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PROJECTIVE REPRESENTATIONS OF A QUIVER WITH THREE VERTICES AND TWO EDGES AS $R[x]$-MODULES

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Abstract. In this paper we show that the projective properties of representations of a quiver $Q = \bullet \to \bullet \to \bullet$ as left $R[x]$-modules. We show that if P is a projective left R-module then $0 \longrightarrow 0 \longrightarrow P[x]$ is a projective representation of a quiver Q as $R[x]$-modules, but $P[x] \longrightarrow 0 \longrightarrow 0$ is not a projective representation of a quiver Q as $R[x]$-modules, if $P \neq 0$. And we show a representation $0 \longrightarrow P[x] \xrightarrow{id} P[x]$ of a quiver Q is a projective representation, if P is a projective left R-module, but $P[x] \xrightarrow{id} P[x] \longrightarrow 0$ is not a projective representation of a quiver Q as $R[x]$-modules, if $P \neq 0$. Then we show a representation $P[x] \xrightarrow{id} P[x] \xrightarrow{id} P[x]$ of a quiver Q is a projective representation, if P is a projective left R-module.

1. Introduction

A quiver is just a directed graph with vertices and edges (arrows) [1]. We may consider many different types of quivers. We allow multiple edges and multiple arrows, and edges going from a vertex back to the same vertex. Originally a representation of the quiver assigned a vector space to each vertex - and a linear map to each edge (or arrow) - with the linear map going from the vector space assigned to the initial vertex to

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the one assigned to the terminal vertex. For example, a representation of the quiver $Q = \bullet \to \bullet \to \bullet$ is $V_1 \xrightarrow{f} V_2 \xrightarrow{g} V_3$, V_1, V_2 and V_3 are vector spaces and f, g are linear maps (morphisms). Then we extend this representation to the left R-modules, a representation of the quiver $Q = \bullet \to \bullet \to \bullet$ is $M_1 \xrightarrow{\alpha} M_2 \xrightarrow{\beta} M_3$, M_1, M_2 and M_3 are left R-modules and α, β are R-linear maps. Throughout this paper a (the) quiver Q means a (the) quiver $Q = \bullet \to \bullet \to \bullet$.

If M is a left R-module, then the polynomial $M[x]$ is a left $R[x]$-module defined by

$$r(m_0 + m_1x + m_2x^2 \cdots + m_ix^i) = rm_0 + rm_1x + rm_2x^2 + \cdots + rm_ix^i$$

$$x(m_0 + m_1x + m_2x^2 \cdots + m_ix^i) = m_0x + m_1x^2 + m_2x^3 + \cdots + m_ix^{i+1}.$$

We call $M[x]$ a polynomial module. Similarly we can define the power series $M[[x]]$ as a left $R[x]$-module and we call a power series modules.

The following diagram

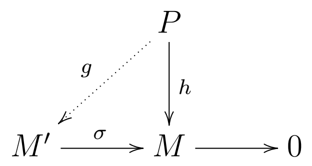
$$\begin{array}{ccccc}
M_1 & \xrightarrow{g_1} & M_2 & \xrightarrow{g_2} & M_3 \\
{\scriptstyle k_1}\downarrow & & {\scriptstyle k_2}\downarrow & & {\scriptstyle k_3}\downarrow \\
N_1 & \xrightarrow[h_1]{} & N_2 & \xrightarrow[h_2]{} & N_3 \\
\downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0
\end{array}$$

is commutative which means $h_1k_1 = k_2g_1, h_2k_2 = k_3g_2$, and $k_3g_2g_1 = h_2h_1k_1 = h_2k_2g_1$.

In [3] a homotopy of a quiver was developed and in [2] a cyclic quiver ring was studies. The theory of projective representations were developed in [4] and the theory of injective representations were studied in [5]. Recently, in [7] injective covers and envelopes of representations of linear quivers were studied, and in [6] properties of multiple edges of quivers were studied.

