Gamma(2-\eta)}{\Gamma(3-\eta)} = \frac{2}{2-\eta} \tag{4.2}$$

and

$$g_3 := \frac{\Gamma(4)\Gamma(2-\eta)}{\Gamma(4-\eta)} = \frac{6}{(2-\eta)(3-\eta)}. \tag{4.3}$$

Theorem 4.1 is reduced to Theorem 4.2 for g_2 and g_3 given by (4.2) and (4.3).

THEOREM 4.2. *Let* $g(z) = z+\sum_{n=2}^{\infty} g_n z^n$, $\quad (g_n > 0)$, *and let the function* $\phi(z)$ *be given by* $\phi(z) = 1+\sum_{n=1}^{\infty} B_n z^n$. *If operator* $\Upsilon^{k,a,s}_{\alpha,\beta,\lambda,\delta}f(z)$

given by (1.5) belongs to $S^*_\lambda(A,B,\eta)$*, then*

$$\left|a_3-\mu a_2^2\right| := \begin{cases} \frac{(2-\eta)(3-\eta)}{6}\Big[-\frac{3(2-\eta)}{2(3-\eta)}\frac{\mu(A-B)^2}{[(\lambda-\delta)(\beta-\alpha)+1]^{2k}\left(\frac{1+a}{2+a}\right)^{2s}} \\ +\frac{A(A-B)}{2[2(\lambda-\delta)(\beta-\alpha)+1]^k\left(\frac{1+a}{3+a}\right)^s} \\ -\frac{B(A-B)}{[2(\lambda-\delta)(\beta-\alpha)+1]^k\left(\frac{1+a}{3+a}\right)^s}\Big] & \mu\le\sigma_1; \\ \frac{(2-\eta)(3-\eta)}{6}\Big[\frac{A-B}{2[2(\lambda-\delta)(\beta-\alpha)+1]^k\left(\frac{1+a}{3+a}\right)^s}\Big] & \sigma_1\le\mu\le\sigma_2; \\ \frac{(2-\eta)(3-\eta)}{6}\Big[\frac{3(2-\eta)}{2(3-\eta)}\frac{\mu(A-B)^2}{[(\lambda-\delta)(\beta-\alpha)+1]^{2k}\left(\frac{1+a}{2+a}\right)^{2s}} \\ -\frac{A(A-B)}{2[2(\lambda-\delta)(\beta-\alpha)+1]^k\left(\frac{1+a}{3+a}\right)^s} \\ +\frac{B(A-B)}{[2(\lambda-\delta)(\beta-\alpha)+1]^k\left(\frac{1+a}{3+a}\right)^s}\Big] & \mu\ge\sigma_2, \end{cases}$$

where

$$\sigma_1=\frac{2(3-\eta)}{3(2-\eta)}.\frac{(A-2B-1)[(\lambda-\delta)(\beta-\alpha)+1]^{2k}\left(\frac{1+a}{2+a}\right)^{2s}}{2(A-B)[2(\lambda-\delta)(\beta-\alpha)+1]^k\left(\frac{1+a}{3+a}\right)^s}$$

and

$$\sigma_2=\frac{2(3-\eta)}{3(2-\eta)}.\frac{(A-2B+1)[(\lambda-\delta)(\beta-\alpha)+1]^{2k}\left(\frac{1+a}{2+a}\right)^{2s}}{2(A-B)[2(\lambda-\delta)(\beta-\alpha)+1]^k\left(\frac{1+a}{3+a}\right)^s}.$$

The result is sharp.

5. Fekete-Szegö Problem For $K_\lambda(A,B)$

The main result is the following:

THEOREM 5.1. *Let $\phi(z) = 1 + B_1 z + B_2 z^2 + \ldots$. If $f(z)$ given by (1.1) belongs to $K_\lambda(A, B)$, then*

