Korean J. Math. **26** (2018), No. 2, pp. 215–269
https://doi.org/10.11568/kjm.2018.26.2.215

SOME RESULTS RELATING TO SUM AND PRODUCT THEOREMS OF RELATIVE $(p,q,t)\,L$-TH ORDER AND RELATIVE $(p,q,t)\,L$-TH TYPE OF ENTIRE FUNCTIONS

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Abstract. Orders and types of entire functions have been actively investigated by many authors. In this paper, we investigate some basic properties in connection with sum and product of relative $(p,q,t)\,L$-th order, relative $(p,q,t)\,L$-th type, and relative $(p,q,t)\,L$-th weak type of entire functions with respect to another entire function where $p,q\in\mathbb{N}$ and $t\in\mathbb{N}\cup\{-1,0\}$.

1. Introduction, Definitions and Notations

Let $\mathbb{C}$ be the set of all finite complex numbers and f be an entire function defined on $\mathbb{C}$. The maximum modulus function M_f of $f=\sum\limits_{n=0}^{\infty}a_nz^n$ on $|z|=r$ is defined as $M_f=\max\limits_{|z|=r}|f(z)|$. If f is non-constant entire, then its maximum modulus function $M_f(r)$ is strictly increasing and continuous and therefore there exists its inverse function $M_f^{-1}:(|f(0)|,\infty)\rightarrow(0,\infty)$ with $\lim\limits_{s\rightarrow\infty}M_f^{-1}(s)=\infty$. Further a non-constant entire function f is said to have the Property (A) if for any $\sigma>1$ and for all sufficiently large r, $[M_f(r)]^2\leq M_f(r^{\sigma})$ holds (see [2]). However our notations are standard within the theory of Nevanlinna's value distribution of entire functions and therefore we do not explain

Received March 5, 2018. Revised March 16, 2018. Accepted March 18, 2018.
2010 Mathematics Subject Classification: 30D20, 30D30, 30D35.
Key words and phrases: Entire function, relative $(p,q,t)\,L$-th order, relative $(p,q,t)\,L$-th type, relative $(p,q,t)\,L$ -th weak type, Property (A)..

those in detail as available in [11, 12]. Moreover for $x \in [0, \infty)$ and $k \in \mathbb{N}$, we define $\exp^{[k]} x = \exp\left(\exp^{[k-1]} x\right)$ and $\log^{[k]} x = \log\left(\log^{[k-1]} x\right)$ where $\mathbb{N}$ be the set of all positive integers. We also denote $\log^{[0]} x = x$, $\log^{[-1]} x = \exp x$, $\exp^{[0]} x = x$ and $\exp^{[-1]} x = \log x$.

Considering the above, let us recall that Juneja, Kapoor and Bajpai [6] defined the (p, q)-th order and (p, q)-th lower order of an entire function f respectively as follows:

$$\begin{matrix} \rho_f(p,q) \\ \lambda_f(p,q) \end{matrix} = \lim_{r\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log^{[p]} M_f(r)}{\log^{[q]} r},$$

where p, q are positive integers with $p \geq q$.

The definition of (p, q)-th order (respectively (p, q)-th lower order) as initiated by Juneja, Kapoor and Bajpai [6] extends the notion of generalized order $\rho_f^{[l]}$ (respectively generalized lower order $\lambda_f^{[l]}$) of an entire function f introduced by Sato in [9] for each integer $l \geq 2$ as these correspond to the particular case $\rho_f^{[l]} = \rho_f(l, 1)$ (respectively $\lambda_f^{[l]} = \lambda_f(l, 1)$). If $p = 2$ and $q = 1$ then we write $\rho_f(2, 1) = \rho_f$ (respectively $\lambda_f(2, 1) = \lambda_f$) which is known as order (respectively lower order) of an entire function f.

An entire function for which (p, q)-th order and (p, q)-th lower order are the same is said to be of regular (p, q)-growth. Functions which are not of regular (p, q)-growth are said to be of irregular (p, q)-growth.

Many authors have investigated the growth properties of composition of entire functions and derived so many great results. The field of this investigate may be more influential through the intensive applications of the theories of slowly changing functions which in fact means that $L(ar) \sim L(r)$ as $r \to \infty$ for every positive constant a i.e., $\lim_{r\to\infty} \frac{L(ar)}{L(r)} = 1$ where $L \equiv L(r)$ is a positive continuous function increasing slowly. Considering $L(r) = \log r$ and $a = 10^{30}$, one can easily show that $\lim_{r\to\infty} \frac{L(ar)}{L(r)} = 1$. Somasundaram and Thamizharasi [10] introduced the notions of L-order and L-lower order for entire functions.

Extending the notion of Somasundaram and Thamizharasi [10], one may introduce the definition of $(p, q, t)L$-th order and $(p, q, t)L$-th lower order of an entire function f, where p, q are positive integers with $p \geq$

$q \geq 1$ and $t \in \mathbb{N} \cup \{-1, 0\}$ in the following way:

$$\begin{matrix} \rho_f^L(p,q,t) \\ \lambda_f^L(p,q,t) \end{matrix} = \lim_{r\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log^{[p]} M_f(r)}{\log^{[q]} r + \exp^{[t]} L(r)}\ .$$

If we consider $p = 2$, $q = 1$ and $t = -1$, then the above definitions reduces to the definition of L-order and L-lower order of an entire function f as introduced by Somasundaram and Thamizharasi [10]. Also for an entire function f, if we consider $p = 2$, $q = 1$ and $t = 0$, then we get the definition of L^*-order and L^*-lower order of f respectively. However, if we take $L(r) \equiv 1$, then the above definitions reduces to the (p,q)-th order and (p,q)-th lower order of f as introduced by Juneja et al. [6].

An entire function for which $(p,q,t)L$-th order and $(p,q,t)L$-th lower order are the same is said to be of regular (p,q,t) growth. Functions which are not of regular (p,q,t) growth are said to be of irregular (p,q,t) growth.

Mainly the growth investigation of entire functions has usually been done through its maximum moduli in comparison with those of exponential function. But if one is paying attention to evaluate the growth rates of any entire function with respect to a new entire function , the notions of relative growth indicators [1, 2] will come. Extending this notion, Ruiz et al. [8] introduce the definition of relative (p,q)-th order and relative (p,q)-th lower order of an entire function f with respect to another entire function g respectively in the light of index-pair (detail about index-pair one may see [6–8]) which are as follows:

Definition 1. [8] Let f and g be any two entire functions with index-pairs (m,q) and (m,p) respectively where p,q,m are positive integers such that $m \geq \max(p,q)$. Then the relative (p,q)-th order and relative (p,q)-th lower order of f with respect to g are defined as:

$$\begin{matrix} \rho_g^{(p,q)}(f) \\ \lambda_g^{(p,q)}(f) \end{matrix} = \lim_{r\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log^{[p]} M_g^{-1} M_f(r)}{\log^{[q]} r}\ .$$

For details about relative (p,q)-th order and relative (p,q)-th lower order of f with respect to g, one may see [8].

In order to make some progress in the study of relative order, now we introduce the idea of relative (p,q,t) L-th order and relative (p,q,t) L-th lower order of an entire function f with respect to another entire function g respectively in the following way:

Definition 2. Let f and g be any two entire functions. Then relative $(p,q,t)L$-th order denoted as $\rho_g^{(p,q,t)L}(f)$ and relative $(p,q,t)L$-th lower order denoted as $\lambda_g^{(p,q,t)L}(f)$ of an entire function f with respect to another entire function g are define by

$$\begin{matrix}\rho_g^{(p,q,t)L}(f) \\ \lambda_g^{(p,q,t)L}(f)\end{matrix} = \lim_{r\to\infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\log^{[p]} M_g^{-1} M_f(r)}{\log^{[q]} r + \exp^{[t]} L(r)},$$

where $p,q \in \mathbb{N}$ and $t \in \mathbb{N} \cup \{-1,0\}$.

An entire function f for which relative $(p,q,t)L$-th order and relative $(p,q,t)L$-th lower order with respect to another entire function g are the same is called a function of regular relative (p,q,t) growth with respect to g. Otherwise, f is said to be irregular relative (p,q,t) growth with respect to g.

Now to compare the relative growth of two entire functions having same non zero finite relative $(p,q,t)\,L$-th order with respect to another entire function, one may introduce the concepts of relative $(p,q,t)\,L$-th type and relative $(p,q,t)\,L$-th lower type in the following manner:

Definition 3. Let f and g be any two entire functions with $0 < \rho_g^{(p,q,t)L}(f) < \infty$ where $p,q \in \mathbb{N}$ and $t \in \mathbb{N} \cup \{-1,0\}$, then the relative $(p,q,t)\,L$-th type and relative $(p,q,t)\,L$-th lower type denoted respectively by $\sigma_g^{(p,q,t)L}(f)$ and $\overline{\sigma}_g^{(p,q,t)L}(f)$ of f with respect to g are respectively defined as follows:

$$\begin{matrix}\sigma_g^{(p,q,t)L}(f) \\ \overline{\sigma}_g^{(p,q,t)L}(f)\end{matrix} = \lim_{r\to\infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\log^{[p-1]} M_g^{-1} M_f(r)}{\left[\log^{[q-1]} r \cdot \exp^{[t+1]} L(r)\right]^{\rho_g^{(p,q,t)L}(f)}}.$$

Analogously to determine the relative growth of two entire functions having same non zero finite relative $(p,q,t)\,L$-th lower order with respect to another entire function, one may introduce the definition of relative $(p,q,t)\,L$-th weak type in the following way:

Definition 4. Let f and g be any two entire functions with $0 < \lambda_g^{(p,q,t)L}(f) < \infty$ where $p,q \in \mathbb{N}$ and $t \in \mathbb{N} \cup \{-1,0\}$, then the relative $(p,q,t)\,L$-th weak type denoted by $\tau_g^{(p,q,t)L}(f)$ of f with respect to g is defined as follows:

$$\tau_g^{(p,q,t)L}(f) = \varlimsup_{r\to\infty} \frac{\log^{[p-1]} M_g^{-1} M_f(r)}{\left[\log^{[q-1]} r \cdot \exp^{[t+1]} L(r)\right]^{\lambda_g^{(p,q,t)L}(f)}}.$$

Also one may define the growth indicator $\overline{\tau}_g^{(p,q,t)L}(f)$ of f with respect to g in the following manner

$$\overline{\tau}_g^{(p,q,t)L}(f) = \varliminf_{r\to\infty} \frac{\log^{[p-1]} M_g^{-1} M_f(r)}{\left[\log^{[q-1]} r \cdot \exp^{[t+1]} L(r)\right]^{\lambda_g^{(p,q,t)L}(f)}},$$

where $p, q \in \mathbb{N}$ and $t \in \mathbb{N} \cup \{-1, 0\}$.

Here, in this paper, we aim at investigating some basic properties of relative (p, q, t) L-th order, relative (p, q, t) L-th type and relative (p, q, t) L-th weak type of a entire function with respect to another entire function where $p, q \in \mathbb{N}$ and $t \in \mathbb{N} \cup \{-1, 0\}$ under somewhat different conditions which in fact extend some results of [3] and [4]. Throughout this paper, we assume that all the growth indicators are all nonzero finite.

2. Lemmas

In this section we present some lemmas which will be needed in the sequel.

Lemma 1. *[2] Suppose that f be an entire function, $\alpha > 1, 0 < \beta < \alpha$, $s > 1$ and $0 < \mu < \lambda$. Then*

$$M_f(\alpha r) > \beta M_f(r).$$

Lemma 2. *[2] Let f be an entire function which satisfies the Property (A) then for any positive integer n and for all sufficiently large r,*

$$[M_f(r)]^n \le M_f\left(r^\delta\right)$$

holds where $\delta > 1$.

Lemma 3. *([5],p. 18) Let f be an entire function. Then for all sufficiently large values of r,*

$$T_f(r) \le \log M_f(r) \le 3T_f(2r).$$

3. Main Results

In this section we present some results which will be needed in the sequel.

Theorem 1. *Let f_1, f_2 and g_1 be any three entire functions such that at least f_1 or f_2 is of regular relative (p, q, t) growth with respect to g_1 where $p, q \in \mathbb{N}$ and $t \in \mathbb{N} \cup \{-1, 0\}$. Then*

$$\lambda_{g_1}^{(p,q,t)L}(f_1 \pm f_2) \le \max\left\{\lambda_{g_1}^{(p,q,t)L}(f_1), \lambda_{g_1}^{(p,q,t)L}(f_2)\right\} .$$

The equality holds when $\lambda_{g_1}^{(p,q,t)L}(f_i) > \lambda_{g_1}^{(p,q,t)L}(f_j)$ with at least f_j is of regular relative (p, q, t) growth with respect to g_1 where $i = j = 1, 2$ and $i \ne j$.

Proof. If $\lambda_{g_1}^{(p,q,t)L}(f_1 \pm f_2) = 0$ then the result is obvious. So we suppose that $\lambda_{g_1}^{(p,q,t)L}(f_1 \pm f_2) > 0$. We can clearly assume that $\lambda_{g_1}^{(p,q,t)L}(f_k)$ is finite for $k = 1, 2$.

Further let $\max\left\{\lambda_{g_1}^{(p,q,t)L}(f_1), \lambda_{g_1}^{(p,q,t)L}(f_2)\right\} = \Delta$ and f_2 is of regular relative (p, q, t) growth with respect to g_1.

Now for any arbitrary $\varepsilon > 0$ from the definition of $\lambda_{g_1}^{(p,q,t)L}(f_1)$, we have for a sequence values of r tending to infinity that

$$M_{f_1}(r) \le M_{g_1}\left[\exp^{[p]}\left[\left(\lambda_{g_1}^{(p,q,t)L}(f_1) + \varepsilon\right)\left[\log^{[q]} r + \exp^{[t]} L(r)\right]\right]\right]$$

$$\text{(1)} \qquad i.e.,\ M_{f_1}(r) \le M_{g_1}\left[\exp^{[p]}\left[(\Delta + \varepsilon)\left[\log^{[q]} r + \exp^{[t]} L(r)\right]\right]\right] .$$

Also for any arbitrary $\varepsilon > 0$ from the definition of $\rho_{g_1}^{(p,q,t)L}(f_2)\left(= \lambda_{g_1}^{(p,q,t)L}(f_2)\right)$, we obtain for all sufficiently large values of r that

$$\text{(2)} \quad M_{f_2}(r) \le M_{g_1}\left[\exp^{[p]}\left[\left(\lambda_{g_1}^{(p,q,t)L}(f_2) + \varepsilon\right)\left[\log^{[q]} r + \exp^{[t]} L(r)\right]\right]\right]$$

$$\text{(3)} \qquad i.e.,\ M_{f_2}(r) \le M_{g_1}\left[\exp^{[p]}\left[(\Delta + \varepsilon)\left[\log^{[q]} r + \exp^{[t]} L(r)\right]\right]\right] .$$

So in view of (1) and (3), we obtain for a sequence values of r tending to infinity that

$$\text{(4)} \quad M_{f_1 \pm f_2}(r) <$$
$$M_{f_1}(r) + M_{f_2}(r) < 2M_{g_1}\left[\exp^{[p]}\left[(\Delta + \varepsilon)\left[\log^{[q]} r + \exp^{[t]} L(r)\right]\right]\right] .$$

Therefore in view of Lemma 1 (a), we obtain from (4) for a sequence values of r tending to infinity that

$$\frac{1}{2}M_{f_1\pm f_2}(r) < M_{g_1}\left[\exp^{[p]}\left[(\Delta+\varepsilon)\left[\log^{[q]} r+\exp^{[t]} L(r)\right]\right]\right]$$

$$i.e.,\ M_{f_1\pm f_2}\left(\frac{r}{3}\right) < M_{g_1}\left[\exp^{[p]}\left[(\Delta+\varepsilon)\left[\log^{[q]} r+\exp^{[t]} L(r)\right]\right]\right]$$

$$i.e.,\ \frac{\log^{[p]} M_{g_1}^{-1} M_{f_1\pm f_2}\left(\frac{r}{3}\right)}{\log^{[q]}\left(\frac{r}{3}\right)+\exp^{[t]} L\left(\frac{r}{3}\right)+O(1)} < (\Delta+\varepsilon)\ .$$

Since $\varepsilon > 0$ is arbitrary, we get from above

$$\lambda_{g_1}^{(p,q,t)L}(f_1\pm f_2) \le \Delta = \max\left\{\lambda_{g_1}^{(p,q,t)L}(f_1), \lambda_{g_1}^{(p,q,t)L}(f_2)\right\}\ .$$

Similarly, if we consider that f_1 is of regular relative (p,q,t) growth with respect to g_1 or both f_1 and f_2 are of regular relative (p,q,t) growth with respect to g_1, then one can easily verify that

$$\lambda_{g_1}^{(p,q,t)L}(f_1\pm f_2) \le \Delta = \max\left\{\lambda_{g_1}^{(p,q,t)L}(f_1), \lambda_{g_1}^{(p,q,t)L}(f_2)\right\}\ . \tag{5}$$

Further without loss of generality, let $\lambda_{g_1}^{(p,q,t)L}(f_1) < \lambda_{g_1}^{(p,q,t)L}(f_2)$, f_1 is of regular relative (p,q,t) growth with respect to g_1 and $f = f_1 \pm f_2$. Then Then in view of (5) we get that $\lambda_{g_1}^{(p,q,t)L}(f) \le \lambda_{g_1}^{(p,q,t)L}(f_2)$. As, $f_2 = \pm(f - f_1)$ and in this case we obtain that $\lambda_{g_1}^{(p,q,t)L}(f_2) \le \max\left\{\lambda_{g_1}^{(p,q,t)L}(f), \lambda_{g_1}^{(p,q,t)L}(f_1)\right\}$. As we assume that $\lambda_{g_1}^{(p,q,t)L}(f_1) < \lambda_{g_1}^{(p,q,t)L}(f_2)$, therefore we have $\lambda_{g_1}^{(p,q,t)L}(f_2) \le \lambda_{g_1}^{(p,q,t)L}(f)$ and hence $\lambda_{g_1}^{(p,q,t)L}(f) = \lambda_{g_1}^{(p,q,t)L}(f_2) = \max\left\{\lambda_{g_1}^{(p,q,t)L}(f_1), \lambda_{g_1}^{(p,q,t)L}(f_2)\right\}$. Therefore, $\lambda_{g_1}^{(p,q,t)L}(f_1\pm f_2) = \lambda_{g_1}^{(p,q,t)L}(f_i) \mid i = 1,2$ provided $\lambda_{g_1}^{(p,q,t)L}(f_1) \ne \lambda_{g_1}^{(p,q,t)L}(f_2)$. Thus the theorem follows. □

Theorem 2. *Let f_1, f_2 and g_1 be any three entire functions such that such that $\rho_{g_1}^{(p,q,t)L}(f_1)$ and $\rho_{g_2}^{(p,q,t)L}(f_1)$ exists where $p,q \in \mathbb{N}$ and $t \in \mathbb{N}\cup\{-1,0\}$. Then*

$$\rho_{g_1}^{(p,q,t)L}(f_1\pm f_2) \le \max\left\{\rho_{g_1}^{(p,q,t)L}(f_1), \rho_{g_1}^{(p,q,t)L}(f_2)\right\}\ .$$

The equality holds when $\rho_{g_1}^{(p,q,t)L}(f_1) \ne, \rho_{g_1}^{(p,q,t)L}(f_2)$.

We omit the proof of Theorem 2 as it can easily be carried out in the line of Theorem 1.

Theorem 3. *Let f_1, g_1 and g_2 be any three entire functions such that $\lambda_{g_1}^{(p,q,t)L}(f_1)$ and $\lambda_{g_2}^{(p,q,t)L}(f_1)$ exists where $p,q \in \mathbb{N}$ and $t \in \mathbb{N} \cup \{-1,0\}$. Then*

$$\lambda_{g_1 \pm g_2}^{(p,q,t)L}(f_1) \geq \min\left\{\lambda_{g_1}^{(p,q,t)L}(f_1), \lambda_{g_2}^{(p,q,t)L}(f_1)\right\} .$$

The equality holds when $\lambda_{g_1}^{(p,q,t)L}(f_1) \neq \lambda_{g_2}^{(p,q,t)L}(f_1)$.

Proof. If $\lambda_{g_1 \pm g_2}^{(p,q,t)L}(f_1) = \infty$ then the result is obvious. So we suppose that $\lambda_{g_1 \pm g_2}^{(p,q,t)L}(f_1) < \infty$.

We can clearly assume that $\lambda_{g_k}^{(p,q,t)L}(f_1)$ is finite for $k = 1,2$.

Further let $\Psi = \min\left\{\lambda_{g_1}^{(p,q,t)L}(f_1), \lambda_{g_2}^{(p,q,t)L}(f_1)\right\}$.