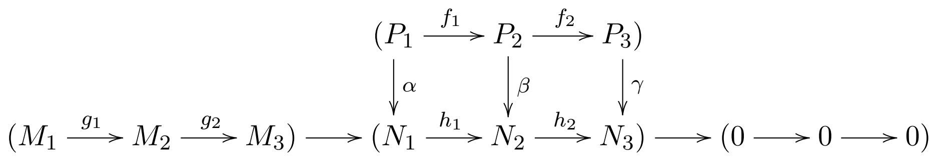
Definition 1.1. [9] A left R-module P is said to be *projective* if given any surjective linear map $\sigma : M' \to M$ and any linear map $h : P \to M$

, there is a linear map $g : P \to M'$ such that $\sigma \circ g = h$. That is



can always be completed to a commutative diagram.

Definition 1.2. Let $M_1, M_2, M_3, N_1, N_2, N_3, P_1, P_2, P_3$ be left R-modules, and $k_1 : M_1 \to N_1$, $k_2 : M_2 \to N_2$, $k_3 : M_3 \to N_3$ be onto R-linear maps. A representation $P_1 \xrightarrow{f_1} P_2 \xrightarrow{f_2} P_3$ of a quiver $Q = \bullet \to \bullet \to \bullet$ is called a *projective representation* if every diagram of representations



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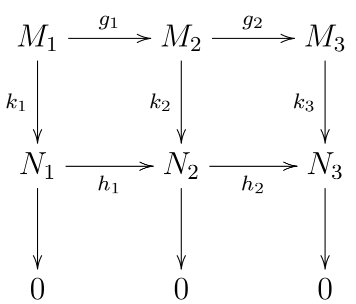
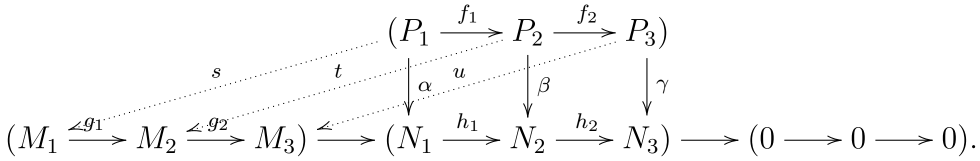


can be completed to a commutative diagram as follows :



2. Projective representation of a quiver Q as $R[x]$-modules

THEOREM 2.1. If P is a projective left R-module then $0 \longrightarrow 0 \longrightarrow P[x]$ is a projective representation of a quiver Q as $R[x]$-modules.

Proof. Let $M_1, M_2, M_3, N_1, N_2, N_3$ be left $R[x]$-modules, and $k_1 : M_1 \to N_1$, $k_2 : M_2 \to N_2$, $k_3 : M_3 \to N_3$ be onto $R[x]$-linear maps, and $f : P[x] \to N_3$ be an $R[x]$-linear map. And consider the following diagrams

$$\begin{array}{ccccc} M_1 & \xrightarrow{g_1} & M_2 & \xrightarrow{g_2} & M_3 \\ {\scriptstyle k_1}\downarrow & & {\scriptstyle k_2}\downarrow & & {\scriptstyle k_3}\downarrow \\ N_1 & \xrightarrow[h_1]{} & N_2 & \xrightarrow[h_2]{} & N_3 \\ \downarrow & & \downarrow & & \downarrow \\ 0 & & 0 & & 0 \end{array}$$

and

$$\begin{array}{ccccccc} & & (0 & \longrightarrow 0 & \longrightarrow & P[x]) & \\ & & \downarrow & \downarrow & & \downarrow{\scriptstyle f} & \\ (M_1 \xrightarrow{g_1} M_2 \xrightarrow{g_2} M_3) & \longrightarrow & (N_1 & \xrightarrow{h_1} N_2 & \xrightarrow{h_2} & N_3) & \longrightarrow (0 \longrightarrow 0 \longrightarrow 0). \end{array}$$

Since P is a projective left R-module, there exists an R-linear map $t : P \to M_3$ such that $k_3 t = f|_P$. Define $\bar{t} : P[x] \to M_3$ by $\bar{t}(p_0 + p_1 x + \cdots + p_n x^n) = t(p_0) + t(p_1)x + \cdots + t(p_n)x^n$. Then

$$\begin{aligned} & k_3\bar{t}(p_0 + p_1 x + \cdots + p_n x^n) \\ & = k_3(t(p_0) + t(p_1)x + \cdots + t(p_n)x^n) \\ & = (k_3 t)(p_0) + (k_3 t)(p_1)x + \cdots + (k_3 t)(p_n)x^n \\ & = f|_P(p_0) + f|_P(p_1)x + \cdots + f|_P(p_n)x^n \\ & = f(p_0 + p_1 x + \cdots + p_n x^n). \end{aligned}$$