$$\left|a_3 - \mu a_2^2\right| := \begin{cases} \dfrac{-\mu(A-B)^2}{4[(\lambda-\delta)(\beta-\alpha)+1]^{2k}\left(\frac{1+a}{2+a}\right)^{2s}} + \\ \dfrac{A(A-B)}{6[2(\lambda-\delta)(\beta-\alpha)+1]^{k}\left(\frac{1+a}{3+a}\right)^{s}} - \\ \dfrac{B(A-B)}{3[2(\lambda-\delta)(\beta-\alpha)+1]^{k}(\frac{1+a}{3+a})^{s}} & \mu \le \sigma_1; \\ \dfrac{A-B}{6[2(\lambda-\delta)(\beta-\alpha)+1]^{k}\left(\frac{1+a}{3+a}\right)^{s}} & \sigma_1 \le \mu \le \sigma_2; \\ \dfrac{\mu(A-B)^2}{4[(\lambda-\delta)(\beta-\alpha)+1]^{2k}\left(\frac{1+a}{2+a}\right)^{2s}} - \\ \dfrac{A(A-B)}{6[2(\lambda-\delta)(\beta-\alpha)+1]^{k}\left(\frac{1+a}{3+a}\right)^{s}} + \\ \dfrac{B(A-B)}{3[2(\lambda-\delta)(\beta-\alpha)+1]^{k}\left(\frac{1+a}{3+a}\right)^{s}} & \mu \ge \sigma_2, \end{cases}$$

where

$$\sigma_1 = \frac{2(A-2B-1)[(\lambda-\delta)(\beta-\alpha)+1]^{2k}\left(\frac{1+a}{2+a}\right)^{2s}}{3(A-B)[2(\lambda-\delta)(\beta-\alpha)+1]^{k}\left(\frac{1+a}{3+a}\right)^{s}}$$

and

$$\sigma_2 = \frac{2(A-2B+1)[(\lambda-\delta)(\beta-\alpha)+1]^{2k}\left(\frac{1+a}{2+a}\right)^{2s}}{3(A-B)[2(\lambda-\delta)(\beta-\alpha)+1]^{k}\left(\frac{1+a}{3+a}\right)^{s}}.$$

The result is sharp.

We can give proof of Theorem 5.1by using the same technique as in Theorem 3.1.

Remark 5.2. If $\sigma_1 \le \mu \le \sigma_2$, then in view of Lemma 2.1, Theorem 5.1 can be improved. Let σ_3 be given by

$$\sigma_3 := \frac{2(A-2B)[(\lambda-\delta)(\beta-\alpha)+1]^{2k}\left(\frac{1+a}{2+a}\right)^{2s}}{3(A-B)[2(\lambda-\delta)(\beta-\alpha)+1]^{k}\left(\frac{1+a}{3+a}\right)^{s}}.$$

If $\sigma_1 \leq \mu \leq \sigma_3$, then

$$\left|a_3 - \mu {a_2}^2\right| + \Big[\frac{3\mu(A-B)[2(\lambda-\delta)(\beta-\alpha)+1]^k\left(\frac{1+a}{3+a}\right)^s}{3(A-B)[2(\lambda-\delta)(\beta-\alpha)+1]^k\left(\frac{1+a}{3+a}\right)^s} - \frac{2(A-2B-1)[(\lambda-\delta)(\beta-\alpha)+1]^{2k}\left(\frac{1+a}{2+a}\right)^{2s}}{3(A-B)[2(\lambda-\delta)(\beta-\alpha)+1]^k\left(\frac{1+a}{3+a}\right)^s}\Big] \times |a_2|^2 \leq \frac{(A-B)}{6[2(\lambda-\delta)(\beta-\alpha)+1]^k\left(\frac{1+a}{3+a}\right)^s}.$$

If $\sigma_3 \leq \mu \leq \sigma_2$, then

$$\left|a_3 - \mu {a_2}^2\right| - \Big[\frac{3\mu(A-B)[2(\lambda-\delta)(\beta-\alpha)+1]^k\left(\frac{1+a}{3+a}\right)^s}{3(A-B)[2(\lambda-\delta)(\beta-\alpha)+1]^k\left(\frac{1+a}{3+a}\right)^s} - \frac{2(A-2B+1)[(\lambda-\delta)(\beta-\alpha)+1]^{2k}\left(\frac{1+a}{2+a}\right)^{2s}}{3(A-B)[2(\lambda-\delta)(\beta-\alpha)+1]^k\left(\frac{1+a}{3+a}\right)^s}\Big] \times |a_2|^2 \leq \frac{(A-B)}{6[2(\lambda-\delta)(\beta-\alpha)+1]^k\left(\frac{1+a}{3+a}\right)^s}.$$

We can give proof of Remark 5.2 by using the same technique as in Remark 3.2.