Now for any arbitrary $\varepsilon > 0$ from the definition of $\lambda_{g_k}^{(p,q,t)L}(f_1)$, we have for all sufficiently large values of r that

$$M_{g_k}\left[\exp^{[p]}\left[\left(\lambda_{g_k}^{(p,q,t)L}(f_1) - \varepsilon\right)\left[\log^{[q]} r + \exp^{[t]} L(r)\right]\right]\right] \leq M_{f_1}(r) \quad \text{where } k = 1,2 \tag{6}$$

$$\text{i.e, } M_{g_k}\left[\exp^{[p]}\left[(\Psi - \varepsilon)\left[\log^{[q]} r + \exp^{[t]} L(r)\right]\right]\right] \leq M_{f_1}(r) \quad \text{where } k = 1,2$$

Now in view of the first part of Lemma 1(a), we obtain from above for all sufficiently large values of r that

$$\begin{aligned}
&M_{g_1 \pm g_2}\left[\exp^{[p]}\left[(\Psi - \varepsilon)\left[\log^{[q]} r + \exp^{[t]} L(r)\right]\right]\right] \\
&< M_{g_1}\left[\exp^{[p]}\left[(\Psi - \varepsilon)\left[\log^{[q]} r + \exp^{[t]} L(r)\right]\right]\right] + \\
&M_{g_2}\left[\exp^{[p]}\left[(\Psi - \varepsilon)\left[\log^{[q]} r + \exp^{[t]} L(r)\right]\right]\right]
\end{aligned}$$

$$\text{i.e., } M_{g_1 \pm g_2}\left[\exp^{[p]}\left[(\Psi - \varepsilon)\left[\log^{[q]} r + \exp^{[t]} L(r)\right]\right]\right] < 2M_{f_1}(r)$$

$$\text{i.e., } M_{g_1 \pm g_2}\left[\exp^{[p]}\left[(\Psi - \varepsilon)\left[\log^{[q]} r + \exp^{[t]} L(r)\right]\right]\right] < M_{f_1}(3r)$$

$$\text{i.e., } \frac{\log^{[p]} M_{g_1 \pm g_2}^{-1} M_{f_1}(3r)}{\log^{[q]}(3r) + \exp^{[t]} L(3r) + O(1)} > \Psi - \varepsilon .$$

Since $\varepsilon > 0$ is arbitrary, we get from above that

$$\lambda_{g_1 \pm g_2}^{(p,q,t)L}(f_1) \geq \Psi = \min\left\{\lambda_{g_1}^{(p,q)}(f_1), \lambda_{g_2}^{(p,q)}(f_1)\right\} . \tag{7}$$

Now without loss of generality, we may consider that $\lambda_{g_1}^{(p,q,t)L}(f_1) < \lambda_{g_2}^{(p,q,t)L}(f_1)$ and $g = g_1 \pm g_2$. Then in view of (7) we get that $\lambda_g^{(p,q,t)L}(f_1)$

$\geq \lambda_{g_1}^{(p,q,t)L}(f_1)$. Further, $g_1 = (g \pm g_2)$ and in this case we obtain that $\lambda_{g_1}^{(p,q,t)L}(f_1) \geq \min\left\{\lambda_{g}^{(p,q,t)L}(f_1), \lambda_{g_2}^{(p,q,t)L}(f_1)\right\}$. As we assume that $\lambda_{g_1}^{(p,q,t)L}(f_1) < \lambda_{g_2}^{(p,q,t)L}(f_1)$, therefore we have $\lambda_{g_1}^{(p,q,t)L}(f_1) \geq \lambda_{g}^{(p,q,t)L}(f_1)$ and hence $\lambda_{g}^{(p,q,t)L}(f_1) = \lambda_{g_1}^{(p,q,t)L}(f_1) = \min\left\{\lambda_{g_1}^{(p,q,t)L}(f_1), \lambda_{g_2}^{(p,q,t)L}(f_1)\right\}$. Therefore, $\lambda_{g_1 \pm g_2}^{(p,q,t)L}(f_1) = \lambda_{g_i}^{(p,q,t)L}(f_1) \mid i = 1,2$ provided $\lambda_{g_1}^{(p,q,t)L}(f_1) \neq \lambda_{g_2}^{(p,q,t)L}(f_1)$. Thus the theorem is established. □

Theorem 4. *Let f_1, g_1 and g_2 be any three entire functions such that f_1 is of regular relative (p,q,t) growth with respect to at least any one of g_1 or g_2 where $p,q \in \mathbb{N}$ and $t \in \mathbb{N} \cup \{-1,0\}$. Then*

$$\rho_{g_1 \pm g_2}^{(p,q,t)L}(f_1) \geq \min\left\{\rho_{g_1}^{(p,q,t)L}(f_1), \rho_{g_2}^{(p,q,t)L}(f_1)\right\} .$$

The equality holds when $\rho_{g_i}^{(p,q,t)L}(f_1) < \rho_{g_j}^{(p,q,t)L}(f_1)$ with at least f_1 is of regular relative (p,q,t) growth with respect to g_j where $i = j = 1,2$ and $i \neq j$.

We omit the proof of Theorem 4 as it can easily be carried out in the line of Theorem 3.

Theorem 5. *Let f_1, f_2, g_1 and g_2 be any four entire functions. Then for any $p,q \in \mathbb{N}$ and $t \in \mathbb{N} \cup \{-1,0\}$,*

$$\rho_{g_1 \pm g_2}^{(p,q,t)L}(f_1 \pm f_2)$$
$$\leq \max\left[\min\left\{\rho_{g_1}^{(p,q,t)L}(f_1), \rho_{g_2}^{(p,q,t)L}(f_1)\right\}, \min\left\{\rho_{g_1}^{(p,q,t)L}(f_2), \rho_{g_2}^{(p,q,t)L}(f_2)\right\}\right]$$

when the following two conditions holds:
(i) $\rho_{g_i}^{(p,q,t)L}(f_1) < \rho_{g_j}^{(p,q,t)L}(f_1)$ with at least f_1 is of regular relative (p,q,t) growth with respect to g_j for $i = 1, 2$, $j = 1,2$ and $i \neq j$; and
(ii) $\rho_{g_i}^{(p,q,t)L}(f_2) < \rho_{g_j}^{(p,q,t)L}(f_2)$ with at least f_2 is of regular relative (p,q,t) growth with respect to g_j for $i = 1, 2$, $j = 1,2$ and $i \neq j$.
The equality holds when $\rho_{g_1}^{(p,q,t)L}(f_i) < \rho_{g_1}^{(p,q,t)L}(f_j)$ and $\rho_{g_2}^{(p,q,t)L}(f_i) < \rho_{g_2}^{(p,q,t)L}(f_j)$ holds simultaneously for $i = 1,2$; $j = 1,2$ and $i \neq j$.

Proof. Let the conditions (i) and (ii) of the theorem hold. Therefore in view of Theorem 2 and Theorem 4 we get that

$$\max\left[\min\left\{\rho_{g_1}^{(p,q,t)L}(f_1), \rho_{g_2}^{(p,q,t)L}(f_1)\right\}, \min\left\{\rho_{g_1}^{(p,q,t)L}(f_2), \rho_{g_2}^{(p,q,t)L}(f_2)\right\}\right]$$

$$= \max\left[\rho_{g_1\pm g_2}^{(p,q,t)L}(f_1), \rho_{g_1\pm g_2}^{(p,q,t)L}(f_2)\right]$$

$$\geq \rho_{g_1\pm g_2}^{(p,q,t)L}(f_1 \pm f_2)\ . \tag{8}$$

Since $\rho_{g_1}^{(p,q,t)L}(f_i) < \rho_{g_1}^{(p,q,t)L}(f_j)$ and $\rho_{g_2}^{(p,q,t)L}(f_i) < \rho_{g_2}^{(p,q,t)L}(f_j)$ hold simultaneously for $i = 1, 2$; $j = 1, 2$ and $i \neq j$, we obtain that

either $\min\left\{\rho_{g_1}^{(p,q,t)L}(f_1), \rho_{g_2}^{(p,q,t)L}(f_1)\right\} > \min\left\{\rho_{g_1}^{(p,q,t)L}(f_2), \rho_{g_2}^{(p,q,t)L}(f_2)\right\}$ or

$\min\left\{\rho_{g_1}^{(p,q,t)L}(f_2), \rho_{g_2}^{(p,q,t)L}(f_2)\right\} > \min\left\{\rho_{g_1}^{(p,q,t)L}(f_1), \rho_{g_2}^{(p,q,t)L}(f_1)\right\}$ holds.

Now in view of the conditions (i) and (ii) of the theorem, it follows from above thateither $\rho_{g_1\pm g_2}^{(p,q,t)L}(f_1) > \rho_{g_1\pm g_2}^{(p,q,t)L}(f_2)$ or $\rho_{g_1\pm g_2}^{(p,q,t)L}(f_2) > \rho_{g_1\pm g_2}^{(p,q,t)L}(f_1)$ which is the condition for holding equality in (8).

Hence the theorem follows. □

Theorem 6. *Let f_1, f_2, g_1 and g_2 be any four entire functions. Then for any $p, q \in \mathbb{N}$ and $t \in \mathbb{N} \cup \{-1, 0\}$,*

$$\lambda_{g_1\pm g_2}^{(p,q,t)L}(f_1 \pm f_2)$$
$$\geq \min\left[\max\left\{\lambda_{g_1}^{(p,q,t)L}(f_1), \lambda_{g_1}^{(p,q,t)L}(f_2)\right\}, \max\left\{\lambda_{g_2}^{(p,q,t)L}(f_1), \lambda_{g_2}^{(p,q,t)L}(f_2)\right\}\right]$$

when the following two conditions holds:
(i) $\rho_{g_1}^{(p,q,t)L}(f_i) > \rho_{g_1}^{(p,q,t)L}(f_j)$ with at least f_j is of regular relative (p, q, t) growth with respect to g_1 for $i = 1, 2$, $j = 1, 2$ and $i \neq j$; and
(ii) $\rho_{g_2}^{(p,q,t)L}(f_i) > \rho_{g_2}^{(p,q,t)L}(f_j)$ with at least f_j is of regular relative (p, q, t) growth with respect to g_2 for $i = 1, 2$, $j = 1, 2$ and $i \neq j$.
The equality holds when $\rho_{g_i}^{(p,q,t)L}(f_1) < \rho_{g_j}^{(p,q,t)L}(f_1)$ and $\rho_{g_i}^{(p,q,t)L}(f_2) < \rho_{g_j}^{(p,q,t)L}(f_2)$ hold simultaneously for $i = 1, 2$; $j = 1, 2$ and $i \neq j$.

Proof. Suppose that the conditions (i) and (ii) of the theorem holds. Therefore in view of Theorem 1 and Theorem 3, we obtain that

$$\min\left[\max\left\{\lambda_{g_1}^{(p,q,t)L}(f_1), \lambda_{g_1}^{(p,q,t)L}(f_2)\right\}, \max\left\{\lambda_{g_2}^{(p,q,t)L}(f_1), \lambda_{g_2}^{(p,q,t)L}(f_2)\right\}\right]$$

$$= \min\left[\lambda_{g_1}^{(p,q,t)L}(f_1 \pm f_2), \lambda_{g_2}^{(p,q,t)L}(f_1 \pm f_2)\right]$$

$$\geq \lambda_{g_1\pm g_2}^{(p,q,t)L}(f_1 \pm f_2)\ . \tag{9}$$

Since $\rho_{g_i}^{(p,q,t)L}(f_1) < \rho_{g_j}^{(p,q,t)L}(f_1)$ and $\rho_{g_i}^{(p,q,t)L}(f_2) < \rho_{g_j}^{(p,q,t)L}(f_2)$ holds simultaneously for $i = 1, 2$; $j = 1, 2$ and $i \neq j$, we get that

either $\max\left\{\lambda_{g_1}^{(p,q,t)L}(f_1), \lambda_{g_1}^{(p,q,t)L}(f_2)\right\} < \max\left\{\lambda_{g_2}^{(p,q,t)L}(f_1), \lambda_{g_2}^{(p,q,t)L}(f_2)\right\}$ or

$\max\left\{\lambda_{g_2}^{(p,q,t)L}(f_1),\lambda_{g_2}^{(p,q,t)L}(f_2)\right\} < \max\left\{\lambda_{g_1}^{(p,q,t)L}(f_1),\lambda_{g_1}^{(p,q,t)L}(f_2)\right\}$ holds.

Since condition (i) and (ii) of the theorem holds, it follows from above that either $\lambda_{g_1}^{(p,q,t)L}(f_1 \pm f_2) < \lambda_{g_2}^{(p,q,t)L}(f_1 \pm f_2)$ or $\lambda_{g_2}^{(p,q,t)L}(f_1 \pm f_2) < \lambda_{g_1}^{(p,q,t)L}(f_1 \pm f_2)$ which is the condition for holding equality in (9).

Hence the theorem follows. □

THEOREM 7. *Let f_1, f_2 and g_1 be any three entire functions such that at least f_1 or f_2 is of regular relative (p,q,t) growth with respect to g_1 where $p,q \in \mathbb{N}$ and $t \in \mathbb{N}\cup\{-1,0\}$. Also let g_1 satisfy the Property (A). Then*

$$\lambda_{g_1}^{(p,q,t)L}(f_1 \cdot f_2) \leq \max\left\{\lambda_{g_1}^{(p,q,t)L}(f_1),\lambda_{g_1}^{(p,q,t)L}(f_2)\right\} .$$

The equality holds when $\lambda_{g_1}^{(p,q,t)L}(f_i) > \lambda_{g_1}^{(p,q,t)L}(f_j)$ with at least f_j is of regular relative (p,q,t) growth with respect to g_1 where $i = j = 1,2$ and $i \neq j$.

Proof. Let $\lambda_{g_1}^{(p,q,t)L}(f_1 \cdot f_2) > 0$. Otherwise if $\lambda_{g_1}^{(p,q,t)L}(f_1 \cdot f_2) = 0$, then the result is obvious. Let us consider that f_2 is of regular relative (p,q,t) growth with respect to g_1. Also suppose that $\max\left\{\lambda_{g_1}^{(p,q,t)L}(f_1),\lambda_{g_1}^{(p,q,t)L}(f_2)\right\}$ $= \Delta$. We can clearly assume that $\lambda_{g_1}^{(p,q,t)L}(f_k)$ is finite for $k = 1,2$.

Now for any arbitrary $\frac{\varepsilon}{2} > 0$, it follows from the definition of $\rho_{g_1}^{(p,q,t)L}(f_1)$, for a sequence values of r tending to infinity that

$$M_{f_1}(r) \leq M_{g_1}\left[\exp^{[p]}\left[\left(\lambda_{g_1}^{(p,q,t)L}(f_1)+\frac{\varepsilon}{2}\right)\left[\log^{[q]} r+\exp^{[t]} L(r)\right]\right]\right]$$

$$\text{(10)} \quad i.e.,\ M_{f_1}(r) \leq M_{g_1}\left[\exp^{[p]}\left[\left(\Delta+\frac{\varepsilon}{2}\right)\left[\log^{[q]} r+\exp^{[t]} L(r)\right]\right]\right] .$$

Also for any arbitrary $\frac{\varepsilon}{2} > 0$, we obtain from the definition of $\rho_{g_1}^{(p,q,t)L}(f_2)$ $\left(= \lambda_{g_1}^{(p,q,t)L}(f_2)\right)$, for all sufficiently large values of r that

$$M_{f_2}(r) \leq M_{g_1}\left[\exp^{[p]}\left[\left(\lambda_{g_1}^{(p,q,t)L}(f_2)+\frac{\varepsilon}{2}\right)\left[\log^{[q]} r+\exp^{[t]} L(r)\right]\right]\right]$$

$$\text{(11)} \quad i.e.,\ M_{f_2}(r) \leq M_{g_1}\left[\exp^{[p]}\left[\left(\Delta+\frac{\varepsilon}{2}\right)\left[\log^{[q]} r+\exp^{[t]} L(r)\right]\right]\right] .$$

Observe that

$$\frac{\Delta+\varepsilon}{\Delta+\frac{\varepsilon}{2}} > 1 .$$

Therefore we consider the expression $\frac{\exp^{[p-1]}\left[(\Delta+\varepsilon)\left[\log^{[q]} r+\exp^{[t]} L(r)\right]\right]}{\exp^{[p-1]}\left[\left(\Delta+\frac{\varepsilon}{2}\right)\left[\log^{[q]} r+\exp^{[t]} L(r)\right]\right]}$ for all sufficiently large values of r. Thus for any $\delta > 1$, it follows from the above expression for all sufficiently large values of r, say $r \geq r_1 \geq r_0$ that

$$\frac{\exp^{[p-1]}\left[(\Delta+\varepsilon)\left[\log^{[q]} r_0+\exp^{[t]} L\left(r_0\right)\right]\right]}{\exp^{[p-1]}\left[\left(\Delta+\frac{\varepsilon}{2}\right)\left[\log^{[q]} r_0+\exp^{[t]} L\left(r_0\right)\right]\right]}=\delta\ . \tag{12}$$

Since $T_{f_1 \cdot f_2}(r) \leq T_{f_1}(r)+T_{f_2}(r)$ for all large r, therefore in view of Lemma 3 we get that

$$\frac{1}{3} \log M_{f_1 \cdot f_2}\left(\frac{r}{2}\right) \leq \log M_{f_1}(r)+\log M_{f_2}(r)\ .$$

Now from (10), (11) and in view of above, we have for a sequence values of r tending to infinity that

$$\log M_{f_1 \cdot f_2}\left(\frac{r}{2}\right)<6 \log M_{g_1}\left[\exp^{[p]}\left[\left(\Delta+\frac{\varepsilon}{2}\right)\left[\log^{[q]} r+\exp^{[t]} L(r)\right]\right]\right]$$

$$M_{f_1 \cdot f_2}\left(\frac{r}{2}\right)<\left[M_{g_1}\left[\exp^{[p]}\left[\left(\Delta+\frac{\varepsilon}{2}\right)\left[\log^{[q]} r+\exp^{[t]} L(r)\right]\right]\right]\right]^6\ .$$

Also in view of Lemma 2, we obtain from above for a sequence values of r tending to infinity that

$$M_{f_1 \cdot f_2}\left(\frac{r}{2}\right)<M_{g_1}\left[\exp^{[p]}\left[\left(\Delta+\frac{\varepsilon}{2}\right)\left[\log^{[q]} r+\exp^{[t]} L(r)\right]\right]^{\delta}\right],$$

since g_1 has the Property (A) and $\delta > 1$. Therefore in view of (12), it follows from above for a sequence values of r tending to infinity that

$$M_{f_1 \cdot f_2}\left(\frac{r}{2}\right)<M_{g_1}\left[\exp^{[p]}\left[(\Delta+\varepsilon)\left[\log^{[q]} r+\exp^{[t]} L(r)\right]\right]\right]\ .$$

So from above we get for a sequence values of r tending to infinity that

$$\frac{\log^{[p]} M_{g_1}^{-1} M_{f_1 \cdot f_2}\left(\frac{r}{2}\right)}{\log^{[q]}\left(\frac{r}{2}\right)+\exp^{[t]} L\left(\frac{r}{2}\right)+O(1)} \leq(\Delta+\varepsilon)\ .$$

Since $\varepsilon > 0$ is arbitrary, we get from above that

$$\lambda_{g_1}^{(p,q,t)L}\left(f_1 \cdot f_2\right) \leq \Delta=\max \left\{\lambda_{g_1}^{(p,q,t)L}\left(f_1\right), \lambda_{g_1}^{(p,q,t)L}\left(f_2\right)\right\}\ .$$

Similarly, if we consider that f_1 is of regular relative (p, q) growth with respect to g_1 or both f_1 and f_2 are of regular relative (p, q) growth with respect to g_1, then also one can easily verify that

$$\lambda_{g_1}^{(p,q,t)L}(f_1 \cdot f_2) \leq \Delta = \max\left\{\lambda_{g_1}^{(p,q,t)L}(f_1), \lambda_{g_1}^{(p,q,t)L}(f_2)\right\} .$$

Now without loss of generality, let $\lambda_{g_1}^{(p,q,t)L}(f_1) < \lambda_{g_1}^{(p,q,t)L}(f_2)$ and $f = f_1 \cdot f_2$. Then $\lambda_{g_1}^{(p,q,t)L}(f) \leq \lambda_{g_1}^{(p,q,t)L}(f_2)$. Further, $f_2 = \frac{f}{f_1}$ and $T_{f_1}(r) = T_{\frac{1}{f_1}}(r) + O(1)$. Therefore $T_{f_2}(r) \leq T_f(r) + T_{f_1}(r) + O(1)$ and in this case we obtain that $\lambda_{g_1}^{(p,q,t)L}(f_2) \leq \max\left\{\lambda_{g_1}^{(p,q,t)L}(f), \lambda_{g_1}^{(p,q,t)L}(f_1)\right\}$. As we assume that $\lambda_{g_1}^{(p,q,t)L}(f_1) < \lambda_{g_1}^{(p,q,t)L}(f_2)$, therefore we have $\lambda_{g_1}^{(p,q,t)L}(f_2) \leq \lambda_{g_1}^{(p,q,t)L}(f)$ and hence $\lambda_{g_1}^{(p,q,t)L}(f) = \lambda_{g_1}^{(p,q,t)L}(f_2) = \max\left\{\lambda_{g_1}^{(p,q,t)L}(f_1), \lambda_{g_1}^{(p,q,t)L}(f_2)\right\}$. Therefore, $\lambda_{g_1}^{(p,q,t)L}(f_1 \cdot f_2) = \lambda_{g_1}^{(p,q,t)L}(f_i) \mid i = 1, 2$ provided $\lambda_{g_1}^{(p,q,t)L}(f_1) \neq \lambda_{g_1}^{(p,q,t)L}(f_2)$. Hence the theorem follows. □

Next we prove the result for the quotient $\frac{f_1}{f_2}$, provided $\frac{f_1}{f_2}$ is entire.