So we have $k_3\bar{t} = f$. Therefore, we can complete the following diagram

$$
\begin{array}{ccccccccccc}
 & & & & (0 & \longrightarrow & 0 & \longrightarrow & P[x]) & & \\
 & 0 & & 0 & \downarrow & \bar{t} & \downarrow & & \downarrow f & & \\
(M_1 \xrightarrow{g_1} M_2 \xrightarrow{g_2} M_3) & \longrightarrow & & & (N_1 & \xrightarrow{h_1} & N_2 & \xrightarrow{h_2} & N_3) & \longrightarrow & (0 \longrightarrow 0 \longrightarrow 0)
\end{array}
$$

as a commutative diagram. Hence, $0 \to 0 \to P[x]$ is a projective representation of a quiver Q as $R[x]$-modules.

□

REMARK 2.2. We see that $P[x] \longrightarrow 0 \longrightarrow 0$ is not a projective representation of a quiver Q as $R[x]$-modules if $P \neq 0$, because the following diagram

$$
\begin{array}{ccccc}
 & & (P[x] \longrightarrow 0 \longrightarrow 0) & & \\
 & & \downarrow id \qquad \downarrow \qquad \downarrow & & \\
(P[x] \xrightarrow{id} P[x] \longrightarrow 0) & \longrightarrow & (P[x] \longrightarrow 0 \longrightarrow 0) & \longrightarrow & (0 \longrightarrow 0 \longrightarrow 0)
\end{array}
$$

can not be completed as a commutative diagram with

$$
\begin{array}{ccccc}
P[x] & \xrightarrow{id} & P[x] & \longrightarrow & 0 \\
\downarrow id & & \downarrow & & \downarrow \\
P[x] & \longrightarrow & 0 & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0.
\end{array}
$$

Similarly, $P[[x]] \longrightarrow 0 \longrightarrow 0$ is not a projective representation of a quiver Q as $R[x]$-modules if $P \neq 0$.

COROLLARY 2.3. If P is a projective left R-module then $0 \longrightarrow 0 \longrightarrow P[[x]]$ is a projective representation of a quiver Q as $R[x]$-modules.

Let Z_n denotes the ring of integers modulo n.

EXAMPLE 2.4. Let $R = Z_6$, then $P = Z_2$ is a projective Z_6-module. $0 \longrightarrow 0 \longrightarrow Z_2[x]$ is a projective representation of a quiver Q as $Z_6[x]$-modules.

THEOREM 2.5. A representation $0 \longrightarrow P[x] \xrightarrow{id} P[x]$ of a quiver Q is a projective representation, if P is a projective left R-module.

Proof. Let $f : P[x] \longrightarrow N_2$ be an $R[x]$-linear map and $k_1 : M_1 \longrightarrow N$, $k_2 : M_2 \longrightarrow N_2$ $k_3 : M_3 \longrightarrow N_3$ be onto $R[x]$-linear maps and choose $h_2 f : P[x] \longrightarrow N_3$ as an $R[x]$-linear map. Consider the following diagrams

$$\begin{array}{ccccc}
M_1 & \xrightarrow{g_1} & M_2 & \xrightarrow{g_2} & M_3 \\
{\scriptstyle k_1}\downarrow & & {\scriptstyle k_2}\downarrow & & {\scriptstyle k_3}\downarrow \\
N_1 & \xrightarrow[h_1]{} & N_2 & \xrightarrow[h_2]{} & N_3 \\
\downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0
\end{array}$$

and

$$\begin{array}{ccccc}
 & & (0 \longrightarrow P[x] \xrightarrow{id} P[x]) & & \\
 & & \downarrow \qquad \downarrow{\scriptstyle f} \qquad \downarrow{\scriptstyle h_2 f} & & \\
(M_1 \xrightarrow{g_1} M_2 \xrightarrow{g_2} M_3) & \longrightarrow & (N_1 \xrightarrow{h_1} N_2 \xrightarrow{h_2} N_3) & \longrightarrow & (0 \longrightarrow 0 \longrightarrow 0).
\end{array}$$