6. Applications To Functions Defined By Fractional Derivatives

In this section, we use Definition 4.1 to define the class $K_\lambda(A, B, \eta)$. The class $K_\lambda(A, B, \eta)$ consists the functions of $f \in A$ for which $\Omega^\eta f \in K_\lambda(A, B)$.

It can be noted that, when

$$g(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\eta)}{\Gamma(n+1-\eta)} z^n,$$

then, $K_\lambda(A, B, \eta)$ is the special case of class $K_\lambda(A, B, g)$.

Let

$$g(z) = z + \sum_{n=2}^{\infty} g_n z^n, \quad (g_n > 0).$$

Since

$$\Upsilon^{k,a,s}_{\alpha,\beta,\lambda,\delta} f(z) = z + \sum_{n=2}^{\infty} [1 + (\lambda - \delta)(\beta - \alpha)(n-1)]^k \left(\frac{1+a}{n+a}\right)^s a_n z^n \in K_\lambda(A, B, g),$$

if and only if

$$\Upsilon^{k,a,s}_{\alpha,\beta,\lambda,\delta} (f * g)(z) = z + \sum_{n=2}^{\infty} [1 + (\lambda - \delta)(\beta - \alpha)(n-1)]^k \left(\frac{1+a}{n+a}\right)^s a_n g_n z^n \in K_\lambda(A, B). \tag{6.1}$$

The estimation of the coefficient for the functions in the class $K_\lambda(A, B, g)$, can be obtained from the estimation of the corresponding functions of f in the class $K_\lambda(A, B)$. Theorem 6.1 can be obtained after an obvious change of the parameter μ by implementing Theorem 5.1 for the operator (6.1).

Theorem 6.1. *Let $g(z) = z + \sum_{n=2}^{\infty} g_n z^n$, $(g_n > 0)$, and let the function $\phi(z)$ be given by $\phi(z) = 1 + \sum_{n=1}^{\infty} B_n z^n$. If operator $\Upsilon^{k,a,s}_{\alpha,\beta,\lambda,\delta} f(z)$*

given by (1.5) belongs to $K_\lambda(A,B,g)$, then

$$\left|a_3-\mu a_2^2\right| := \begin{cases} \dfrac{1}{g_3}\Big[\dfrac{-\mu(A-B)^2 g_3}{4[(\lambda-\delta)(\beta-\alpha)+1]^{2k}\left(\frac{1+a}{2+a}\right)^{2s} g_2^2} \\ +\dfrac{A(A-B)}{6[2(\lambda-\delta)(\beta-\alpha)+1]^k\left(\frac{1+a}{3+a}\right)^s} \\ -\dfrac{B(A-B)}{3[2(\lambda-\delta)(\beta-\alpha)+1]^k\left(\frac{1+a}{3+a}\right)^s}\Big] & \mu \le \sigma_1; \\ \dfrac{1}{g_3}\Big[\dfrac{A-B}{6[2(\lambda-\delta)(\beta-\alpha)+1]^k\left(\frac{1+a}{3+a}\right)^s}\Big] & \sigma_1 \le \mu \le \sigma_2; \\ \dfrac{1}{g_3}\Big[\dfrac{\mu(A-B)^2 g_3}{4[(\lambda-\delta)(\beta-\alpha)+1]^{2k}\left(\frac{1+a}{2+a}\right)^{2s} g_2^2} \\ -\dfrac{A(A-B)}{6[2(\lambda-\delta)(\beta-\alpha)+1]^k\left(\frac{1+a}{3+a}\right)^s} \\ +\dfrac{B(A-B)}{3[2(\lambda-\delta)(\beta-\alpha)+1]^k\left(\frac{1+a}{3+a}\right)^s}\Big] & \mu \ge \sigma_2, \end{cases}$$

where

$$\sigma_1 = \frac{g_2^2}{g_3}.\frac{2(A-2B-1)[(\lambda-\delta)(\beta-\alpha)+1]^{2k}\left(\frac{1+a}{2+a}\right)^{2s}}{3(A-B)[2(\lambda-\delta)(\beta-\alpha)+1]^k\left(\frac{1+a}{3+a}\right)^s}$$

and

$$\sigma_2 = \frac{g_2^2}{g_3}.\frac{2(A-2B+1)[(\lambda-\delta)(\beta-\alpha)+1]^{2k}\left(\frac{1+a}{2+a}\right)^{2s}}{3(A-B)[2(\lambda-\delta)(\beta-\alpha)+1]^k\left(\frac{1+a}{3+a}\right)^s}.$$

The result is sharp.