THEOREM 8. *Let f_1, f_2 and g_1 be any three entire functions such that at least f_1 or f_2 is of regular relative (p, q, t) growth with respect to g_1 where $p, q \in \mathbb{N}$ and $t \in \mathbb{N} \cup \{-1, 0\}$. Also let g_1 satisfy the Property (A). Then*

$$\lambda_{g_1}^{(p,q,t)L}\left(\frac{f_1}{f_2}\right) \leq \max\left\{\lambda_{g_1}^{(p,q,t)L}(f_1), \lambda_{g_1}^{(p,q,t)L}(f_2)\right\} .$$

The equality holds when at least f_2 is of regular relative (p, q, t) growth with respect to g_1 and $\lambda_{g_1}^{(p,q,t)L}(f_1) \neq \lambda_{g_1}^{(p,q,t)L}(f_2)$.

Proof. Since $T_{f_2}(r) = T_{\frac{1}{f_2}}(r) + O(1)$ and $T_{\frac{f_1}{f_2}}(r) \leq T_{f_1}(r) + T_{\frac{1}{f_2}}(r)$, we get in view of Theorem 7 that

$$\lambda_{g_1}^{(p,q,t)L}\left(\frac{f_1}{f_2}\right) \leq \max\left\{\lambda_{g_1}^{(p,q,t)L}(f_1), \lambda_{g_1}^{(p,q,t)L}(f_2)\right\} . \tag{13}$$

Now in order to prove the equality conditions, we discuss the following two cases:

Case I. Suppose $\frac{f_1}{f_2}$ $(= h)$ satisfies the following condition

$$\lambda_{g_1}^{(p,q,t)L}(f_1) < \lambda_{g_1}^{(p,q,t)L}(f_2),$$

and f_2 is of regular relative (p, q, t) growth with respect to g_1.

Now if possible, let $\lambda_{g_1}^{(p,q,t)L}\left(\frac{f_1}{f_2}\right) < \lambda_{g_1}^{(p,q,t)L}(f_2)$. Therefore from $f_1 = h \cdot f_2$ we get that $\lambda_{g_1}^{(p,q,t)L}(f_1) = \lambda_{g_1}^{(p,q,t)L}(f_2)$ which is a contradiction. Therefore $\lambda_{g_1}^{(p,q,t)L}\left(\frac{f_1}{f_2}\right) \geq \lambda_{g_1}^{(p,q,t)L}(f_2)$ and in view of (13), we get that

$$\lambda_{g_1}^{(p,q,t)L}\left(\frac{f_1}{f_2}\right) = \lambda_{g_1}^{(p,q,t)L}(f_2) .$$

Case II. Suppose $\frac{f_1}{f_2}\,(= h)$ satisfies the following condition

$$\lambda_{g_1}^{(p,q,t)L}(f_1) > \lambda_{g_1}^{(p,q,t)L}(f_2),$$

and f_2 is of regular relative (p,q,t) growth with respect to g_1.

Now from $f_1 = h \cdot f_2$ we get that either $\lambda_{g_1}^{(p,q,t)L}(f_1) \leq \lambda_{g_1}^{(p,q,t)L}\left(\frac{f_1}{f_2}\right)$ or $\lambda_{g_1}^{(p,q,t)L}(f_1) \leq \lambda_{g_1}^{(p,q,t)L}(f_2)$. But according to our assumption $\lambda_{g_1}^{(p,q,t)L}(f_1) \nleq \lambda_{g_1}^{(p,q,t)L}(f_2)$. Therefore $\lambda_{g_1}^{(p,q,t)L}\left(\frac{f_1}{f_2}\right) \geq \lambda_{g_1}^{(p,q,t)L}(f_1)$ and in view of (13), we get that

$$\lambda_{g_1}^{(p,q,t)L}\left(\frac{f_1}{f_2}\right) = \lambda_{g_1}^{(p,q,t)L}(f_1) .$$

Thus the theorem follows. □

Now we state the following theorem which can easily be carried out in the line of Theorem 7 and Theorem 8 and therefore its proof is omitted.

Theorem 9. *Let f_1, f_2 and g_1 be any three entire functions such that such that $\rho_{g_1}^{(p,q,t)L}(f_1)$ and $\rho_{g_2}^{(p,q,t)L}(f_1)$ exists where $p, q \in \mathbb{N}$ and $t \in \mathbb{N} \cup \{-1, 0\}$. Also let g_1 satisfy the Property (A). Then*

$$\rho_{g_1}^{(p,q,t)L}(f_1 \cdot f_2) \leq \max\left\{\rho_{g_1}^{(p,q,t)L}(f_1), \rho_{g_1}^{(p,q,t)L}(f_2)\right\} .$$

The equality holds when $\rho_{g_1}^{(p,q,t)L}(f_1) \neq \rho_{g_1}^{(p,q,t)L}(f_2)$.
Similar results hold for the quotient $\frac{f_1}{f_2}$, provided $\frac{f_1}{f_2}$ is entire.

Theorem 10. *Let f_1, g_1 and g_2 be any three entire functions such that $\lambda_{g_1}^{(p,q,t)L}(f_1)$ and $\lambda_{g_2}^{(p,q,t)L}(f_1)$ exists where $p, q \in \mathbb{N}$ and $t \in \mathbb{N} \cup \{-1, 0\}$. Also let $g_1 \cdot g_2$ satisfy the Property (A). Then*

$$\lambda_{g_1 \cdot g_2}^{(p,q,t)L}(f_1) \geq \min\left\{\lambda_{g_1}^{(p,q,t)L}(f_1), \lambda_{g_2}^{(p,q,t)L}(f_1)\right\} .$$

The equality holds when $\lambda_{g_i}^{(p,q,t)L}(f_1) < \lambda_{g_j}^{(p,q,t)L}(f_1)$ where $i = j = 1, 2$ and $i \neq j$ and g_i satisfy the Property (A).

Similar results hold for the quotient $\frac{g_1}{g_2}$, *provided* $\frac{g_1}{g_2}$ *is entire and satisfy the Property (A). The equality holds when* $\lambda_{g_1}^{(p,q,t)L}(f_1) \neq \lambda_{g_2}^{(p,q,t)L}(f_1)$ *and* g_1 *satisfy the Property (A).*

Proof. Let $\lambda_{g_1 \cdot g_2}^{(p,q,t)L}(f_1) < \infty$. Otherwise if $\lambda_{g_1 \cdot g_2}^{(p,q,t)L}(f_1) = \infty$ then the result is obvious. Also suppose that $\min\left\{\lambda_{g_1}^{(p,q,t)L}(f_1), \lambda_{g_2}^{(p,q,t)L}(f_1)\right\} = \Psi$. We can clearly assume that $\lambda_{g_k}^{(p,q,t)L}(f_1)$ is finite for $k = 1, 2$.

Now for any arbitrary $\varepsilon > 0$, with $\varepsilon < \Psi$, we obtain for all sufficiently large values of r that

$$M_{g_k}\left[\exp^{[p]}\left[\left(\lambda_{g_k}^{(p,q,t)L}(f_1) - \frac{\varepsilon}{2}\right)\left[\log^{[q]} r + \exp^{[t]} L(r)\right]\right]\right] \leq M_{f_1}(r) \quad \text{where } k = 1, 2$$

$$\text{(14)} \qquad i.e., \; M_{g_k}\left[\exp^{[p]}\left[\left(\Psi - \frac{\varepsilon}{2}\right)\left[\log^{[q]} r + \exp^{[t]} L(r)\right]\right]\right] \leq M_{f_1}(r) \quad \text{where } k = 1, 2 \; .$$

Observe that

$$\frac{\Psi - \frac{\varepsilon}{2}}{\Psi - \varepsilon} > 1 \; .$$

Now we consider the expression $\frac{\exp^{[p-1]}\left[\left(\Psi - \frac{\varepsilon}{2}\right)\left[\log^{[q]} r + \exp^{[t]} L(r)\right]\right]}{\exp^{[p-1]}\left[(\Psi - \varepsilon)\left[\log^{[q]} r + \exp^{[t]} L(r)\right]\right]}$ for all sufficiently large values of r. Thus for any $\delta > 1$, it follows from the above expression for all sufficiently large values of r, say $r \geq r_1 \geq r_0$ that

$$\text{(15)} \qquad \frac{\exp^{[p-1]}\left[\left(\Psi - \frac{\varepsilon}{2}\right)\left[\log^{[q]} r_0 + \exp^{[t]} L(r_0)\right]\right]}{\exp^{[p-1]}\left[(\Psi - \varepsilon)\left[\log^{[q]} r_0 + \exp^{[t]} L(r_0)\right]\right]} = \delta \; .$$

Since $T_{g_1 \cdot g_2}(r) \leq T_{g_1}(r) + T_{g_2}(r)$ for all large r, therefore in view of Lemma 3 we get that

$$\frac{1}{3}\log M_{g_1 \cdot g_2}\left(\frac{r}{2}\right) \leq \log M_{g_1}(r) + \log M_{g_2}(r) \; .$$

Now from (14) and in view of above, we have for all sufficiently large values of r that

$$\log M_{g_1 \cdot g_2}\left(\frac{1}{2}\exp^{[p]}\left[\left(\Psi - \frac{\varepsilon}{2}\right)\left[\log^{[q]} r + \exp^{[t]} L(r)\right]\right]\right) \leq 6 \log M_{f_1}(r)$$

$$i.e., \left[M_{g_1\cdot g_2}\left(\frac{1}{2}\exp^{[p]}\left[\left(\Psi-\frac{\varepsilon}{2}\right)\left[\log^{[q]} r+\exp^{[t]} L\left(r\right)\right]\right]\right)\right]^{\frac{1}{6}} \leq M_{f_1}\left(r\right) \ .$$

Also in view of Lemma 2, we obtain from above for all sufficiently large values of r that

$$M_{g_1\cdot g_2}\left(\left[\frac{1}{2}\exp^{[p]}\left[\left(\Psi-\frac{\varepsilon}{2}\right)\left[\log^{[q]} r+\exp^{[t]} L\left(r\right)\right]\right]\right]^{\frac{1}{\delta}}\right) \leq M_{f_1}\left(r\right),$$

since $g_1 \cdot g_2$ has the Property (A) and $\delta > 1$.

Therefore in view of (15), it follows from above for all sufficiently large values of r that

$$M_{g_1\cdot g_2}\left(\left(\frac{1}{2}\right)^{\frac{1}{\delta}}\exp^{[p]}\left[\left(\Psi-\varepsilon\right)\left[\log^{[q]} r+\exp^{[t]} L\left(r\right)\right]\right]\right) < M_{f_1}\left(r\right) \ .$$

So from above we get for all sufficiently large values of r that

$$\frac{\log^{[p]} M^{-1}_{g_1\cdot g_2} M_{f_1}\left(r\right)+O(1)}{\left[\log^{[q]} r+\exp^{[t]} L\left(r\right)\right]} > \left(\Psi-\varepsilon\right) \ .$$

Since $\varepsilon > 0$ is arbitrary, we get from above that

$$(16) \qquad \lambda^{(p,q,t)L}_{g_1\cdot g_2}\left(f_1\right) \geq \Psi = \min\left\{\lambda^{(p,q,t)L}_{g_1}\left(f_1\right), \lambda^{(p,q,t)L}_{g_2}\left(f_1\right)\right\} \ .$$

Now without loss of generality, we may consider that $\lambda^{(p,q,t)L}_{g_1}\left(f_1\right) < \lambda^{(p,q,t)L}_{g_2}\left(f_1\right)$ and $g = g_1 \cdot g_2$. Then $\lambda^{(p,q,t)L}_{g}\left(f_1\right) \geq \lambda^{(p,q,t)L}_{g_1}\left(f_1\right)$. Further, $g_1 = \frac{g}{g_2}$ and and $T_{g_2}\left(r\right) = T_{\frac{1}{g_2}}\left(r\right) + O(1)$. Therefore $T_{g_1}\left(r\right) \leq T_{g}\left(r\right) + T_{g_2}\left(r\right) + O(1)$ and in this case we obtain that $\lambda^{(p,q,t)L}_{g_1}\left(f_1\right) \geq \min\left\{\lambda^{(p,q,t)L}_{g}\left(f_1\right), \lambda^{(p,q,t)L}_{g_2}\left(f_1\right)\right\}$. As we assume that $\lambda^{(p,q,t)L}_{g_1}\left(f_1\right) < \lambda^{(p,q,t)L}_{g_2}\left(f_1\right)$, so we have $\lambda^{(p,q,t)L}_{g_1}\left(f_1\right) \geq \lambda^{(p,q,t)L}_{g}\left(f_1\right)$ and hence $\lambda^{(p,q,t)L}_{g}\left(f_1\right) = \lambda^{(p,q,t)L}_{g_1}\left(f_1\right) = \min\left\{\lambda^{(p,q,t)L}_{g_1}\left(f_1\right), \lambda^{(p,q,t)L}_{g_2}\left(f_1\right)\right\}$. Therefore, $\lambda^{(p,q,t)L}_{g_1\cdot g_2}\left(f_1\right) = \lambda^{(p,q,t)L}_{g_i}\left(f_1\right) \mid i = 1, 2$ provided $\lambda^{(p,q,t)L}_{g_1}\left(f_1\right) < \lambda^{(p,q,t)L}_{g_2}\left(f_1\right)$, $g_1 \cdot g_2$ and g_1 are satisfy the Property (A).

Hence the first part of the theorem follows.

Now we prove our results for the quotient $\frac{g_1}{g_2}$, provided $\frac{g_1}{g_2}$ is entire and $\lambda^{(p,q,t)L}_{g_1}\left(f_1\right) \neq \lambda^{(p,q,t)L}_{g_2}\left(f_1\right)$.

Since $T_{g_2}(r) = T_{\frac{1}{g_2}}(r) + O(1)$ and $T_{\frac{g_1}{g_2}}(r) \leq T_{g_1}(r) + T_{\frac{1}{g_2}}(r)$, we get in view of (16) that

$$(17)\qquad \lambda^{(p,q,t)L}_{\frac{g_1}{g_2}}(f_1) \geq \Psi = \min\left\{\lambda^{(p,q,t)L}_{g_1}(f_1), \lambda^{(p,q,t)L}_{g_2}(f_1)\right\}\ .$$

Now in order to prove the equality conditions, we discuss the following two cases:

Case I. Suppose $\frac{g_1}{g_2}\,(=h)$ satisfies the following condition

$$\lambda^{(p,q,t)L}_{g_1}(f_1) > \lambda^{(p,q,t)L}_{g_2}(f_1)\ .$$

Now if possible, let $\lambda^{(p,q,t)L}_{\frac{g_1}{g_2}}(f_1) > \lambda^{(p,q,t)L}_{g_2}(f_1)$. Therefore from $g_1 = h \cdot g_2$ we get that $\lambda^{(p,q,t)L}_{g_1}(f_1) = \lambda^{(p,q,t)L}_{g_2}(f_1)$, which is a contradiction. Therefore $\lambda^{(p,q,t)L}_{\frac{g_1}{g_2}}(f_1) \leq \lambda^{(p,q,t)L}_{g_2}(f_1)$ and in view of (17), we get that

$$\lambda^{(p,q,t)L}_{\frac{g_1}{g_2}}(f_1) = \lambda^{(p,q,t)L}_{g_2}(f_1)\ .$$

Case II. Suppose that $\frac{g_1}{g_2}\,(=h)$ satisfies the following condition

$$\lambda^{(p,q,t)L}_{g_1}(f_1) < \lambda^{(p,q,t)L}_{g_2}(f_1)\ .$$

Therefore from $g_1 = h \cdot g_2$, we get that either $\lambda^{(p,q,t)L}_{g_1}(f_1) \geq \lambda^{((p,q,t)L)}_{\frac{g_1}{g_2}}(f_1)$ or $\lambda^{(p,q,t)L}_{g_1}(f_1) \geq \lambda^{(p,q,t)L}_{g_2}(f_1)$. But according to our assumption $\lambda^{(p,q,t)L}_{g_1}(f_1) \ngeq \lambda^{(p,q,t)L}_{g_2}(f_1)$. Therefore $\lambda^{(p,q,t)L}_{\frac{g_1}{g_2}}(f_1) \leq \lambda^{(p,q,t)L}_{g_1}(f_1)$ and in view of (17), we get that

$$\lambda^{(p,q,t)L}_{\frac{g_1}{g_2}}(f_1) = \lambda^{(p,q,t)L}_{g_1}(f_1)\ .$$

Hence the theorem follows. □

Theorem 11. *Let f_1, g_1 and g_2 be any three entire functions such that $\rho^{(p,q,t)L}_{g_1}(f_1)$ and $\rho^{(p,q,t)L}_{g_2}(f_1)$ exists where $p, q \in \mathbb{N}$ and $t \in \mathbb{N} \cup \{-1, 0\}$. Further let f_1 is of regular relative (p,q,t) growth with respect to at least any one of g_1 or g_2. Also let $g_1 \cdot g_2$ satisfy the Property (A). Then*

$$\rho^{(p,q,t)L}_{g_1 \cdot g_2}(f_1) \geq \min\left\{\rho^{(p,q,t)L}_{g_1}(f_1), \rho^{(p,q,t)L}_{g_2}(f_1)\right\}\ .$$

The equality holds when $\rho^{(p,q,t)L}_{g_i}(f_1) < \rho^{(p,q,t)L}_{g_j}(f_1)$ with at least f_1 is of regular relative (p,q,t) growth with respect to g_j where $i = j = 1, 2$ and $i \neq j$ and g_i satisfy the Property (A).

Theorem 12. *Let f_1, g_1 and g_2 be any three entire functions such that $\rho_{g_1}^{(p,q,t)L}(f_1)$ and $\rho_{g_2}^{(p,q,t)L}(f_1)$ exists where $p,q\in\mathbb{N}$ and $t\in\mathbb{N}\cup\{-1,0\}$. Further let f_1 is of regular relative (p,q,t) growth with respect to at least any one of g_1 or g_2. Also let $\frac{g_1}{g_2}$ is entire and satisfy the Property (A). Then*

$$\rho_{\frac{g_1}{g_2}}^{(p,q,t)L}(f_1)\geq\min\left\{\rho_{g_1}^{(p,q,t)L}(f_1),\rho_{g_2}^{(p,q,t)L}(f_1)\right\}.$$

The equality holds when $\rho_{g_1}^{(p,q,t)L}(f_1)\neq\rho_{g_2}^{(p,q,t)L}(f_1)$, at least f_1 is of regular relative (p,q,t) growth with respect to g_2 and g_1 satisfy the Property (A).

We omit the proof of Theorem 11 and Theorem 12 as those can easily be carried out in the line of Theorem 10.

Now we state the following four theorems without their proofs as those can easily be carried out in the line of Theorem 5 and Theorem 6 respectively.