Then since P is a projective left R-module, there exists $t : P \to M_2$ such that $k_2 t = f|_P$.
Define $\bar{t} : P[x] \to M_2$ by $\bar{t}(n_0 + n_1 x + \cdots + n_i x^i) = t(n_0) + t(n_1)x + \cdots + t(n_i)x^i$. Then

$$\begin{aligned}
& k_2\bar{t}(n_0 + n_1 x + \cdots + n_i x^i) \\
& = k_2(t(n_0) + t(n_1)x + \cdots + t(n_i)x^i) \\
& = (k_2 t)(n_0) + (h_2 t)(n_1)x + \cdots + (h_2 t)(n_i)x^i \\
& = f|_P(n_0) + f|_P(n_1)x + \cdots + f|_P(n_i)x^i \\
& = f(n_0 + n_1 x + \cdots + n_i x^i).
\end{aligned}$$

Now let $g_2\bar{t} : P[x] \to M_3$ be an $R[x]$-linear map, then $k_3 g_2 \bar{t} = h_2 k_2 \bar{t} = h_2 f$. So we have the following commutative diagram

$$\begin{array}{ccccc} & & (0 \longrightarrow P[x] \xrightarrow{id} P[x]) & & \\ & {\scriptstyle 0,\ \bar{t},\ g_2\bar{t}} & \downarrow \quad \downarrow f \quad \downarrow h_2 f & & \\ (M_1 \underset{g_1}{\longrightarrow} M_2 \underset{g_2}{\longrightarrow} M_3) & \longrightarrow & (N_1 \xrightarrow{h_1} N_2 \xrightarrow{h_2} N_3) & \longrightarrow & (0 \longrightarrow 0 \longrightarrow 0). \end{array}$$

Hence, P is a projective left R-module implies a representation $0 \longrightarrow P[x] \xrightarrow{id} P[x]$ is a projective representation of a quiver Q.

□

Remark 2.6. We see that $P[x] \xrightarrow{id} P[x] \longrightarrow 0$ is not a projective representation of a quiver Q as $R[x]$-modules if $P \neq 0$, because the following diagram

$$\begin{array}{ccccc} & & (P[x] \xrightarrow{id} P[x] \longrightarrow 0) & & \\ & & \downarrow id \qquad \downarrow \qquad \downarrow & & \\ (P[x] \xrightarrow{id} P[x] \xrightarrow{id} P[x]) & \longrightarrow & (P[x] \xrightarrow{id} P[x] \longrightarrow 0) & \longrightarrow & (0 \longrightarrow 0 \longrightarrow 0) \end{array}$$

can not be completed as a commutative diagram with

$$\begin{array}{ccccc} P[x] & \xrightarrow{id} & P[x] & \xrightarrow{id} & P[x] \\ id \downarrow & & \downarrow id & & \downarrow \\ P[x] & \xrightarrow{id} & P[x] & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow \\ 0 & & 0 & & 0. \end{array}$$

Similarly, $P[[x]] \xrightarrow{id} P[[x]] \longrightarrow 0$ is not a projective representation of a quiver Q as $R[x]$-modules if $P \neq 0$.

Corollary 2.7. If P is a projective left R-module then $0 \longrightarrow P[[x]] \xrightarrow{id} P[[x]]$ is a projective representation of a quiver Q as $R[x]$-modules.

Example 2.8. Let $R = Z_6$, then $P = Z_2$ is a projective Z_6-module. $0 \longrightarrow Z_2[x] \xrightarrow{id} Z_2[x]$ is a projective representation of a quiver Q as $Z_6[x]$-modules.

Theorem 2.9. A representation $P[x] \xrightarrow{id} P[x] \xrightarrow{id} P[x]$ of a quiver Q is a projective representation, if P is a projective left R-module.