Theorem 6.1 reduced to Theorem 6.2 for g_2 and g_3 given by (4.2) and (4.3).

Theorem 6.2. *Let $g(z) = z + \sum_{n=2}^{\infty} g_n z^n$, $(g_n > 0)$, and let the function $\phi(z)$ be given by $\phi(z) = 1+\sum_{n=1}^{\infty} B_n z^n$. If operator $\Upsilon_{\alpha,\beta,\lambda,\delta}^{k,a,s} f(z)$*

given by (1.5) belongs to $K_\lambda(A,B,\eta)$, then

$$\left|a_3-\mu a_2^2\right| := \begin{cases} \frac{(2-\eta)(3-\eta)}{6}\Big[-\frac{3(2-\eta)}{2(3-\eta)}\frac{\mu(A-B)^2}{4[(\lambda-\delta)(\beta-\alpha)+1]^{2k}\left(\frac{1+a}{2+a}\right)^{2s}} \\ +\frac{A(A-B)}{6[2(\lambda-\delta)(\beta-\alpha)+1]^{k}\left(\frac{1+a}{3+a}\right)^{s}}- \\ \frac{B(A-B)}{3[2(\lambda-\delta)(\beta-\alpha)+1]^{k}\left(\frac{1+a}{3+a}\right)^{s}}\Big] & \mu\le\sigma_1; \\ \frac{(2-\eta)(3-\eta)}{6}\Big[\frac{A-B}{6[2(\lambda-\delta)(\beta-\alpha)+1]^{k}\left(\frac{1+a}{3+a}\right)^{s}}\Big] & \sigma_1\le\mu\le\sigma_2; \\ \frac{(2-\eta)(3-\eta)}{6}\Big[\frac{3(2-\eta)}{2(3-\eta)}\frac{\mu(A-B)^2}{4[(\lambda-\delta)(\beta-\alpha)+1]^{2k}\left(\frac{1+a}{2+a}\right)^{2s}} \\ -\frac{A(A-B)}{6[2(\lambda-\delta)(\beta-\alpha)+1]^{k}\left(\frac{1+a}{3+a}\right)^{s}}+ \\ \frac{B(A-B)}{3[2(\lambda-\delta)(\beta-\alpha)+1]^{k}\left(\frac{1+a}{3+a}\right)^{s}}\Big] & \mu\ge\sigma_2, \end{cases}$$

where

$$\sigma_1=\frac{2(3-\eta)}{3(2-\eta)}.\frac{2(A-2B-1)[(\lambda-\delta)(\beta-\alpha)+1]^{2k}\left(\frac{1+a}{2+a}\right)^{2s}}{3(A-B)[2(\lambda-\delta)(\beta-\alpha)+1]^{k}\left(\frac{1+a}{3+a}\right)^{s}}$$

$$\sigma_2=\frac{2(3-\eta)}{3(2-\eta)}.\frac{2(A-2B+1)[(\lambda-\delta)(\beta-\alpha)+1]^{2k}\left(\frac{1+a}{2+a}\right)^{2s}}{3(A-B)[2(\lambda-\delta)(\beta-\alpha)+1]^{k}\left(\frac{1+a}{3+a}\right)^{s}}.$$

The result is sharp.

Competing interests

The authors declare that they have no competing interests.

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K. A. Challab
School of Mathematical Sciences
Faculty of Science and Technology
Universiti Kebangsaan Malaysia
Bangi 43600 Selangor D. Ehsan, Malaysia
E-mail: khalid_math1363@yahoo.com

M. Darus
School of Mathematical Sciences
Faculty of Science and Technology
Universiti Kebangsaan Malaysia
Bangi 43600 Selangor D. Ehsan, Malaysia
E-mail: maslina@ukm.edu.my

F. Ghanim
Department of Mathematics
College of Sciences
University of Sharjah
Sharjah, United Arab Emirates
E-mail: fgahmed@sharjah.ac.ae