Theorem 13. *Let f_1, f_2, g_1 and g_2 be any four entire functions. Also let $g_1\cdot g_2$ satisfy the Property (A). Then for any $p,q\in\mathbb{N}$ and $t\in\mathbb{N}\cup\{-1,0\}$,*

$$\rho_{g_1\cdot g_2}^{(p,q,t)L}(f_1\cdot f_2)$$
$$\leq\max\left[\min\left\{\rho_{g_1}^{(p,q,t)L}(f_1),\rho_{g_2}^{(p,q,t)L}(f_1)\right\},\min\left\{\rho_{g_1}^{(p,q,t)L}(f_2),\rho_{g_2}^{(p,q,t)L}(f_2)\right\}\right],$$

when the following two conditions holds:
(i) $\rho_{g_i}^{(p,q,t)L}(f_1)<\rho_{g_j}^{(p,q,t)L}(f_1)$ with at least f_1 is of regular relative (p,q,t) growth with respect to g_j and g_i satisfy the Property (A) for $i=1,2$, $j=1,2$ and $i\neq j$ and
(ii) $\rho_{g_i}^{(p,q,t)L}(f_2)<\rho_{g_j}^{(p,q,t)L}(f_2)$ with at least f_2 is of regular relative (p,q,t) growth with respect to g_j and g_i satisfy the Property (A) for $i=1,2$, $j=1,2$ and $i\neq j$;
The equality holds when $\rho_{g_1}^{(p,q,t)L}(f_i)<\rho_{g_1}^{(p,q,t)L}(f_j)$ and $\rho_{g_2}^{(p,q)}(f_i)<\rho_{g_2}^{(p,q,t)L}(f_j)$ holds simultaneously for $i=1,2$; $j=1,2$ and $i\neq j$.

Theorem 14. *Let f_1, f_2, g_1 and g_2 be any four entire functions. Also let $g_1\cdot g_2$, g_1 and g_2 be satisfy the Property (A). Then for any $p,q\in\mathbb{N}$ and $t\in\mathbb{N}\cup\{-1,0\}$,*

$$\lambda_{g_1\cdot g_2}^{(p,q,t)L}(f_1\cdot f_2)$$
$$\geq\min\left[\max\left\{\lambda_{g_1}^{(p,q,t)L}(f_1),\lambda_{g_1}^{(p,q,t)L}(f_2)\right\},\max\left\{\lambda_{g_2}^{(p,q,t)L}(f_1),\lambda_{g_2}^{(p,q,t)L}(f_2)\right\}\right]$$

when the following two conditions holds:
(i) $\lambda_{g_1}^{(p,q,t)L}(f_i) > \lambda_{g_1}^{(p,q,t)L}(f_j)$ *with at least* f_j *is of regular relative* (p,q,t) *growth with respect to* g_1 *for* $i = 1, 2,$ $j = 1, 2$ *and* $i \neq j$*; and*
(ii) $\lambda_{g_2}^{(p,q,t)L}(f_i) > \lambda_{g_2}^{(p,q,t)L}(f_j)$ *with at least* f_j *is of regular relative* (p,q,t) *growth with respect to* g_2 *for* $i = 1, 2,$ $j = 1, 2$ *and* $i \neq j$.
The equality holds when $\lambda_{g_i}^{(p,q,t)L}(f_1) < \lambda_{g_j}^{(p,q,t)L}(f_1)$ *and* $\lambda_{g_i}^{(p,q,t)L}(f_2) < \lambda_{g_j}^{(p,q,t)L}(f_2)$ *holds simultaneously for* $i = 1, 2;$ $j = 1, 2$ *and* $i \neq j$.

Theorem 15. *Let* f_1, f_2, g_1 *and* g_2 *be any four entire functions. Also let* $\frac{g_1}{g_2}$ *satisfy the Property (A). Then for any* $p, q \in \mathbb{N}$ *and* $t \in \mathbb{N} \cup \{-1, 0\}$,

$$\rho_{\frac{g_1}{g_2}}^{(p,q,t)L}\left(\frac{f_1}{f_2}\right)$$
$$\leq \max\left[\min\left\{\rho_{g_1}^{(p,q,t)L}(f_1), \rho_{g_2}^{(p,q)}(f_1)\right\}, \min\left\{\rho_{g_1}^{(p,q,t)L}(f_2), \rho_{g_2}^{(p,q)}(f_2)\right\}\right]$$

when the following two conditions holds:
(i) *At least* f_1 *is of regular relative* (p,q) *growth with respect to* g_2 *and* $\rho_{g_1}^{(p,q,t)L}(f_1) \neq \rho_{g_2}^{(p,q,t)L}(f_1)$*; and*
(ii) *At least* f_2 *is of regular relative* (p,q) *growth with respect to* g_2 *and* $\rho_{g_1}^{(p,q,t)L}(f_2) \neq \rho_{g_2}^{(p,q,t)L}(f_2)$.
The equality holds when $\rho_{g_1}^{(p,q,t)L}(f_i) < \rho_{g_1}^{(p,q,t)L}(f_j)$ *and* $\rho_{g_2}^{(p,q,t)L}(f_i) < \rho_{g_2}^{(p,q,t)L}(f_j)$ *holds simultaneously for* $i = 1, 2;$ $j = 1, 2$ *and* $i \neq j$.

Theorem 16. *Let* f_1, f_2, g_1 *and* g_2 *be any four entire functions such that* $\frac{f_1}{f_2}$ *and* $\frac{g_1}{g_2}$ *are also entire functions. Also let* $\frac{g_1}{g_2}$, g_1 *and* g_2 *are satisfy the Property (A). Then for any* $p, q \in \mathbb{N}$ *and* $t \in \mathbb{N} \cup \{-1, 0\}$,

$$\lambda_{\frac{g_1}{g_2}}^{(p,q,t)L}\left(\frac{f_1}{f_2}\right)$$
$$\geq \min\left[\max\left\{\lambda_{g_1}^{(p,q,t)L}(f_1), \lambda_{g_1}^{(p,q,t)L}(f_2)\right\}, \max\left\{\lambda_{g_2}^{(p,q,t)L}(f_1), \lambda_{g_2}^{(p,q,t)L}(f_2)\right\}\right]$$

when the following two conditions hold:
(i) *At least* f_2 *is of regular relative* (p,q,t) *growth with respect to* g_1 *and* $\lambda_{g_1}^{(p,q,t)L}(f_1) \neq \lambda_{g_1}^{(p,q,t)L}(f_2)$*; and*
(ii) *At least* f_2 *is of regular relative* (p,q,t) *growth with respect to* g_2 *and* $\lambda_{g_2}^{(p,q,t)L}(f_1) \neq \lambda_{g_2}^{(p,q,t)L}(f_2)$.
The sign of equality holds when $\lambda_{g_i}^{(p,q,t)L}(f_1) < \lambda_{g_j}^{(p,q,t)L}(f_1)$ *and* $\lambda_{g_i}^{(p,q,t)L}(f_2) < \lambda_{g_j}^{(p,q,t)L}(f_2)$ *holds simultaneously for* $i = 1, 2;$ $j = 1, 2$ *and* $i \neq j$.

Next we find out the sum and product theorems of relative (p,q,t) L-th type (respectively relative (p,q,t) L-th lower type) and relative (p,q,t) L-th weak type of entire function with respect to an entire function taking into consideration of the above theorems.

THEOREM 17. *Let f_1, f_2, g_1 and g_2 be any four entire functions. Also let $\rho_{g_1}^{(p,q,t)L}(f_1)$, $\rho_{g_1}^{(p,q,t)L}(f_2)$, $\rho_{g_2}^{(p,q,t)L}(f_1)$ and $\rho_{g_2}^{(p,q,t)L}(f_2)$ are all non zero and finite where $p,q \in \mathbb{N}$ and $t \in \mathbb{N} \cup \{-1,0\}$.*
(A) *If $\rho_{g_1}^{(p,q,t)L}(f_i) > \rho_{g_1}^{(p,q,t)L}(f_j)$ for $i = j = 1,2$ and $i \neq j$, then*

$$\begin{aligned} \sigma_{g_1}^{(p,q,t)L}(f_1 \pm f_2) &= \sigma_{g_1}^{(p,q,t)L}(f_i) \mid i = 1,2 \text{ and} \\ \overline{\sigma}_{g_1}^{(p,q,t)L}(f_1 \pm f_2) &= \overline{\sigma}_{g_1}^{(p,q,t)L}(f_i) \mid i = 1,2 \ . \end{aligned}$$

(B) *If $\rho_{g_i}^{(p,q,t)L}(f_1) < \rho_{g_j}^{(p,q,t)L}(f_1)$ with at least f_1 is of regular relative (p,q,t) growth with respect to g_j for $i = j = 1,2$ and $i \neq j$, then*

$$\begin{aligned} \sigma_{g_1 \pm g_2}^{(p,q,t)L}(f_1) &= \sigma_{g_i}^{(p,q,t)L}(f_1) \mid i = 1,2 \text{ and} \\ \overline{\sigma}_{g_1 \pm g_2}^{(p,q,t)L}(f_1) &= \overline{\sigma}_{g_i}^{(p,q,t)L}(f_1) \mid i = 1,2 \ . \end{aligned}$$

(C) *Assume the functions f_1, f_2, g_1 and g_2 satisfy the following conditions:*
(i) $\rho_{g_i}^{(p,q,t)L}(f_1) < \rho_{g_j}^{(p,q,t)L}(f_1)$ with at least f_1 is of regular relative (p,q,t) growth with respect to g_j for $i = 1, 2$, $j = 1,2$ and $i \neq j$;
(ii) $\rho_{g_i}^{(p,q,t)L}(f_2) < \rho_{g_j}^{(p,q,t)L}(f_2)$ with at least f_2 is of regular relative (p,q,t) growth with respect to g_j for $i = 1, 2$, $j = 1,2$ and $i \neq j$;
(iii) $\rho_{g_1}^{(p,q,t)L}(f_i) > \rho_{g_1}^{(p,q,t)L}(f_j)$ and $\rho_{g_2}^{(p,q,t)L}(f_i) > \rho_{g_2}^{(p,q,t)L}(f_j)$ holds simultaneously for $i = 1,2$; $j = 1,2$ and $i \neq j$;
*(iv) $\rho_{g_m}^{(p,q,t)L}(f_l) =$
$\max\left[\min\left\{\rho_{g_1}^{(p,q,t)L}(f_1), \rho_{g_2}^{(p,q,t)L}(f_1)\right\}, \min\left\{\rho_{g_1}^{(p,q,t)L}(f_2), \rho_{g_2}^{(p,q,t)L}(f_2)\right\}\right] \mid$
$l = m = 1,2$;*
then we have

$$\sigma_{g_1 \pm g_2}^{(p,q,t)L}(f_1 \pm f_2) = \sigma_{g_m}^{(p,q,t)L}(f_l) \mid l = m = 1,2$$

and

$$\overline{\sigma}_{g_1 \pm g_2}^{(p,q,t)L}(f_1 \pm f_2) = \overline{\sigma}_{g_m}^{(p,q,t)L}(f_l) \mid l = m = 1,2 \ .$$

Proof. From the definition of relative (p,q,t) L-th type and relative (p,q,t) L-th lower type of entire function, we have for all sufficiently large values of r that

(18) $M_{f_k}(r) \leq$

$$M_{g_l}\left[\exp^{[p-1]}\left\{\left(\sigma_{g_l}^{(p,q,t)L}(f_k)+\varepsilon\right)\left[\log^{[q-1]} r\cdot\exp^{[t+1]} L(r)\right]^{\rho_{g_l}^{(p,q,t)L}(f_k)}\right\}\right],$$

(19) $M_{f_k}(r)\geq$

$$M_{g_l}\left[\exp^{[p-1]}\left\{\left(\overline{\sigma}_{g_l}^{(p,q,t)L}(f_k)-\varepsilon\right)\left[\log^{[q-1]} r\cdot\exp^{[t+1]} L(r)\right]^{\rho_{g_l}^{(p,q,t)L}(f_k)}\right\}\right]$$

and for a sequence of values of r tending to infinity, we obtain that

(20) $M_{f_k}(r)\geq$

$$M_{g_l}\left[\exp^{[p-1]}\left\{\left(\sigma_{g_l}^{(p,q,t)L}(f_k)-\varepsilon\right)\left[\log^{[q-1]} r\cdot\exp^{[t+1]} L(r)\right]^{\rho_{g_l}^{(p,q,t)L}(f_k)}\right\}\right]$$

and

(21) $M_{f_k}(r)\leq$

$$M_{g_l}\left[\exp^{[p-1]}\left\{\left(\overline{\sigma}_{g_l}^{(p,q,t)L}(f_k)+\varepsilon\right)\left[\log^{[q-1]} r\cdot\exp^{[t+1]} L(r)\right]^{\rho_{g_l}^{(p,q,t)L}(f_k)}\right\}\right],$$

where $\varepsilon>0$ is any arbitrary positive number $k=1,\,2$ and $l=1,2.$

Case I. Suppose that $\rho_{g_1}^{(p,q,t)L}(f_1)>\rho_{g_1}^{(p,q,t)L}(f_2)$ hold. Also let $\varepsilon\,(>0)$ be arbitrary. Now from (18), we get for all sufficiently large values of r that

$M_{f_1\pm f_2}(r)\leq$

$$M_{g_1}\left[\exp^{[p-1]}\left\{\left(\sigma_{g_1}^{(p,q,t)L}(f_1)+\varepsilon\right)\left[\log^{[q-1]} r\cdot\exp^{[t+1]} L(r)\right]^{\rho_{g_1}^{(p,q,t)L}(f_1)}\right\}\right](1+\omega_1),$$

where $\omega_1=\dfrac{M_{g_1}\left[\exp^{[p-1]}\left\{\left(\sigma_{g_1}^{(p,q,t)L}(f_2)+\varepsilon\right)\left[\log^{[q-1]} r\cdot\exp^{[t+1]} L(r)\right]^{\rho_{g_1}^{(p,q,t)L}(f_2)}\right\}\right]}{M_{g_1}\left[\exp^{[p-1]}\left\{\left(\sigma_{g_1}^{(p,q,t)L}(f_1)+\varepsilon\right)\left[\log^{[q-1]} r\cdot\exp^{[t+1]} L(r)\right]^{\rho_{g_1}^{(p,q,t)L}(f_1)}\right\}\right]}$, and

in view of $\rho_{g_1}^{(p,q,t)L}(f_1)>\rho_{g_1}^{(p,q,t)L}(f_2)$, and for all sufficiently large values of r, we can make the term $\omega_1$sufficiently small . Hence for any $\alpha=1+\varepsilon_1$, it follows from above inequality for all sufficiently large values of r that

$M_{f_1\pm f_2}(r)\leq$

$$M_{g_1}\left[\exp^{[p-1]}\left\{\left(\sigma_{g_1}^{(p,q,t)L}(f_1)+\varepsilon\right)\left[\log^{[q-1]} r\cdot\exp^{[t+1]} L(r)\right]^{\rho_{g_1}^{(p,q,t)L}(f_1)}\right\}\right](1+\varepsilon_1)$$

i.e., $M_{f_1\pm f_2}(r)\leq$

$$M_{g_1}\left[\exp^{[p-1]}\left\{\left(\sigma_{g_1}^{(p,q,t)L}(f_1)+\varepsilon\right)\left[\log^{[q-1]} r\cdot\exp^{[t+1]} L(r)\right]^{\rho_{g_1}^{(p,q,t)L}(f_1)}\right\}\right]\cdot\alpha .$$

Since $\varepsilon > 0$ is arbitrary, therefore by making $\alpha \to 1+$, we obtain in view of Theorem 2 , $\rho_{g_1}^{(p,q,t)L}(f_1) > \rho_{g_1}^{(p,q,t)L}(f_2)$, and for all sufficiently large values of r that

$$\overline{\lim_{r\to\infty}} \frac{\log^{[p-1]} M_{g_1}^{-1} M_{f_1\pm f_2}(r)}{\left[\log^{[q-1]} r\cdot\exp^{[t+1]} L(r)\right]^{\rho_{g_1}^{(p,q,t)L}(f_1\pm f_2)}} \le \sigma_{g_1}^{(p,q,t)L}(f_1)$$

$$i.e.,\ \sigma_{g_1}^{(p,q,t)L}(f_1\pm f_2) \le \sigma_{g_1}^{(p,q,t)L}(f_1)\ . \tag{22}$$

Now we may consider that $f = f_1 \pm f_2$. Since $\rho_{g_1}^{(p,q,t)L}(f_1) > \rho_{g_1}^{(p,q,t)L}(f_2)$ hold. Then $\sigma_{g_1}^{(p,q,t)L}(f) = \sigma_{g_1}^{(p,q,t)L}(f_1 \pm f_2) \le \sigma_{g_1}^{(p,q,t)L}(f_1)$. Further, let $f_1 = (f \pm f_2)$. Therefore in view of Theorem 2 and $\rho_{g_1}^{(p,q,t)L}(f_1) > \rho_{g_1}^{(p,q,t)L}(f_2)$, we obtain that $\rho_{g_1}^{(p,q,t)L}(f) > \rho_{g_1}^{(p,q,t)L}(f_2)$ holds. Therefore in view of (22), $\sigma_{g_1}^{(p,q,t)L}(f_1) \le \sigma_{g_1}^{(p,q,t)L}(f) = \sigma_{g_1}^{(p,q,t)L}(f_1 \pm f_2)$. Hence $\sigma_{g_1}^{(p,q,t)L}(f) = \sigma_{g_1}^{(p,q,t)L}(f_1) \Rightarrow \sigma_{g_1}^{(p,q,t)L}(f_1 \pm f_2) = \sigma_{g_1}^{(p,q,t)L}(f_1)$.

Similarly, if we consider $\rho_{g_1}^{(p,q,t)L}(f_1) < \rho_{g_1}^{(p,q,t)L}(f_2)$, then one can easily verify that $\sigma_{g_1}^{(p,q,t)L}(f_1 \pm f_2) = \sigma_{g_1}^{(p,q,t)L}(f_2)$.

Case II. Let us consider that $\rho_{g_1}^{(p,q,t)L}(f_1) > \rho_{g_1}^{(p,q,t)L}(f_2)$ hold. Also let $\varepsilon\,(>0)$ are arbitrary.

Now from (18) and (21), we get for a sequence of values of r tending to infinity that

$$M_{f_1\pm f_2}(r_n) \le$$
$$M_{g_1}\left[\exp^{[p-1]}\left\{\left(\overline{\sigma}_{g_1}^{(p,q,t)L}(f_1)+\varepsilon\right)\left[\log^{[q-1]} r_n\cdot\exp^{[t+1]} L(r_n)\right]^{\rho_{g_1}^{(p,q,t)L}(f_1)}\right\}\right](1+\omega_2),$$

where $\omega_2 = \dfrac{M_{g_1}\left[\exp^{[p-1]}\left\{\left(\sigma_{g_1}^{(p,q,t)L}(f_2)+\varepsilon\right)\left[\log^{[q-1]} r_n\cdot\exp^{[t+1]} L(r_n)\right]^{\rho_{g_1}^{(p,q,t)L}(f_2)}\right\}\right]}{M_{g_1}\left[\exp^{[p-1]}\left\{\left(\overline{\sigma}_{g_1}^{(p,q,t)L}(f_1)+\varepsilon\right)\left[\log^{[q-1]} r_n\cdot\exp^{[t+1]} L(r_n)\right]^{\rho_{g_1}^{(p,q,t)L}(f_1)}\right\}\right]}$,

and in view of $\rho_{g_1}^{(p,q,t)L}(f_1) > \rho_{g_1}^{(p,q,t)L}(f_2)$, we can make the term ω_2 sufficiently small by taking n sufficiently large and therefore using the similar technique for as executed in the proof of Case I we get from above inequality that $\overline{\sigma}_{g_1}^{(p,q,t)L}(f_1 \pm f_2) = \overline{\sigma}_{g_1}^{(p,q,t)L}(f_1)$ when $\rho_{g_1}^{(p,q,t)L}(f_1) > \rho_{g_1}^{(p,q,t)L}(f_2)$ hold.

Likewise, if we consider $\rho_{g_1}^{(p,q,t)L}(f_1) < \rho_{g_1}^{(p,q,t)L}(f_2)$, then one can easily verify that $\overline{\sigma}_{g_1}^{(p,q,t)L}(f_1 \pm f_2) = \overline{\sigma}_{g_1}^{(p,q,t)L}(f_2)$.

Thus combining Case I and Case II, we obtain the first part of the theorem.

Case III. Let us consider that $\rho_{g_1}^{(p,q,t)L}(f_1) < \rho_{g_2}^{(p,q,t)L}(f_1)$ with at least f_1 is of regular relative (p,q,t) growth with respect to g_2. We can make the term $\omega_3 = \frac{M_{g_2}\left[\exp^{[p-1]}\left\{\left(\sigma_{g_1}^{(p,q,t)L}(f_1)-\varepsilon\right)\left[\log^{[q-1]} r_n\cdot\exp^{[t+1]} L(r_n)\right]^{\rho_{g_1}^{(p,q,t)L}(f_1)}\right\}\right]}{M_{g_2}\left[\exp^{[p-1]}\left\{\left(\overline{\sigma}_{g_2}^{(p,q,t)L}(f_1)-\varepsilon\right)\left[\log^{[q-1]} r_n\cdot\exp^{[t+1]} L(r_n)\right]^{\rho_{g_2}^{(p,q,t)L}(f_1)}\right\}\right]}$ sufficiently small by taking n sufficiently large, since $\rho_{g_1}^{(p,q,t)L}(f_1) < \rho_{g_2}^{(p,q,t)L}(f_1)$. Hence $\omega_3 < \varepsilon_1$.