Proof. Let $f : P[x] \longrightarrow N_1$ be an $R[x]$-linear map and $k_1 : M_1 \longrightarrow N$, $k_2 : M_2 \longrightarrow N_2$ $k_3 : M_3 \longrightarrow N_3$ be onto $R[x]$-linear maps and choose $h_1 f : P[x] \longrightarrow N_2$ and $h_2 h_1 f : P[x] \longrightarrow N_3$ as $R[x]$-linear maps. And consider the following diagrams

$$\begin{array}{ccccc}
M_1 & \xrightarrow{g_1} & M_2 & \xrightarrow{g_2} & M_3 \\
{\scriptstyle k_1}\downarrow & & {\scriptstyle k_2}\downarrow & & {\scriptstyle k_3}\downarrow \\
N_1 & \xrightarrow[h_1]{} & N_2 & \xrightarrow[h_2]{} & N_3 \\
\downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0
\end{array}$$

and

$$\begin{array}{ccc}
 & (P[x] \xrightarrow{id} P[x] \xrightarrow{id} P[x]) & \\
 & \downarrow{\scriptstyle f} \qquad \downarrow{\scriptstyle h_1 f} \qquad \downarrow{\scriptstyle h_2 h_1 f} & \\
(M_1 \xrightarrow{g_1} M_2 \xrightarrow{g_2} M_3) \longrightarrow & (N_1 \xrightarrow{h_1} N_2 \xrightarrow{h_2} N_3) & \longrightarrow (0 \longrightarrow 0 \longrightarrow 0).
\end{array}$$

Then since P is a projective left R-module, there exists $t : P \to M_1$ such that $k_1 t = f|_P$.
Define $\bar{t} : P[x] \to M_1$ by $\bar{t}(n_0 + n_1 x + \cdots + n_i x^i) = t(n_0) + t(n_1)x + \cdots + t(n_i)x^i$. Then

$$\begin{aligned}
&k_1\bar{t}(n_0 + n_1 x + \cdots + n_i x^i) \\
&= k_1(t(n_0) + t(n_1)x + \cdots + t(n_i)x^i) \\
&= (k_1 t)(n_0) + (k_1 t)(n_1)x + \cdots + (k_1 t)(n_i)x^i \\
&= f|_P(n_0) + f|_P(n_1)x + \cdots + f|_P(n_i)x^i \\
&= f(n_0 + n_1 x + \cdots + n_i x^i).
\end{aligned}$$

Now let $g_1\bar{t} : P[x] \to M_2$ be an $R[x]$-linear map, then $k_2 g_1 \bar{t} = h_1 k_1 \bar{t} = h_1 f$. And let $g_2 g_1 \bar{t} : P[x] \to M_3$ be an $R[x]$-linear map. Then $k_3 g_2 g_1 \bar{t} = h_2 k_2 g_1 \bar{t} = h_2 h_1 k_1 \bar{t} = h_2 h_1 f$. So we have the following commutative diagram

$$\begin{array}{ccccccccccc} & & & & (P[x] & \xrightarrow{id} & P[x] & \xrightarrow{id} & P[x]) & & \\ & \bar{t} & & g_1\bar{t} & \downarrow f & g_2g_1\bar{t} & \downarrow h_1f & & \downarrow h_2h_1f & & \\ (M_1 & \xrightarrow{g_1} & M_2 & \xrightarrow{g_2} & M_3) \longrightarrow (N_1 & \xrightarrow{h_1} & N_2 & \xrightarrow{h_2} & N_3) & \longrightarrow & (0 \longrightarrow 0 \longrightarrow 0). \end{array}$$

Hence, P is a projective left R-module implies a representation $P[x] \xrightarrow{id} P[x] \xrightarrow{id} P[x]$ is a projective representation of a quiver Q.

□

COROLLARY 2.10. If P is a projective left R-module then $P[[x]] \xrightarrow{id} P[[x]] \xrightarrow{id} P[[x]]$ is a projective representation of a quiver Q as $R[x]$-modules.

EXAMPLE 2.11. Let $R = Z_6$, then $P = Z_2$ is a projective Z_6-module. $Z_2[x] \xrightarrow{id} Z_2[x] \xrightarrow{id} Z_2[x]$ is a projective representation of a quiver Q as $Z_6[x]$-modules.

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