Now

$$M_{g_1\pm g_2}\left(\exp^{[p-1]}\left\{\left(\sigma_{g_1}^{(p,q,t)L}(f_1)-\varepsilon\right)\left[\log^{[q-1]} r_n\cdot\exp^{[t+1]} L(r_n)\right]^{\rho_{g_1}^{(p,q,t)L}(f_1)}\right\}\right) \leq$$
$$M_{g_1}\left[\exp^{[p-1]}\left\{\left(\sigma_{g_1}^{(p,q,t)L}(f_1)-\varepsilon\right)\left[\log^{[q-1]} r_n\cdot\exp^{[t+1]} L(r_n)\right]^{\rho_{g_1}^{(p,q,t)L}(f_1)}\right\}\right] +$$
$$M_{g_2}\left[\exp^{[p-1]}\left\{\left(\sigma_{g_1}^{(p,q,t)L}(f_1)-\varepsilon\right)\left[\log^{[q-1]} r_n\cdot\exp^{[t+1]} L(r_n)\right]^{\rho_{g_1}^{(p,q,t)L}(f_1)}\right\}\right].$$

Therefore for any $\alpha = 1+\varepsilon_1$, we obtain in view of $\omega_3 < \varepsilon_1$, (19) and (20) for a sequence of values of r tending to infinity that

$$M_{g_1\pm g_2}\left(\exp^{[p-1]}\left\{\left(\sigma_{g_1}^{(p,q,t)L}(f_1)-\varepsilon\right)\left[\log^{[q-1]} r_n\cdot\exp^{[t+1]} L(r_n)\right]^{\rho_{g_1}^{(p,q,t)L}(f_1)}\right\}\right)$$
$$\leq \alpha M_{f_1}(r_n)$$

Now making $\alpha \to 1+$, we obtain from above for a sequence of values of r tending to infinity that

$$\left(\sigma_{g_1}^{(p,q,t)L}(f_1)-\varepsilon\right)\left[\log^{[q-1]} r_n\cdot\exp^{[t+1]} L(r_n)\right]^{\rho_{g_1\pm g_2}^{(p,q,t)L}(f_1)} < \log^{[p-1]} M_{g_1\pm g_2}^{-1} M_{f_1}(r_n)$$

Since $\varepsilon > 0$ is arbitrary, we find that

$$\sigma_{g_1\pm g_2}^{(p,q,t)L}(f_1) \geq \sigma_{g_1}^{(p,q,t)L}(f_1) \quad . \tag{23}$$

Now we may consider that $g = g_1 \pm g_2$. Also $\rho_{g_1}^{(p,q,t)L}(f_1) < \rho_{g_2}^{(p,q,t)L}(f_1)$ and at least f_1 is of regular relative (p,q,t) growth with respect to

g_2. Then $\sigma_g^{(p,q,t)L}(f_1) = \sigma_{g_1 \pm g_2}^{(p,q,t)L}(f_1) \geq \sigma_{g_1}^{(p,q,t)L}(f_1)$. Further let $g_1 = (g \pm g_2)$. Therefore in view of Theorem 4 and $\rho_{g_1}^{(p,q,t)L}(f_1) < \rho_{g_2}^{(p,q,t)L}(f_1)$, we obtain that $\rho_g^{(p,q,t)L}(f_1) < \rho_{g_2}^{(p,q,t)L}(f_1)$ as at least f_1 is of regular relative (p,q,t) growth with respect to g_2. Hence in view of (23), $\sigma_{g_1}^{(p,q,t)L}(f_1) \geq \sigma_g^{(p,q,t)L}(f_1) = \sigma_{g_1 \pm g_2}^{(p,q,t)L}(f_1)$. Therefore $\sigma_g^{(p,q,t)L}(f_1) = \sigma_{g_1}^{(p,q,t)L}(f_1)$ $\Rightarrow \sigma_{g_1 \pm g_2}^{(p,q,t)L}(f_1) = \sigma_{g_1}^{(p,q,t)L}(f_1)$.

Similarly if we consider $\rho_{g_1}^{(p,q,t)L}(f_1) > \rho_{g_2}^{(p,q,t)L}(f_1)$ with at least f_1 is of regular relative (p,q,t) growth with respect to g_1, then $\sigma_{g_1 \pm g_2}^{(p,q,t)L}(f_1) = \sigma_{g_2}^{(p,q,t)L}(f_1)$.

Case IV. In this case suppose that $\rho_{g_1}^{(p,q,t)L}(f_1) < \rho_{g_2}^{(p,q,t)L}(f_1)$ with at least f_1 is of regular relative (p,q,t) growth with respect to g_2. we can also make the term $\omega_4 = \dfrac{M_{g_2}\left[\exp^{[p-1]}\left\{\left(\overline{\sigma}_{g_1}^{(p,q,t)L}(f_1)-\varepsilon\right)\left[\log^{[q-1]} r \cdot \exp^{[t+1]} L(r)\right]^{\rho_{g_1}^{(p,q,t)L}(f_1)}\right\}\right]}{M_{g_2}\left[\exp^{[p-1]}\left\{\left(\overline{\sigma}_{g_2}^{(p,q,t)L}(f_1)-\varepsilon\right)\left[\log^{[q-1]} r \cdot \exp^{[t+1]} L(r)\right]^{\rho_{g_2}^{(p,q,t)L}(f_1)}\right\}\right]}$ sufficiently small by taking r sufficiently large as $\rho_{g_1}^{(p,q,t)L}(f_1) < \rho_{g_2}^{(p,q,t)L}(f_1)$. So $\omega_4 < \varepsilon_1$ for sufficiently large r. Therefore in view of (19), we obtain for all sufficiently large values of r that

$$M_{g_1 \pm g_2}\left(\exp^{[p-1]}\left\{\left(\overline{\sigma}_{g_1}^{(p,q,t)L}(f_1) - \varepsilon\right)\left[\log^{[q-1]} r \cdot \exp^{[t+1]} L(r)\right]^{\rho_{g_1}^{(p,q,t)L}(f_1)}\right\}\right) \leq$$
$$M_{g_1}\left[\exp^{[p-1]}\left\{\left(\overline{\sigma}_{g_1}^{(p,q,t)L}(f_1) - \varepsilon\right)\left[\log^{[q-1]} r \cdot \exp^{[t+1]} L(r)\right]^{\rho_{g_1}^{(p,q,t)L}(f_1)}\right\}\right] +$$
$$M_{g_2}\left[\exp^{[p-1]}\left\{\left(\overline{\sigma}_{g_1}^{(p,q,t)L}(f_1) - \varepsilon\right)\left[\log^{[q-1]} r \cdot \exp^{[t+1]} L(r)\right]^{\rho_{g_1}^{(p,q,t)L}(f_1)}\right\}\right].$$

Therefore from above it follows for all sufficiently large values of r that

$$M_{g_1 \pm g_2}\left(\exp^{[p-1]}\left\{\left(\overline{\sigma}_{g_1}^{(p,q,t)L}(f_1) - \varepsilon\right)\left[\log^{[q-1]} r \cdot \exp^{[t+1]} L(r)\right]^{\rho_{g_1}^{(p,q,t)L}(f_1)}\right\}\right)$$
$$\leq (1+\varepsilon_1) M_{f_1}(r) ., \tag{24}$$

and therefore using the similar technique for as executed in the proof of Case III we get from (24) that $\overline{\sigma}_{g_1 \pm g_2}^{(p,q,t)L}(f_1) = \overline{\sigma}_{g_1}^{(p,q,t)L}(f_1)$ where $\rho_{g_1}^{(p,q,t)L}(f_1) < \rho_{g_2}^{(p,q,t)L}(f_1)$ and at least f_1 is of regular relative (p,q,t) growth with respect to g_2.

Likewise if we consider $\rho_{g_1}^{(p,q,t)L}(f_1) > \rho_{g_2}^{(p,q,t)L}(f_1)$ with at least f_1 is of regular relative (p,q,t) growth with respect to g_1, then $\overline{\sigma}_{g_1 \pm g_2}^{(p,q,t)L}(f_1) = \overline{\sigma}_{g_2}^{(p,q,t)L}(f_1)$.

Thus combining Case III and Case IV, we obtain the second part of the theorem.

The third part of the theorem is a natural consequence of Theorem 5 and the first part and second part of the theorem. Hence its proof is omitted. □

THEOREM 18. *Let f_1, f_2, g_1 and g_2 be any four entire functions. Also let $\lambda_{g_1}^{(p,q,t)L}(f_1)$, $\lambda_{g_1}^{(p,q,t)L}(f_2)$, $\lambda_{g_2}^{(p,q,t)L}(f_1)$ and $\lambda_{g_2}^{(p,q,t)L}(f_2)$ are all non zero and finite where $p, q \in \mathbb{N}$ and $t \in \mathbb{N} \cup \{-1, 0\}$.*

(A) *If $\lambda_{g_1}^{(p,q,t)L}(f_i) > \lambda_{g_1}^{(p,q,t)L}(f_j)$ with at least f_j is of regular relative (p,q,t) growth with respect to g_1 for $i = j = 1,2$ and $i \neq j$, then*

$$\tau_{g_1}^{(p,q,t)L}(f_1 \pm f_2) = \tau_{g_1}^{(p,q,t)L}(f_i) \mid i = 1,2 \text{ and}$$
$$\overline{\tau}_{g_1}^{(p,q,t)L}(f_1 \pm f_2) = \overline{\tau}_{g_1}^{(p,q,t)L}(f_i) \mid i = 1,2 \; .$$

(B) *If $\lambda_{g_i}^{(p,q,t)L}(f_1) < \lambda_{g_j}^{(p,q,t)L}(f_1)$ for $i = j = 1,2$ and $i \neq j$, then*

$$\tau_{g_1 \pm g_2}^{(p,q,t)L}(f_1) = \tau_{g_i}^{(p,q,t)L}(f_1) \mid i = 1,2 \text{ and}$$
$$\overline{\tau}_{g_1 \pm g_2}^{(p,q,t)L}(f_1) = \overline{\tau}_{g_i}^{(p,q,t)L}(f_1) \mid i = 1,2 \; .$$

(C) *Assume the functions f_1, f_2, g_1 and g_2 satisfy the following conditions:*
(i) $\rho_{g_1}^{(p,q,t)L}(f_i) > \rho_{g_1}^{(p,q,t)L}(f_j)$ with at least f_j is of regular relative (p,q,t) growth with respect to g_1 for $i = j = 1,2$ and $i \neq j$;
(ii) $\rho_{g_2}^{(p,q,t)L}(f_i) > \rho_{g_2}^{(p,q,t)L}(f_j)$ with at least f_j is of regular relative (p,q,t) growth with respect to g_2 for $i = j = 1,2$ and $i \neq j$;
(iii) $\rho_{g_i}^{(p,q,t)L}(f_1) < \rho_{g_j}^{(p,q,t)L}(f_1)$ and $\rho_{g_i}^{(p,q,t)L}(f_2) < \rho_{g_j}^{(p,q,t)L}(f_2)$ holds simultaneously for $i = j = 1,2$ and $i \neq j$;
*(iv) $\lambda_{g_m}^{(p,q,t)L}(f_l) =$
$\min\left[\max\left\{\lambda_{g_1}^{(p,q,t)L}(f_1), \lambda_{g_1}^{(p,q,t)L}(f_2)\right\}, \max\left\{\lambda_{g_2}^{(p,q,t)L}(f_1), \lambda_{g_2}^{(p,q,t)L}(f_2)\right\}\right] \mid$
$l = m = 1,2$;*
then we have

$$\tau_{g_1 \pm g_2}^{(p,q,t)L}(f_1 \pm f_2) = \tau_{g_m}^{(p,q,t)L}(f_l) \mid l = m = 1,2$$

and

$$\overline{\tau}_{g_1 \pm g_2}^{(p,q,t)L}(f_1 \pm f_2) = \overline{\tau}_{g_m}^{(p,q,t)L}(f_l) \mid l = m = 1,2 \; .$$

Proof. For any arbitrary positive number $\varepsilon(>0)$, we have for all sufficiently large values of r that

(25) $M_{f_k}(r) \leq$

$$M_{g_l}\left[\exp^{[p-1]}\left\{\left(\overline{\tau}_{g_l}^{(p,q,t)L}(f_k)+\varepsilon\right)\left[\log^{[q-1]} r\cdot\exp^{[t+1]} L(r)\right]^{\lambda_{g_l}^{(p,q,t)L}(f_k)}\right\}\right],$$

(26) $M_{f_k}(r) \geq$

$$M_{g_l}\left[\exp^{[p-1]}\left\{\left(\tau_{g_l}^{(p,q,t)L}(f_k)-\varepsilon\right)\left[\log^{[q-1]} r\cdot\exp^{[t+1]} L(r)\right]^{\lambda_{g_l}^{(p,q,t)L}(f_k)}\right\}\right]$$

and for a sequence of values of r tending to infinity we obtain that

(27) $M_{f_k}(r) \geq$

$$M_{g_l}\left[\exp^{[p-1]}\left\{\left(\overline{\tau}_{g_l}^{(p,q,t)L}(f_k)-\varepsilon\right)\left[\log^{[q-1]} r\cdot\exp^{[t+1]} L(r)\right]^{\lambda_{g_l}^{(p,q,t)L}(f_k)}\right\}\right]$$

and

(28) $M_{f_k}(r) \leq$

$$M_{g_l}\left[\exp^{[p-1]}\left\{\left(\tau_{g_l}^{(p,q,t)L}(f_k)+\varepsilon\right)\left[\log^{[q-1]} r\cdot\exp^{[t+1]} L(r)\right]^{\lambda_{g_l}^{(p,q,t)L}(f_k)}\right\}\right],$$

where $k=1,2$ and $l=1,2$.

Case I. Let $\lambda_{g_1}^{(p,q,t)L}(f_1) > \lambda_{g_1}^{(p,q,t)L}(f_2)$ with at least f_2 is of regular relative (p,q,t) growth with respect to g_1. Also let $\varepsilon\,(>0)$ be arbitrary. Now from (25) and (28), we get for a sequence $\{r_n\}$ of values of r tending to infinity that

$$M_{f_1\pm f_2}(r_n) \leq (1+\omega_5)$$
$$M_{g_1}\left[\exp^{[p-1]}\left\{\left(\tau_{g_1}^{(p,q,t)L}(f_1)+\varepsilon\right)\left[\log^{[q-1]} r_n\cdot\exp^{[t+1]} L(r_n)\right]^{\lambda_{g_1}^{(p,q,t)L}(f_1)}\right\}\right]$$

where $\omega_5 = \dfrac{M_{g_1}\left[\exp^{[p-1]}\left\{\left(\overline{\tau}_{g_1}^{(p,q,t)L}(f_2)+\varepsilon\right)\left[\log^{[q-1]} r_n\cdot\exp^{[t+1]} L(r_n)\right]^{\lambda_{g_1}^{(p,q,t)L}(f_2)}\right\}\right]}{M_{g_1}\left[\exp^{[p-1]}\left\{\left(\tau_{g_1}^{(p,q,t)L}(f_1)+\varepsilon\right)\left[\log^{[q-1]} r_n\cdot\exp^{[t+1]} L(r_n)\right]^{\lambda_{g_1}^{(p,q,t)L}(f_1)}\right\}\right]}$ and in view of $\lambda_{g_1}^{(p,q,t)L}(f_1) > \lambda_{g_1}^{(p,q,t)L}(f_2)$, we can make the term ω_5 sufficiently small by taking n sufficiently large. Thus with the help of Lemma 1 (a) and Theorem 1 and using the similar technique of Case I of Theorem 17, we get from above inequality that

$$\tau_{g_1}^{(p,q,t)L}(f_1\pm f_2) \leq \tau_{g_1}^{(p,q,t)L}(f_1)\ . \tag{29}$$

Further, we may consider that $f = f_1 \pm f_2$. Also suppose that $\lambda_{g_1}^{(p,q,t)L}(f_1) > \lambda_{g_1}^{(p,q,t)L}(f_2)$ and at least f_2 is of regular relative (p,q,t) growth with respect to g_1. Then $\tau_{g_1}^{(p,q,t)L}(f) = \tau_{g_1}^{(p,q,t)L}(f_1 \pm f_2) \leq \tau_{g_1}^{(p,q,t)L}(f_1)$. Now let $f_1 = (f \pm f_2)$. Therefore in view of Theorem 1, $\lambda_{g_1}^{(p,q,t)L}(f_1) > \lambda_{g_1}^{(p,q,t)L}(f_2)$ and at least f_2 is of regular relative (p,q,t) growth with respect to g_1, we obtain that $\lambda_{g_1}^{(p,q,t)L}(f) > \lambda_{g_1}^{(p,q,t)L}(f_2)$ holds. Hence in view of (29), $\tau_{g_1}^{(p,q,t)L}(f_1) \leq \tau_{g_1}^{(p,q,t)L}(f) = \tau_{g_1}^{(p,q,t)L}(f_1 \pm f_2)$. Therefore $\tau_{g_1}^{(p,q,t)L}(f) = \tau_{g_1}^{(p,q,t)L}(f_1) \Rightarrow \tau_{g_1}^{(p,q,t)L}(f_1 \pm f_2) = \tau_{g_1}^{(p,q,t)L}(f_1)$.

Similarly, if we consider $\lambda_{g_1}^{(p,q,t)L}(f_1) < \lambda_{g_1}^{(p,q,t)L}(f_2)$ with at least f_1 is of regular relative (p,q,t) growth with respect to g_1 then one can easily verify that $\tau_{g_1}^{(p,q,t)L}(f_1 \pm f_2) = \tau_{g_1}^{(p,q,t)L}(f_2)$.

Case II. Let us consider that $\lambda_{g_1}^{(p,q,t)L}(f_1) > \lambda_{g_1}^{(p,q,t)L}(f_2)$ with at least f_2 is of regular relative (p,q,t) growth with respect to g_1. Also let $\varepsilon\,(>0)$ be arbitrary. Now from (25), we get for all sufficiently large values of r that

$$M_{f_1 \pm f_2}(r) \leq (1+\omega_6)$$

$$M_{g_1}\left[\exp^{[p-1]}\left\{\left(\overline{\tau}_{g_1}^{(p,q,t)L}(f_1)+\varepsilon\right)\left[\log^{[q-1]} r \cdot \exp^{[t+1]} L(r)\right]^{\lambda_{g_1}^{(p,q,t)L}(f_1)}\right\}\right],$$

where $\omega_6 = \dfrac{M_{g_1}\left[\exp^{[p-1]}\left\{\left(\overline{\tau}_{g_1}^{(p,q,t)L}(f_2)+\varepsilon\right)\left[\log^{[q-1]} r \cdot \exp^{[t+1]} L(r)\right]^{\lambda_{g_i}^{(p,q,t)L}(f_2)}\right\}\right]}{M_{g_1}\left[\exp^{[p-1]}\left\{\left(\overline{\tau}_{g_1}^{(p,q,t)L}(f_1)+\varepsilon\right)\left[\log^{[q-1]} r \cdot \exp^{[t+1]} L(r)\right]^{\lambda_{g_1}^{(p,q,t)L}(f_1)}\right\}\right]}$ and in view of $\lambda_{g_1}^{(p,q,t)L}(f_1) > \lambda_{g_1}^{(p,q,t)L}(f_2)$, we can make the term ω_6 sufficiently small by taking r sufficiently large and therefore for similar reasoning of Case I we get from above inequality that $\overline{\tau}_{g_1}^{(p,q,t)L}(f_1 \pm f_2) = \overline{\tau}_{g_1}^{(p,q,t)L}(f_1)$ when $\lambda_{g_1}^{(p,q,t)L}(f_1) > \lambda_{g_1}^{(p,q,t)L}(f_2)$ and at least f_2 is of regular relative (p,q,t) growth with respect to g_1.

Likewise, if we consider $\lambda_{g_1}^{(p,q,t)L}(f_1) < \lambda_{g_1}^{(p,q,t)L}(f_2)$ with at least f_1 is of regular relative (p,q,t) growth with respect to g_1 then one can easily verify that $\overline{\tau}_{g_1}^{(p,q,t)L}(f_1 \pm f_2) = \overline{\tau}_{g_1}^{(p,q,t)L}(f_2)$

Thus combining Case I and Case II, we obtain the first part of the theorem.

Case III. Let us consider that $\lambda_{g_1}^{(p,q,t)L}(f_1) < \lambda_{g_2}^{(p,q,t)L}(f_1)$. Therefore we can make the term

$\omega_7 = \frac{M_{g_2}\left[\exp^{[p-1]}\left\{\left(\tau_{g_1}^{(p,q,t)L}(f_1)-\varepsilon\right)\left[\log^{[q-1]} r_n\cdot\exp^{[t+1]} L(r_n)\right]^{\lambda_{g_1}^{(p,q,t)L}(f_1)}\right\}\right]}{M_{g_2}\left[\exp^{[p-1]}\left\{\left(\tau_{g_2}^{(p,q,t)L}(f_1)-\varepsilon\right)\left[\log^{[q-1]} r_n\cdot\exp^{[t+1]} L(r_n)\right]^{\lambda_{g_2}^{(p,q,t)L}(f_1)}\right\}\right]}$ sufficiently small by taking r sufficiently large since $\lambda_{g_1}^{(p,q,t)L}(f_1) < \lambda_{g_2}^{(p,q,t)L}(f_1)$. So $\omega_7 < \varepsilon_1$. Therefore, in view of (26), we get for all sufficiently large values of r that

$$M_{g_1\pm g_2}\left(\exp^{[p-1]}\left\{\left(\tau_{g_1}^{(p,q,t)L}(f_1)-\varepsilon\right)\left[\log^{[q-1]} r_n\cdot\exp^{[t+1]} L(r_n)\right]^{\lambda_{g_1}^{(p,q,t)L}(f_1)}\right\}\right) \leq$$

$$M_{g_1}\left[\exp^{[p-1]}\left\{\left(\tau_{g_1}^{(p,q,t)L}(f_1)-\varepsilon\right)\left[\log^{[q-1]} r_n\cdot\exp^{[t+1]} L(r_n)\right]^{\lambda_{g_1}^{(p,q,t)L}(f_1)}\right\}\right]+$$

$$M_{g_2}\left[\exp^{[p-1]}\left\{\left(\tau_{g_1}^{(p,q,t)L}(f_1)-\varepsilon\right)\left[\log^{[q-1]} r_n\cdot\exp^{[t+1]} L(r_n)\right]^{\lambda_{g_1}^{(p,q,t)L}(f_1)}\right\}\right].$$

So from above we have for all sufficiently large values of r that

$$M_{g_1\pm g_2}\left(\exp^{[p-1]}\left\{\left(\tau_{g_1}^{(p,q,t)L}(f_1)-\varepsilon\right)\left[\log^{[q-1]} r_n\cdot\exp^{[t+1]} L(r_n)\right]^{\lambda_{g_1}^{(p,q,t)L}(f_1)}\right\}\right)$$

$$\leq (1+\varepsilon_1)\, M_{f_1}(r)\ . \tag{30}$$

Now with the help of Lemma 1 (a) and Theorem 3 and using the similar technique of Case III of Theorem 17, we get from (30) that

$$\tau_{g_1\pm g_2}^{(p,q,t)L}(f_1) \geq \tau_{g_1}^{((p,q,t)L)}(f_1)\ . \tag{31}$$

Further, we may consider that $g = g_1 \pm g_2$. As $\lambda_{g_1}^{(p,q,t)L}(f_1) < \lambda_{g_2}^{(p,q,t)L}(f_1)$, so $\tau_{g}^{(p,q,t)L}(f_1) = \tau_{g_1\pm g_2}^{(p,q,t)L}(f_1) \geq \tau_{g_1}^{(p,q,t)L}(f_1)$. Further let $g_1 = (g \pm g_2)$. Therefore in view of Theorem 3 and $\lambda_{g_1}^{(p,q,t)L}(f_1) < \lambda_{g_2}^{(p,q,t)L}(f_1)$ we obtain that $\lambda_{g}^{(p,q,t)L}(f_1) < \lambda_{g_2}^{(p,q,t)L}(f_1)$ holds. Hence in view of (31) $\tau_{g_1}^{(p,q,t)L}(f_1) \geq \tau_{g}^{(p,q,t)L}(f_1) = \tau_{g_1\pm g_2}^{(p,q,t)L}(f_1)$. Therefore $\tau_{g}^{(p,q,t)L}(f_1) = \tau_{g_1}^{(p,q,t)L}(f_1) \Rightarrow \tau_{g_1\pm g_2}^{(p,q,t)L}(f_1) = \tau_{g_1}^{(p,q,t)L}(f_1)$.

Likewise, if we consider that $\lambda_{g_1}^{(p,q,t)L}(f_1) > \lambda_{g_2}^{(p,q,t)L}(f_1)$, then one can easily verify that $\tau_{g_1\pm g_2}^{(p,q,t)L}(f_1) = \tau_{g_2}^{(p,q,t)L}(f_1)$.

Case IV. In this case further we consider $\lambda_{g_1}^{(p,q,t)L}(f_1) < \lambda_{g_2}^{(p,q,t)L}(f_1)$. Further we can make the term

$$\omega_8 = \frac{M_{g_2}\left[\exp^{[p-1]}\left\{\left(\overline{\tau}_{g_1}^{(p,q,t)L}(f_1)-\varepsilon\right)\left[\log^{[q-1]} r_n\cdot\exp^{[t+1]} L(r_n)\right]^{\lambda_{g_1}^{(p,q,t)L}(f_1)}\right\}\right]}{M_{g_2}\left[\exp^{[p-1]}\left\{\left(\tau_{g_2}^{(p,q,t)L}(f_1)-\varepsilon\right)\left[\log^{[q-1]} r_n\cdot\exp^{[t+1]} L(r_n)\right]^{\lambda_{g_2}^{(p,q,t)L}(f_1)}\right\}\right]}$$ sufficiently small by taking n sufficiently large, since $\lambda_{g_1}^{(p,q,t)L}(f_1) < \lambda_{g_2}^{(p,q,t)L}(f_1)$. Therefore $\omega_8 < \varepsilon_1$ for sufficiently large n. Therefore now from (26) and (27), we obtain for a sequence $\{r_n\}$ of values of r tending to infinity that

$$M_{g_1\pm g_2}\left(\exp^{[p-1]}\left\{\left(\overline{\tau}_{g_1}^{(p,q,t)L}(f_1)-\varepsilon\right)\left[\log^{[q-1]} r_n\cdot\exp^{[t+1]} L(r_n)\right]^{\lambda_{g_1}^{(p,q,t)L}(f_1)}\right\}\right) \leq$$
$$M_{g_1}\left[\exp^{[p-1]}\left\{\left(\overline{\tau}_{g_1}^{(p,q,t)L}(f_1)-\varepsilon\right)\left[\log^{[q-1]} r_n\cdot\exp^{[t+1]} L(r_n)\right]^{\lambda_{g_1}^{(p,q,t)L}(f_1)}\right\}\right] +$$
$$M_{g_2}\left[\exp^{[p-1]}\left\{\left(\overline{\tau}_{g_1}^{(p,q,t)L}(f_1)-\varepsilon\right)\left[\log^{[q-1]} r_n\cdot\exp^{[t+1]} L(r_n)\right]^{\lambda_{g_1}^{(p,q,t)L}(f_1)}\right\}\right].$$

Therefore from above we get for a sequence $\{r_n\}$ of values of r tending to infinity that

$$M_{g_1\pm g_2}\left(\exp^{[p-1]}\left\{\left(\overline{\tau}_{g_1}^{(p,q,t)L}(f_1)-\varepsilon\right)\left[\log^{[q-1]} r_n\cdot\exp^{[t+1]} L(r_n)\right]^{\lambda_{g_1}^{(p,q,t)L}(f_1)}\right\}\right)$$
$$\leq (1+\varepsilon_1)\, M_{f_1}(r), \tag{32}$$

and therefore using the similar technique for as executed in the proof of Case IV of Theorem 17, we get from (32) that $\overline{\tau}_{g_1\pm g_2}^{(p,q,t)L}(f_1) = \overline{\tau}_{g_1}^{(p,q,t)L}(f_1)$ when $\lambda_{g_1}^{(p,q,t)L}(f_1) < \lambda_{g_2}^{(p,q,t)L}(f_1)$.

Similarly, if we consider that $\lambda_{g_1}^{(p,q,t)L}(f_1) > \lambda_{g_2}^{(p,q,t)L}(f_1)$, then one can easily verify that $\overline{\tau}_{g_1\pm g_2}^{(p,q,t)L}(f_1) = \overline{\tau}_{g_2}^{(p,q,t)L}(f_1)$.

Thus combining Case III and Case IV, we obtain the second part of the theorem.

The proof of the third part of the Theorem is omitted as it can be carried out in view of Theorem 6 and the above cases. □

In the next two theorems we reconsider the equalities in Theorem 1 to Theorem 4 under somewhat different conditions.

THEOREM 19. *Let f_1, f_2, g_1 and g_2 be any four entire functions. Also let $p,q \in \mathbb{N}$ and $t \in \mathbb{N}\cup\{-1,0\}$.*
(A) *The following condition is assumed to be satisfied:*

(i) Either $\sigma_{g_1}^{(p,q,t)L}(f_1) \neq \sigma_{g_1}^{(p,q,t)L}(f_2)$ or $\overline{\sigma}_{g_1}^{(p,q,t)L}(f_1) \neq \overline{\sigma}_{g_1}^{(p,q,t)L}(f_2)$ holds, then

$$\rho_{g_1}^{(p,q,t)L}(f_1 \pm f_2) = \rho_{g_1}^{(p,q,t)L}(f_1) = \rho_{g_1}^{(p,q,t)L}(f_2) \ .$$

(B) *The following conditions are assumed to be satisfied:*
(i) Either $\sigma_{g_1}^{(p,q,t)L}(f_1) \neq \sigma_{g_2}^{(p,q,t)L}(f_1)$ or $\overline{\sigma}_{g_1}^{(p,q,t)L}(f_1) \neq \overline{\sigma}_{g_2}^{(p,q,t)L}(f_1)$ holds;
(ii) f_1 is of regular relative (p,q,t) growth with respect to at least any one of g_1 or g_2, then

$$\rho_{g_1 \pm g_2}^{(p,q,t)L}(f_1) = \rho_{g_1}^{(p,q,t)L}(f_1) = \rho_{g_2}^{(p,q,t)L}(f_1) \ .$$

Proof. Let f_1, f_2, g_1 and g_2 be any four entire functions satisfying the conditions of the theorem.

Case I. Suppose that $\rho_{g_1}^{(p,q,t)L}(f_1) = \rho_{g_1}^{(p,q,t)L}(f_2)$ $(0 < \rho_{g_1}^{(p,q,t)L}(f_1), \rho_{g_1}^{(p,q,t)L}(f_2) < \infty)$. Now in view of Theorem 2 it is easy to see that $\rho_{g_1}^{(p,q,t)L}(f_1 \pm f_2) \leq \rho_{g_1}^{(p,q,t)L}(f_1) = \rho_{g_1}^{(p,q,t)L}(f_2)$. If possible let

$$\rho_{g_1}^{(p,q,t)L}(f_1 \pm f_2) < \rho_{g_1}^{(p,q,t)L}(f_1) = \rho_{g_1}^{(p,q,t)L}(f_2) \ . \tag{33}$$

Let $\sigma_{g_1}^{(p,q,t)L}(f_1) \neq \sigma_{g_1}^{(p,q,t)L}(f_2)$. Then in view of the first part of Theorem 17 and (33) we obtain that $\sigma_{g_1}^{(p,q,t)L}(f_1) = \sigma_{g_1}^{(p,q,t)L}(f_1 \pm f_2 \mp f_2) = \sigma_{g_1}^{(p,q,t)L}(f_2)$ which is a contradiction. Hence $\rho_{g_1}^{(p,q,t)L}(f_1 \pm f_2) = \rho_{g_1}^{(p,q,t)L}(f_1) = \rho_{g_1}^{(p,q,t)L}(f_2)$. Similarly with the help of the first part of Theorem 17, one can obtain the same conclusion under the hypothesis $\overline{\sigma}_{g_1}^{(p,q,t)L}(f_1) \neq \overline{\sigma}_{g_1}^{(p,q,t)L}(f_2)$. This proves the first part of the theorem.

Case II. Let us consider that $\rho_{g_1}^{(p,q,t)L}(f_1) = \rho_{g_2}^{(p,q,t)L}(f_1)$ $(0 < \rho_{g_1}^{(p,q,t)L}(f_1), \rho_{g_2}^{(p,q,t)L}(f_1) < \infty)$ and f_1 is of regular relative (p,q,t) growth with respect to at least any one of g_1 or g_2 and $(g_1 \pm g_2)$. Therefore in view of Theorem 4, it follows that $\rho_{g_1 \pm g_2}^{(p,q,t)L}(f_1) \geq \rho_{g_1}^{(p,q,t)L}(f_1) = \rho_{g_2}^{(p,q,t)L}(f_1)$ and if possible let

$$\rho_{g_1 \pm g_2}^{(p,q,t)L}(f_1) > \rho_{g_1}^{(p,q,t)L}(f_1) = \rho_{g_2}^{(p,q,t)L}(f_1) \ . \tag{34}$$

Let us consider that $\sigma_{g_1}^{(p,q,t)L}(f_1) \neq \sigma_{g_2}^{(p,q,t)L}(f_1)$. Then. in view of the proof of the second part of Theorem 17 and (34) we obtain that $\sigma_{g_1}^{(p,q,t)L}(f_1) = \sigma_{g_1 \pm g_2 \mp g_2}^{(p,q,t)L}(f_1) = \sigma_{g_2}^{(p,q,t)L}(f_1)$ which is a contradiction. Hence $\rho_{g_1 \pm g_2}^{(p,q,t)L}(f_1) = \rho_{g_1}^{(p,q,t)L}(f_1) = \rho_{g_2}^{(p,q,t)L}(f_1)$. Also in view of the proof of second part of Theorem 17 one can derive the same conclusion for the condition $\overline{\sigma}_{g_1}^{(p,q,t)L}(f_1) \neq \overline{\sigma}_{g_2}^{(p,q,t)L}(f_1)$ and therefore the second part of the theorem is established. □

Theorem 20. *Let f_1, f_2, g_1 and g_2 be any four entire functions. Also let $p, q \in \mathbb{N}$ and $t \in \mathbb{N} \cup \{-1, 0\}$.*
(A) *The following conditions are assumed to be satisfied:*
(i) $(f_1 \pm f_2)$ is of regular relative (p, q, t) growth with respect to at least any one of g_1 or g_2;
(ii) Either $\sigma_{g_1}^{(p,q,t)L}(f_1 \pm f_2) \neq \sigma_{g_2}^{(p,q,t)L}(f_1 \pm f_2)$ or $\overline{\sigma}_{g_1}^{(p,q,t)L}(f_1 \pm f_2) \neq \overline{\sigma}_{g_2}^{(p,q,t)L}(f_1 \pm f_2)$;
(iii) Either $\sigma_{g_1}^{(p,q,t)L}(f_1) \neq \sigma_{g_1}^{(p,q,t)L}(f_2)$ or $\overline{\sigma}_{g_1}^{(p,q,t)L}(f_1) \neq \overline{\sigma}_{g_1}^{(p,q,t)L}(f_2)$;
(iv) Either $\sigma_{g_2}^{(p,q,t)L}(f_1) \neq \sigma_{g_2}^{(p,q,t)L}(f_2)$ or $\overline{\sigma}_{g_2}^{(p,q,t)L}(f_1) \neq \overline{\sigma}_{g_2}^{(p,q,t)L}(f_2)$; then

$$\rho_{g_1 \pm g_2}^{(p,q,t)L}(f_1 \pm f_2) = \rho_{g_1}^{(p,q,t)L}(f_1) = \rho_{g_1}^{(p,q,t)L}(f_2) = \rho_{g_2}^{(p,q,t)L}(f_1) = \rho_{g_2}^{(p,q,t)L}(f_2) \ .$$

(B) *The following conditions are assumed to be satisfied:*
(i) f_1 and f_2 are of regular relative (p, q, t) growth with respect to at least any one of g_1 or g_2;
(ii) Either $\sigma_{g_1 \pm g_2}^{(p,q,t)L}(f_1) \neq \sigma_{g_1 \pm g_2}^{(p,q,t)L}(f_2)$ or $\overline{\sigma}_{g_1 \pm g_2}^{(p,q,t)L}(f_1) \neq \overline{\sigma}_{g_1 \pm g_2}^{(p,q,t)L}(f_2)$;
(iii) Either $\sigma_{g_1}^{(p,q,t)L}(f_1) \neq \sigma_{g_2}^{(p,q,t)L}(f_1)$ or $\overline{\sigma}_{g_1}^{(p,q,t)L}(f_1) \neq \overline{\sigma}_{g_2}^{(p,q,t)L}(f_1)$;
(iv) Either $\sigma_{g_1}^{(p,q,t)L}(f_2) \neq \sigma_{g_2}^{(p,q,t)L}(f_2)$ or $\overline{\sigma}_{g_1}^{(p,q,t)L}(f_2) \neq \overline{\sigma}_{g_2}^{(p,q,t)L}(f_2)$; then

$$\rho_{g_1 \pm g_2}^{(p,q,t)L}(f_1 \pm f_2) = \rho_{g_1}^{(p,q,t)L}(f_1) = \rho_{g_1}^{(p,q,t)L}(f_2) = \rho_{g_2}^{(p,q,t)L}(f_1) = \rho_{g_2}^{(p,q,t)L}(f_2) \ .$$

We omit the proof of Theorem 20 as it is a natural consequence of Theorem 19.

Theorem 21. *Let f_1, f_2, g_1 and g_2 be any four entire functions.*
(A) *The following conditions are assumed to be satisfied:*
(i) At least any one of f_1 or f_2 is of regular relative (p, q, t) growth with respect to g_1 where $p, q \in \mathbb{N}$ and $t \in \mathbb{N} \cup \{-1, 0\}$;
(ii) Either $\tau_{g_1}^{(p,q,t)L}(f_1) \neq \tau_{g_1}^{(p,q,t)L}(f_2)$ or $\overline{\tau}_{g_1}^{(p,q,t)L}(f_1) \neq \overline{\tau}_{g_1}^{(p,q,t)L}(f_2)$ holds, then

$$\lambda_{g_1}^{(p,q,t)L}(f_1 \pm f_2) = \lambda_{g_1}^{(p,q,t)L}(f_1) = \lambda_{g_1}^{(p,q,t)L}(f_2) \ .$$

(B) *The following conditions are assumed to be satisfied:*
(i) f_1, g_1 and g_2 be any three entire functions such that $\lambda_{g_1}^{(p,q,t)L}(f_1)$ and $\lambda_{g_2}^{(p,q,t)L}(f_1)$ exists where $p, q \in \mathbb{N}$ and $t \in \mathbb{N} \cup \{-1, 0\}$;
(ii) Either $\tau_{g_1}^{(p,q,t)L}(f_1) \neq \tau_{g_2}^{(p,q,t)L}(f_1)$ or $\overline{\tau}_{g_1}^{(p,q,t)L}(f_1) \neq \overline{\tau}_{g_2}^{(p,q,t)L}(f_1)$ holds, then

$$\lambda_{g_1 \pm g_2}^{(p,q,t)L}(f_1) = \lambda_{g_1}^{(p,q,t)L}(f_1) = \lambda_{g_2}^{(p,q,t)L}(f_1) \ .$$

Proof. Let f_1, f_2, g_1 and g_2 be any four entire functions satisfying the conditions of the theorem.

Case I. Let $\lambda_{g_1}^{(p,q,t)L}(f_1) = \lambda_{g_1}^{(p,q,t)L}(f_2)$ $(0 < \lambda_{g_1}^{(p,q,t)L}(f_1), \lambda_{g_1}^{(p,q,t)L}(f_2) < \infty)$ and at least f_1 or f_2 and $(f_1 \pm f_2)$ are of regular relative (p,q,t) growth with respect to g_1. Now, in view of Theorem 1, it is easy to see that $\lambda_{g_1}^{(p,q,t)L}(f_1 \pm f_2) \leq \lambda_{g_1}^{(p,q,t)L}(f_1) = \lambda_{g_1}^{(p,q,t)L}(f_2)$. If possible let

$$(35) \qquad \lambda_{g_1}^{(p,q,t)L}(f_1 \pm f_2) < \lambda_{g_1}^{(p,q,t)L}(f_1) = \lambda_{g_1}^{(p,q,t)L}(f_2) \ .$$

Let $\tau_{g_1}^{(p,q,t)L}(f_1) \neq \tau_{g_1}^{(p,q,t)L}(f_2)$. Then in view of the proof of the first part of Theorem 18 and (35) we obtain that $\tau_{g_1}^{(p,q,t)L}(f_1) = \tau_{g_1}^{(p,q,t)L}(f_1 \pm f_2 \mp f_2)$ $= \tau_{g_1}^{(p,q,t)L}(f_2)$ which is a contradiction. Hence $\lambda_{g_1}^{(p,q,t)L}(f_1 \pm f_2) = \lambda_{g_1}^{(p,q,t)L}(f_1)$ $= \lambda_{g_1}^{(p,q,t)L}(f_2)$. Similarly in view of the proof of the first part of Theorem 18 , one can establish the same conclusion under the hypothesis $\overline{\tau}_{g_1}^{(p,q,t)L}(f_1) \neq \overline{\tau}_{g_1}^{(p,q,t)L}(f_2)$. This proves the first part of the theorem.

Case II. Let us consider that $\lambda_{g_1}^{(p,q,t)L}(f_1) = \lambda_{g_2}^{(p,q,t)L}(f_1)$ $(0 < \lambda_{g_1}^{(p,q,t)L}(f_1), \lambda_{g_2}^{(p,q,t)L}(f_1) < \infty)$. Therefore in view of Theorem 3, it follows that $\lambda_{g_1 \pm g_2}^{(p,q,t)L}(f_1) \geq \lambda_{g_1}^{(p,q,t)L}(f_1) = \lambda_{g_2}^{(p,q,t)L}(f_1)$ and if possible let

$$(36) \qquad \lambda_{g_1 \pm g_2}^{(p,q,t)L}(f_1) > \lambda_{g_1}^{(p,q,t)L}(f_1) = \lambda_{g_2}^{(p,q,t)L}(f_1) \ .$$

Suppose $\tau_{g_1}^{(p,q,t)L}(f_1) \neq \tau_{g_2}^{(p,q,t)L}(f_1)$. Then in view of the second part of Theorem 18 and (36), we obtain that $\tau_{g_1}^{(p,q,t)L}(f_1) = \tau_{g_1 \pm g_2 \mp g_2}^{(p,q,t)L}(f_1) =$ $\tau_{g_2}^{(p,q,t)L}(f_1)$ which is a contradiction. Hence $\lambda_{g_1 \pm g_2}^{(p,q,t)L}(f_1) = \lambda_{g_1}^{(p,q,t)L}(f_1) =$ $\lambda_{g_2}^{(p,q,t)L}(f_1)$. Analogously with the help of the second part of Theorem 18, the same conclusion can also be derived under the condition $\overline{\tau}_{g_1}^{(p,q,t)L}(f_1) \neq \overline{\tau}_{g_2}^{(p,q,t)L}(f_1)$ and therefore the second part of the theorem is established. □

THEOREM 22. *Let f_1, f_2, g_1 and g_2 be any four entire functions.*
(A) *The following conditions are assumed to be satisfied:*
(i) At least any one of f_1 or f_2 is of regular relative (p,q,t) growth with respect to g_1 and g_2 where $p, q \in \mathbb{N}$ and $t \in \mathbb{N} \cup \{-1, 0\}$;
(ii) Either $\tau_{g_1}^{(p,q,t)L}(f_1 \pm f_2) \neq \tau_{g_2}^{(p,q,t)L}(f_1 \pm f_2)$ or $\overline{\tau}_{g_1}^{(p,q,t)L}(f_1 \pm f_2) \neq \overline{\tau}_{g_2}^{(p,q,t)L}(f_1 \pm f_2)$;
(iii) Either $\tau_{g_1}^{(p,q,t)L}(f_1) \neq \tau_{g_1}^{(p,q,t)L}(f_2)$ or $\overline{\tau}_{g_1}^{(p,q,t)L}(f_1) \neq \overline{\tau}_{g_1}^{(p,q,t)L}(f_2)$;

(iv) Either $\tau_{g_2}^{(p,q,t)L}(f_1) \neq \tau_{g_2}^{(p,q,t)L}(f_2)$ or $\overline{\tau}_{g_2}^{(p,q,t)L}(f_1) \neq \overline{\tau}_{g_2}^{(p,q,t)L}(f_2)$*; then*

$$\lambda_{g_1 \pm g_2}^{(p,q,t)L}(f_1 \pm f_2) = \lambda_{g_1}^{(p,q,t)L}(f_1) = \lambda_{g_1}^{(p,q,t)L}(f_2) = \lambda_{g_2}^{(p,q,t)L}(f_1) = \lambda_{g_2}^{(p,q,t)L}(f_2) .$$

(B) *The following conditions are assumed to be satisfied:*
(i) At least any one of f_1 *or* f_2 *are of regular relative* (p,q,t) *growth with respect to* $g_1 \pm g_2$ *where* $p,q \in \mathbb{N}$ *and* $t \in \mathbb{N} \cup \{-1,0\}$*;*
(ii) Either $\tau_{g_1 \pm g_2}^{(p,q,t)L}(f_1) \neq \tau_{g_1 \pm g_2}^{(p,q,t)L}(f_2)$ *or* $\overline{\tau}_{g_1 \pm g_2}^{(p,q,t)L}(f_1) \neq \overline{\tau}_{g_1 \pm g_2}^{(p,q,t)L}(f_2)$ *holds;*
(iii) Either $\tau_{g_1}^{(p,q,t)L}(f_1) \neq \tau_{g_2}^{(p,q,t)L}(f_1)$ *or* $\overline{\tau}_{g_1}^{(p,q,t)L}(f_1) \neq \overline{\tau}_{g_2}^{(p,q,t)L}(f_1)$ *holds;*
(iv) Either $\tau_{g_1}^{(p,q,t)L}(f_2) \neq \tau_{g_2}^{(p,q,t)L}(f_2)$ *or* $\overline{\tau}_{g_1}^{(p,q,t)L}(f_2) \neq \overline{\tau}_{g_2}^{(p,q,t)L}(f_2)$ *holds, then*

$$\lambda_{g_1 \pm g_2}^{(p,q,t)L}(f_1 \pm f_2) = \lambda_{g_1}^{(p,q,t)L}(f_1) = \lambda_{g_1}^{(p,q,t)L}(f_2) = \lambda_{g_2}^{(p,q,t)L}(f_1) = \lambda_{g_2}^{(p,q,t)L}(f_2) .$$

We omit the proof of Theorem 22 as it is a natural consequence of Theorem 21.

Theorem 23. *Let* f_1, f_2, g_1 *and* g_2 *be any four entire functions. Also let* $\rho_{g_1}^{(p,q,t)L}(f_1)$, $\rho_{g_1}^{(p,q,t)L}(f_2)$, $\rho_{g_2}^{(p,q,t)L}(f_1)$ *and* $\rho_{g_2}^{(p,q,t)L}(f_2)$ *are all non zero and finite where* $p,q \in \mathbb{N}$ *and* $t \in \mathbb{N} \cup \{-1,0\}$*.*
(A) *Assume the functions* f_1, f_2 *and* g_1 *satisfy the following conditions:*
(i) $\rho_{g_1}^{(p,q,t)L}(f_i) > \rho_{g_1}^{(p,q,t)L}(f_j)$ *for* $i = j = 1,2$ *and* $i \neq j$*;*
(ii) g_1 *satisfies the Property (A) and* $q > 1$*, then*

$$\begin{aligned} \sigma_{g_1}^{(p,q,t)L}(f_1 \cdot f_2) &= \sigma_{g_1}^{(p,q,t)L}(f_i) \mid i = 1,2 \text{ and} \\ \overline{\sigma}_{g_1}^{(p,q,t)L}(f_1 \cdot f_2) &= \overline{\sigma}_{g_1}^{(p,q,t)L}(f_i) \mid i = 1,2 \quad . \end{aligned}$$

Similarly,

$$\begin{aligned} \sigma_{g_1}^{(p,q,t)L}\left(\frac{f_1}{f_2}\right) &= \sigma_{g_1}^{(p,q,t)L}(f_i) \mid i = 1,2 \text{ and} \\ \overline{\sigma}_{g_1}^{(p,q,t)L}\left(\frac{f_1}{f_2}\right) &= \overline{\sigma}_{g_1}^{(p,q,t)L}(f_i) \mid i = 1,2 \end{aligned}$$

holds provided (i) $\frac{f_1}{f_2}$ *is entire, (ii)* $\rho_{g_1}^{(p,q,t)L}(f_i) > \rho_{g_1}^{(p,q,t)L}(f_j) \mid i = 1,2; j = 1,2; i \neq j$*, (iii)* g_1 *satisfy the Property (A) and (iv)* $q > 1$*.*
(B) *Assume the functions* g_1, g_2 *and* f_1 *satisfy the following conditions:*
(i) $\rho_{g_i}^{(p,q,t)L}(f_1) < \rho_{g_j}^{(p,q,t)L}(f_1)$ *with at least* f_1 *is of regular relative* (p,q,t) *growth with respect to* g_j *for* $i = j = 1,2$ *and* $i \neq j$*, and* g_i *satisfy the Property (A);*

(ii) $g_1 \cdot g_2$ *satisfy the Property (A) and* $p > 1$*, then*

$$\begin{aligned} \sigma_{g_1 \cdot g_2}^{(p,q,t)L}(f_1) &= \sigma_{g_i}^{(p,q,t)L}(f_1) \mid i = 1,2 \text{ and} \\ \overline{\sigma}_{g_1 \cdot g_2}^{(p,q,t)L}(f_1) &= \overline{\sigma}_{g_i}^{(p,q,t)L}(f_1) \mid i = 1,2 \ . \end{aligned}$$

Similarly,

$$\begin{aligned} \sigma_{\frac{g_1}{g_2}}^{(p,q,t)L}(f_1) &= \sigma_{g_i}^{(p,q,t)L}(f_1) \mid i = 1,2 \text{ and} \\ \overline{\sigma}_{\frac{g_1}{g_2}}^{(p,q,t)L}(f_1) &= \overline{\sigma}_{gi}^{(p,q,t)L}(f_1) \mid i = 1,2 \end{aligned}$$

holds provided (i) $\frac{g_1}{g_2}$ *is entire and satisfy the Property (A),* (ii) *At least* f_1 *is of regular relative* (p,q,t) *growth with respect to* g_2*,* (iii) $\rho_{g_i}^{(p,q,t)L}(f_1) < \rho_{g_j}^{(p,q,t)L}(f_1) \mid i = 1,2;\ j = 1,2;\ i \neq j$ *and* (iv) g_1 *satisfy the Property (A).*

(C) *Assume the functions* f_1, f_2*,* g_1 *and* g_2 *satisfy the following conditions:*

(i) $g_1 \cdot g_2$ *satisfy the Property (A),* $p > 1$ *and* $q > 1$*;*

(ii) $\rho_{g_i}^{(p,q,t)L}(f_1) < \rho_{g_j}^{(p,q,t)L}(f_1)$ *with at least* f_1 *is of regular relative* (p,q,t) *growth with respect to* g_j *for* $i = 1,\ 2,\ j = 1,2$ *and* $i \neq j$*;*

(iii) $\rho_{g_i}^{(p,q,t)L}(f_2) < \rho_{g_j}^{(p,q,t)L}(f_2)$ *with at least* f_2 *is of regular relative* (p,q,t) *growth with respect to* g_j *for* $i = 1,\ 2,\ j = 1,2$ *and* $i \neq j$*;*

(iv) $\rho_{g_1}^{(p,q,t)L}(f_i) > \rho_{g_1}^{(p,q,t)L}(f_j)$ *and* $\rho_{g_2}^{(p,q,t)L}(f_i) > \rho_{g_2}^{(p,q,t)L}(f_j)$ *holds simultaneously for* $i = 1,2;\ j = 1,2$ *and* $i \neq j$*;*

(v) $\rho_{g_m}^{(p,q,t)L}(f_l) =$ $\max\left[\min\left\{\rho_{g_1}^{(p,q,t)L}(f_1), \rho_{g_2}^{(p,q,t)L}(f_1)\right\}, \min\left\{\rho_{g_1}^{(p,q,t)L}(f_2), \rho_{g_2}^{(p,q,t)L}(f_2)\right\}\right] \mid$ $l = m = 1,2$*; then*

$$\begin{aligned} &\sigma_{g_1 \cdot g_2}^{(p,q,t)L}(f_1 \cdot f_2) = \sigma_{g_m}^{(p,q,t)L}(f_l) \mid l = m = 1,2 \text{ and} \\ &\overline{\sigma}_{g_1 \cdot g_2}^{(p,q,t)L}(f_1 \cdot f_2) = \overline{\sigma}_{g_m}^{(p,q,t)L}(f_l) \mid l = m = 1,2 \ . \end{aligned}$$

Similarly,

$$\begin{aligned} \sigma_{\frac{g_1}{g_2}}^{(p,q,t)L}\left(\frac{f_1}{f_2}\right) &= \sigma_{g_m}^{(p,q,t)L}(f_l) \mid l = m = 1,2 \text{ and} \\ \overline{\sigma}_{\frac{g_1}{g_2}}^{(p,q,t)L}\left(\frac{f_1}{f_2}\right) &= \overline{\sigma}_{g_m}^{(p,q,t)L}(f_l) \mid l = m = 1,2. \end{aligned}$$

holds provided $\frac{f_1}{f_2}$ *and* $\frac{g_1}{g_2}$ *are entire functions which satisfy the following conditions:*

(i) $\frac{g_1}{g_2}$ *satisfy the Property (A),* $p>1$ *and* $q>1$*;*
(ii) At least f_1 *is of regular relative* (p,q,t) *growth with respect to* g_2 *and* $\rho_{g_1}^{(p,q,t)L}(f_1) \neq \rho_{g_2}^{(p,q,t)L}(f_1)$*;*
(iii) At least f_2 *is of regular relative* (p,q,t) *growth with respect to* g_2 *and* $\rho_{g_1}^{(p,q,t)L}(f_2) \neq \rho_{g_2}^{(p,q,t)L}(f_2)$*;*
(iv) $\rho_{g_1}^{(p,q,t)L}(f_i) < \rho_{g_1}^{(p,q,t)L}(f_j)$ *and* $\rho_{g_2}^{(p,q,t)L}(f_i) < \rho_{g_2}^{(p,q,t)L}(f_j)$ *holds simultaneously for* $i=1,2$*;* $j=1,2$ *and* $i\neq j$*;*
(v) $\rho_{g_m}^{(p,q,t)L}(f_l) =$
$\max\left[\min\left\{\rho_{g_1}^{(p,q,t)L}(f_1), \rho_{g_2}^{(p,q,t)L}(f_1)\right\}, \min\left\{\rho_{g_1}^{(p,q,t)L}(f_2), \rho_{g_2}^{(p,q,t)L}(f_2)\right\}\right] \mid$
$l=m=1,2$.

Proof. Let us consider that $\rho_{g_1}^{(p,q,t)L}(f_1)$, $\rho_{g_1}^{(p,q,t)L}(f_2)$, $\rho_{g_2}^{(p,q,t)L}(f_1)$ and $\rho_{g_2}^{(p,q,t)L}(f_2)$ are all non zero and finite.

Case I. Suppose that $\rho_{g_1}^{(p,q,t)L}(f_1) > \rho_{g_1}^{(p,q,t)L}(f_2)$. Also let g_1 satisfy the Property (A). Now for any arbitrary $\varepsilon>0$, we get from (18) for all sufficiently large values of r that

$$M_{f_1\cdot f_2}(r) \leq$$

$$M_{g_1}\left[\exp^{[p-1]}\left\{\left(\sigma_{g_1}^{(p,q,t)L}(f_1)+\frac{\varepsilon}{2}\right)\left[\log^{[q-1]} r\cdot \exp^{[t+1]} L(r)\right]^{\rho_{g_1}^{(p,q,t)L}(f_1)}\right\}\right]$$

$$\times M_{g_1}\left[\exp^{[p-1]}\left\{\left(\sigma_{g_1}^{(p,q,t)L}(f_2)+\frac{\varepsilon}{2}\right)\left[\log^{[q-1]} r\cdot \exp^{[t+1]} L(r)\right]^{\rho_{g_1}^{(p,q,t)L}(f_2)}\right\}\right].$$

Since $\rho_{g_1}^{(p,q,t)L}(f_1) > \rho_{g_1}^{(p,q,t)L}(f_2)$, we get that

$$\lim_{r\to\infty}\frac{\left(\sigma_{g_1}^{(p,q,t)L}(f_1)+\frac{\varepsilon}{2}\right)\left[\log^{[q-1]} r\cdot \exp^{[t+1]} L(r)\right]^{\rho_{g_1}^{(p,q,t)L}(f_1)}}{\left(\sigma_{g_1}^{(p,q,t)L}(f_2)+\frac{\varepsilon}{2}\right)\left[\log^{[q-1]} r\cdot \exp^{[t+1]} L(r)\right]^{\rho_{g_1}^{(p,q,t)L}(f_2)}} = \infty .$$

Therefore we get for all sufficiently large values of r that

$$M_{g_1}\left[\exp^{[p-1]}\left\{\left(\sigma_{g_1}^{(p,q,t)L}(f_1)+\frac{\varepsilon}{2}\right)\left[\log^{[q-1]} r\cdot \exp^{[t+1]} L(r)\right]^{\rho_{g_1}^{(p,q,t)L}(f_1)}\right\}\right]$$

$$> M_{g_1}\left[\exp^{[p-1]}\left\{\left(\sigma_{g_1}^{(p,q,t)L}(f_2)+\frac{\varepsilon}{2}\right)\left[\log^{[q-1]} r\cdot \exp^{[t+1]} L(r)\right]^{\rho_{g_1}^{(p,q,t)L}(f_2)}\right\}\right]$$

hold and from the above arguments it follows for all sufficiently large values of r that

(37) $M_{f_1 \cdot f_2}(r) <$

$$\left[M_{g_1}\left[\exp^{[p-1]}\left\{ \left(\sigma_{g_1}^{(p,q,t)L}(f_1) + \frac{\varepsilon}{2} \right) \left[\log^{[q-1]} r \cdot \exp^{[t+1]} L(r) \right]^{\rho_{g_1}^{(p,q,t)L}(f_1)} \right\} \right] \right]^2 .$$

Let us observe that

$$\delta_1 := \frac{\sigma_{g_1}^{(p,q,t)L}(f_1) + \varepsilon}{\sigma_{g_1}^{(p,q,t)L}(f_1) + \frac{\varepsilon}{2}} > 1$$

which implies that

$$(38) \quad \frac{\exp^{[p-2]}\left(\sigma_{g_1}^{(p,q,t)L}(f_1) + \varepsilon \right) \left[\log^{[q-1]} r \cdot \exp^{[t+1]} L(r) \right]^{\rho_{g_1}^{(p,q,t)L}(f_1)}}{\exp^{[p-2]}\left(\sigma_{g_1}^{(p,q,t)L}(f_1) + \frac{\varepsilon}{2} \right) \left[\log^{[q-1]} r \cdot \exp^{[t+1]} L(r) \right]^{\rho_{g_1}^{(p,q,t)L}(f_1)}} = \delta\,(\text{say}) > 1 .$$

Since g_1 satisfy the Property (A), in view of Lemma 2 and (38) we obtain from (37) for all sufficiently large values of r that

$M_{f_1 \cdot f_2}(r) <$

$$M_{g_1}\left[\left[\exp^{[p-1]}\left\{ \left(\sigma_{g_1}^{(p,q,t)L}(f_1) + \frac{\varepsilon}{2} \right) \left[\log^{[q-1]} r \cdot \exp^{[t+1]} L(r) \right]^{\rho_{g_1}^{(p,q,t)L}(f_1)} \right\} \right]^{\delta} \right]$$

i.e., $M_{f_1 \cdot f_2}(r) <$

$$M_{g_1}\left[\exp^{[p-1]}\left\{ \left(\sigma_{g_1}^{(p,q,t)L}(f_1) + \varepsilon \right) \left[\log^{[q-1]} r \cdot \exp^{[t+1]} L(r) \right]^{\rho_{g_1}^{(p,q,t)L}(f_1)} \right\} \right] .$$

As $\rho_{g_1}^{(p,q,t)L}(f_1) > \rho_{g_1}^{(p,q,t)L}(f_2)$, so in view of Theorem 9, we get from above for all sufficiently large values of r that

$M_{f_1 \cdot f_2}(r) <$

$$M_{g_1}\left[\exp^{[p-1]}\left\{ \left(\sigma_{g_1}^{(p,q,t)L}(f_1) + \varepsilon \right) \left[\log^{[q-1]} r \cdot \exp^{[t+1]} L(r) \right]^{\rho_{g_1}^{(p,q,t)L}(f_1 \cdot f_2)} \right\} \right] .$$

$$(39) \quad i.e.,\ \sigma_{g_1}^{(p,q,t)L}(f_1 \cdot f_2) \le \sigma_{g_1}^{(p,q,t)L}(f_1) .$$

In order to establish the equality of (39), let us restrict ourselves on the functions g_1 and $f_i \mid i = 1, 2$ such that $q > 1$. Now let h, h_1, h_2 and k

be any four entire functions such that $h=\frac{h_2}{h_1}$ and k satisfy the Property (A). Further without loss of any generality let $\rho_k^{(p,q,t)L}(h_1)<\rho_k^{(p,q,t)L}(h_2)$ where p, q are any two positive integers with $q>1$. Now we know that $T_h(r)=T_{\frac{h_2}{h_1}}(r)\leq T_{h_2}(r)+T_{h_1}(r)$. Therefore in view of Lemma 3, we get (in the line of the construction of the proof as above) for all sufficiently large values of r that

$$\log M_{\frac{h_2}{h_1}}(r)\leq 3\left[T_{h_1}(2r)+T_{h_2}(2r)\right]$$

$$i.e.,\ \left[M_{\frac{h_2}{h_1}}\left(\frac{r}{2}\right)\right]^{\frac{1}{3}}\leq M_{h_1}(r)\cdot M_{h_2}(r)$$

$$i.e.,\ M_{\frac{h_2}{h_1}}\left(\frac{r}{2}\right)<$$

$$\left[M_k\left[\exp^{[p-1]}\left\{\left(\sigma_k^{(p,q,t)L}(h_2)+\frac{\varepsilon}{2}\right)\left[\log^{[q-1]}r\cdot\exp^{[t+1]}L(r)\right]^{\rho_k^{(p,q,t)L}(h_2)}\right\}\right]\right]^6 .$$

Therefore in view of Theorem 9 and (38), we get for all sufficiently large values of r that

$$M_{\frac{h_2}{h_1}}\left(\frac{r}{2}\right)<$$

$$M_k\left[\exp^{[p-1]}\left\{\left(\sigma_k^{(p,q,t)L}(h_2)+\varepsilon\right)\left[\log^{[q-1]}r\cdot\exp^{[t+1]}L(r)\right]^{\rho_k^{(p,q,t)L}(h_2)}\right\}\right] .$$

$$i.e.,\ \sigma_k^{(p,q,t)L}(h)=\sigma_k^{(p,q,t)L}\left(\frac{h_2}{h_1}\right)\leq\sigma_k^{(p,q,t)L}(h_2)\ . \tag{40}$$

Further without loss of any generality, let $f=f_1\cdot f_2$ and $\rho_{g_1}^{(p,q,t)L}(f_2)<\rho_{g_1}^{(p,q,t)L}(f_1)=\rho_{g_1}^{(p,q,t)L}(f)$. Then in view of (39), we obtain that $\sigma_{g_1}^{(p,q,t)L}(f)=\sigma_{g_1}^{(p,q,t)L}(f_1\cdot f_2)\leq\sigma_{g_1}^{(p,q,t)L}(f_1)$. Also $f_1=\frac{f}{f_2}$ and in this case we obtain from (40) that $\sigma_{g_1}^{(p,q,t)L}(f_1)\leq\sigma_{g_1}^{(p,q,t)L}(f)=\sigma_{g_1}^{(p,q,t)L}(f_1\cdot f_2)$. Hence $\sigma_{g_1}^{(p,q,t)L}(f)=\sigma_{g_1}^{(p,q,t)L}(f_1)\Rightarrow\sigma_{g_1}^{(p,q,t)L}(f_1\cdot f_2)=\sigma_{g_1}^{(p,q,t)L}(f_1)$ provided $q>1$.

Similarly, if we consider $\rho_{g_1}^{(p,q,t)L}(f_1)<\rho_{g_1}^{(p,q,t)L}(f_2)$, then one can verify that $\sigma_{g_1}^{(p,q,t)L}(f_1\cdot f_2)=\sigma_{g_1}^{(p,q,t)L}(f_2)$ provided $q>1$.

Next we may suppose that $f=\frac{f_1}{f_2}$ with f_1, f_2 and f are all entire functions.

Sub Case $\mathbf{I_A}$. Let $\rho_{g_1}^{(p,q,t)L}(f_2)<\rho_{g_1}^{(p,q,t)L}(f_1)$. Therefore in view of

Theorem 9, $\rho_{g_1}^{(p,q,t)L}(f_2) < \rho_{g_1}^{(p,q,t)L}(f_1) = \rho_{g_1}^{(p,q,t)L}(f)$. We have $f_1 = f \cdot f_2$. So, $\sigma_{g_1}^{(p,q,t)L}(f_1) = \sigma_{g_1}^{(p,q,t)L}(f) = \sigma_{g_1}^{(p,q,t)L}\left(\frac{f_1}{f_2}\right)$ provided $q > 1$.

Sub Case $\mathbf{I_B}$. Let $\rho_{g_1}^{(p,q,t)L}(f_2) > \rho_{g_1}^{(p,q,t)L}(f_1)$. Therefore in view of Theorem 9, $\rho_{g_1}^{(p,q,t)L}(f_1) < \rho_{g_1}^{(p,q,t)L}(f_2) = \rho_{g_1}^{(p,q,t)L}(f)$. Now in view of (40), we get that $\sigma_{g_1}^{(p,q,t)L}\left(\frac{f_1}{f_2}\right) \leq \sigma_{g_1}^{(p,q,t)L}(f_2)$. Further we have $f_2 = \frac{f_1}{f}$ and in this case $\sigma_{g_1}^{(p,q,t)L}(f_2) \leq \sigma_{g_1}^{(p,q,t)L}\left(\frac{f_1}{f_2}\right)$. So $\sigma_{g_1}^{(p,q,t)L}\left(\frac{f_1}{f_2}\right) = \sigma_{g_1}^{(p,q,t)L}(f_2)$ provided $q > 1$.

Case II. Let $\rho_{g_1}^{(p,q,t)L}(f_1) > \rho_{g_1}^{(p,q,t)L}(f_2)$. Also let g_1 satisfy the Property (A). Now for any arbitrary $\varepsilon > 0$, we obtain from (18) and (21) for a sequence of values of r tending to infinity that

$$M_{f_1 \cdot f_2}(r) \leq$$
$$M_{g_1}\left[\exp^{[p-1]}\left\{\left(\overline{\sigma}_{g_1}^{(p,q,t)L}(f_1) + \frac{\varepsilon}{2}\right)\left[\log^{[q-1]} r \cdot \exp^{[t+1]} L(r)\right]^{\rho_{g_1}^{(p,q,t)L}(f_1)}\right\}\right]$$
$$\times M_{g_1}\left[\exp^{[p-1]}\left\{\left(\sigma_{g_1}^{(p,q,t)L}(f_2) + \frac{\varepsilon}{2}\right)\left[\log^{[q-1]} r \cdot \exp^{[t+1]} L(r)\right]^{\rho_{g_1}^{(p,q,t)L}(f_2)}\right\}\right].$$

Now in view of $\rho_{g_1}^{(p,q,t)L}(f_1) > \rho_{g_1}^{(p,q,t)L}(f_2)$, we get that

$$\lim_{r\to\infty} \frac{\left(\overline{\sigma}_{g_1}^{(p,q,t)L}(f_1) + \frac{\varepsilon}{2}\right)\left[\log^{[q-1]} r \cdot \exp^{[t+1]} L(r)\right]^{\rho(p,q,t)L(f_1)}}{\left(\sigma_{g_1}^{(p,q,t)L}(f_2) + \frac{\varepsilon}{2}\right)\left[\log^{[q-1]} r \cdot \exp^{[t+1]} L(r)\right]^{\rho_{g_1}^{(p,q,t)L}(f_2)}} = \infty .$$

Therefore we get for all sufficiently large values of r that

$$M_{g_1}\left[\exp^{[p-1]}\left\{\left(\overline{\sigma}_{g_1}^{(p,q,t)L}(f_1) + \frac{\varepsilon}{2}\right)\left[\log^{[q-1]} r \cdot \exp^{[t+1]} L(r)\right]^{\rho_{g_1}^{(p,q,t)L}(f_1)}\right\}\right]$$
$$> M_{g_1}\left[\exp^{[p-1]}\left\{\left(\sigma_{g_1}^{(p,q,t)L}(f_2) + \frac{\varepsilon}{2}\right)\left[\log^{[q-1]} r \cdot \exp^{[t+1]} L(r)\right]^{\rho_{g_1}^{(p,q,t)L}(f_2)}\right\}\right]$$

and therefore from the above arguments it follows for a sequence of values of r tending to infinity that

$$M_{f_1 \cdot f_2}(r) <$$
$$\left[M_{g_1}\left[\exp^{[p-1]}\left\{\left(\overline{\sigma}_{g_1}^{(p,q,t)L}(f_1) + \frac{\varepsilon}{2}\right)\left[\log^{[q-1]} r \cdot \exp^{[t+1]} L(r)\right]^{\rho_{g_1}^{(p,q,t)L}(f_1)}\right\}\right]\right]^2 .$$

Now using the similar technique for a sequence of values of r tending to infinity as explored in the proof of Case I, one can easily verify that $\overline{\sigma}_{g_1}^{(p,q,t)L}(f_1 \cdot f_2) = \overline{\sigma}_{g_1}^{(p,q,t)L}(f_1)$ and $\overline{\sigma}_{g_1}^{(p,q,t)L}\left(\frac{f_1}{f_2}\right) = \overline{\sigma}_{g_1}^{(p,q,t)L}(f_i) \mid i = 1, 2$ under the conditions specified in the theorem provided $q > 1$.

Similarly, if we consider $\rho_{g_1}^{(p,q,t)L}(f_1) < \rho_{g_1}^{(p,q,t)L}(f_2)$, then one can verify that $\overline{\sigma}_{g_1}^{(p,q,t)L}(f_1 \cdot f_2) = \overline{\sigma}_{g_1}^{(p,q,t)L}(f_2)$ provided $q > 1$.

Therefore the first part of theorem follows from Case I and Case II.

Case III. Let $\rho_{g_1}^{(p,q,t)L}(f_1) < \rho_{g_2}^{(p,q,t)L}(f_1)$ and $g_1 \cdot g_2$, g_1 are satisfy the Property (A) with at least f_1 is of regular relative (p, q, t) growth with respect to g_2. Now for all sufficiently large values of n and $\rho_{g_1}^{(p,q,t)L}(f_1) < \rho_{g_2}^{(p,q,t)L}(f_1)$, we get that

$$\exp^{[p-1]}\left\{\left(\overline{\sigma}_{g_2}^{(p,q,t)L}(f_1) - \varepsilon\right)\left[\log^{[q-1]} r_n \cdot \exp^{[t+1]} L(r_n)\right]^{\rho_{g_2}^{(p,q,t)L}(f_1)}\right\} >$$

$$\exp^{[p-1]}\left\{\left(\sigma_{g_1}^{(p,q,t)L}(f_1) - \varepsilon\right)\left[\log^{[q-1]} r_n \cdot \exp^{[t+1]} L(r_n)\right]^{\rho_{g_1}^{(p,q,t)L}(f_1)}\right\}$$

holds. Consequently

$$M_{g_2}\left[\exp^{[p-1]}\left\{\left(\overline{\sigma}_{g_2}^{(p,q,t)L}(f_1) - \varepsilon\right)\left[\log^{[q-1]} r_n \cdot \exp^{[t+1]} L(r_n)\right]^{\rho_{g_2}^{(p,q,t)L}(f_1)}\right\}\right] >$$

$$M_{g_2}\left[\exp^{[p-1]}\left\{\left(\sigma_{g_1}^{(p,q,t)L}(f_1) - \varepsilon\right)\left[\log^{[q-1]} r_n \cdot \exp^{[t+1]} L(r_n)\right]^{\rho_{g_1}^{(p,q,t)L}(f_1)}\right\}\right].$$

also holds.

Therefore in view of (19), (20) and above, we obtain for a sequence of values of r tending to infinity that

$$M_{g_1 \cdot g_2}\left[\exp^{[p-1]}\left\{\left(\sigma_{g_1}^{(p,q,t)L}(f_1) - \varepsilon\right)\left[\log^{[q-1]} r_n \cdot \exp^{[t+1]} L(r_n)\right]^{\rho_{g_1}^{(p,q,t)L}(f_1)}\right\}\right]$$

$$\leq [M_{f_1}(r)]^2$$

Since $g_1 \cdot g_2$ has the Property (A), in view of Lemma 2 we obtain from above for a sequence of values of r tending to infinity that

$$M_{g_1\cdot g_2}\left[\left[\exp^{[p-1]}\left\{\left(\sigma_{g_1}^{(p,q,t)L}(f_1)-\varepsilon\right)\left[\log^{[q-1]}r_n\cdot\exp^{[t+1]}L(r_n)\right]^{\rho_{g_1}^{(p,q,t)L}(f_1)}\right\}\right]^{\frac{1}{\delta}}\right]$$
$$\leq M_{f_1}(r)$$

Now making $\delta\rightarrow 1+$ we obtain in view of Theorem 11 and above that

$$\left(\sigma_{g_1}^{(p,q,t)L}(f_1)-\varepsilon\right)\left[\log^{[q-1]}r_n\cdot\exp^{[t+1]}L(r_n)\right]^{\rho_{g_1}^{(p,q,t)L}(f_1)}<$$
$$\log^{[p-1]}M_{g_1\cdot g_2}^{-1}M_{f_1}(r)\ .$$

Since $\varepsilon>0$ is arbitrary, it follows from above that

$$\sigma_{g_1\cdot g_2}^{(p,q,t)L}(f_1)\geq\sigma_{g_1}^{(p,q,t)L}(f_1)\ . \tag{41}$$

In order to establish the equality of (41), let us restrict ourselves on the functions $g_i\mid i=1,2$ and f_1 such that $p>1$. Now let h, h_1, h_2 and k be any four entire functions such that $h=\frac{h_1}{h_2}$, h satisfy the Property (A) and at least k is of regular relative (p,q,t) growth with respect to h_2. Further without loss of any generality let $\rho_{h_1}^{(p,q,t)L}(k)<\rho_{h_2}^{(p,q,t)L}(k)$. Now we know that $T_h(r)=T_{\frac{h_2}{h_1}}(r)\leq T_{h_2}(r)+T_{h_1}(r)$. Therefore in view of Lemma 3 we get (in the line of the construction of the proof as above) for a sequence of values of r tending to infinity that

$$\log M_{\frac{h_2}{h_1}}(r)\leq 3\left[T_{h_1}(2r)+T_{h_2}(2r)\right]$$
$$i.e.,\ \left[M_{\frac{h_2}{h_1}}\left(\frac{r}{2}\right)\right]^{\frac{1}{3}}\leq M_{h_1}(r)\cdot M_{h_2}(r)\ .$$

Therefore in view of Theorem 12 and in the line of the construction of the proof as above we get that

$$i.e.,\ \sigma_h^{(p,q,t)L}(k)=\sigma_{\frac{h_1}{h_2}}^{(p,q,t)L}(k)\geq\sigma_{h_1}^{(p,q,t)L}(k)\,, \tag{42}$$

provided $p>1$.

Further without loss of any generality, let $g=g_1\cdot g_2$ and $\rho_g^{(p,q,t)L}(f_1)=\rho_{g_1}^{(p,q,t)L}(f_1)<\rho_{g_2}^{(p,q,t)L}(f_1)$. Then in view of (41), we obtain that $\sigma_g^{(p,q,t)L}(f_1)=\sigma_{g_1\cdot g_2}^{(p,q,t)L}(f_1)\geq\sigma_{g_1}^{(p,q,t)L}(f_1)$. Also $g_1=\frac{g}{g_2}$ and in this case we obtain from (42) that $\sigma_{g_1}^{(p,q,t)L}(f_1)\geq\ \sigma_g^{(p,q,t)L}(f_1)=\sigma_{g_1\cdot g_2}^{(p,q,t)L}(f_1)$.

Hence $\sigma_{g}^{(p,q,t)L}(f_1)=\sigma_{g_1}^{(p,q,t)L}(f_1)\Rightarrow\sigma_{g_1\cdot g_2}^{(p,q,t)L}(f_1)=\sigma_{g_1}^{(p,q,t)L}(f_1)$ provided $p>1$.

Similarly, if we consider $\rho_{g_1}^{(p,q,t)L}(f_1)>\rho_{g_2}^{(p,q,t)L}(f_1)$ with at least f_1 is of regular relative (p,q,t) growth with respect to g_1 and g_2 satisfy Property (A), then one can verify that $\sigma_{g_1\cdot g_2}^{(p,q,t)L}(f_1)=\sigma_{g_2}^{(p,q,t)L}(f_1)$ provided $p>1$.

Next we may suppose that $g=\frac{g_1}{g_2}$ with g_1, g_2, g are all entire functions satisfying the conditions specified in the theorem.

Sub Case III$_{\mathbf{A}}$. Let $\rho_{g_1}^{(p,q,t)L}(f_1)<\rho_{g_2}^{(p,q,t)L}(f_1)$. Therefore in view of Theorem 12, $\rho_{g}^{(p,q,t)L}(f_1)=\rho_{g_1}^{(p,q,t)L}(f_1)<\rho_{g_2}^{(p,q,t)L}(f_1)$. We have $g_1=g\cdot g_2$. So $\sigma_{g_1}^{(p,q,t)L}(f_1)=\sigma_{g}^{(p,q,t)L}(f_1)=\sigma_{\frac{g_1}{g_2}}^{(p,q,t)L}(f_1)$ provided $p>1$.

Sub Case III$_{\mathbf{B}}$. Let $\rho_{g_1}^{(p,q,t)L}(f_1)>\rho_{g_2}^{(p,q,t)L}(f_1)$. Therefore in view of Theorem 12, $\rho_{g}^{(p,q,t)L}(f_1)=\rho_{g_2}^{(p,q,t)L}(f_1)<\rho_{g_1}^{(p,q,t)L}(f_1)$. Now in view of (42), we get that $\sigma_{\frac{g_1}{g_2}}^{(p,q,t)L}(f_1)\le\sigma_{g_2}^{(p,q,t)L}(f_1)$. Further we have $g_2=\frac{g_1}{g}$ and in this case $\sigma_{g_2}^{(p,q,t)L}(f_1)\le\sigma_{\frac{g_1}{g_2}}^{(p,q,t)L}(f_1)$. So $\sigma_{\frac{g_1}{g_2}}^{(p,q,t)L}(f_1)=\sigma_{g_2}^{(p,q,t)L}(f_1)$ provided $p>1$.

Case IV. Suppose $\rho_{g_1}^{(p,q,t)L}(f_1)<\rho_{g_2}^{(p,q,t)L}(f_1)$ and $g_1\cdot g_2$, g_1 are satisfy the Property (A) with at least f_1 is of regular relative (p,q,t) growth with respect to g_2. Therefore for all sufficiently large values of r and $\rho_{g_1}^{(p,q,t)L}(f_1)<\rho_{g_2}^{(p,q,t)L}(f_1)$

$$\exp^{[p-1]}\left\{\left(\overline{\sigma}_{g_2}^{(p,q,t)L}(f_1)-\varepsilon\right)\left[\log^{[q-1]}r_n\cdot\exp^{[t+1]}L(r_n)\right]^{\rho_{g_2}^{(p,q,t)L}(f_1)}\right\}>$$

$$\exp^{[p-1]}\left\{\left(\overline{\sigma}_{g_1}^{(p,q,t)L}(f_1)-\varepsilon\right)\left[\log^{[q-1]}r_n\cdot\exp^{[t+1]}L(r_n)\right]^{\rho_{g_1}^{(p,q,t)L}(f_1)}\right\}$$

holds. Consequently

$$M_{g_2}\left[\exp^{[p-1]}\left\{\left(\overline{\sigma}_{g_2}^{(p,q,t)L}(f_1)-\varepsilon\right)\left[\log^{[q-1]}r_n\cdot\exp^{[t+1]}L(r_n)\right]^{\rho_{g_2}^{(p,q,t)L}(f_1)}\right\}\right]>$$

$$M_{g_2}\left[\exp^{[p-1]}\left\{\left(\overline{\sigma}_{g_1}^{(p,q,t)L}(f_1)-\varepsilon\right)\left[\log^{[q-1]}r_n\cdot\exp^{[t+1]}L(r_n)\right]^{\rho_{g_1}^{(p,q,t)L}(f_1)}\right\}\right].$$

Hence in view of (19) and from above arguments we obtain for all sufficiently large values of r that

$$M_{g_1 \cdot g_2}\left[\exp^{[p-1]}\left\{\left(\overline{\sigma}_{g_1}^{(p,q,t)L}(f_1)-\varepsilon\right)\left[\log^{[q-1]} r_n \cdot \exp^{[t+1]} L(r_n)\right]^{\rho_{g_1}^{(p,q,t)L}(f_1)}\right\}\right] \leq \left[M_{f_1}(r)\right]^2 .$$

Now using the similar technique for all sufficiently large values of r as explored in the proof of Case III, one can easily verify that $\overline{\sigma}_{g_1 \cdot g_2}^{(p,q,t)L}(f_1) = \overline{\sigma}_{g_1}^{(p,q,t)L}(f_1)$ and $\overline{\sigma}_{\frac{g_1}{g_2}}^{(p,q,t)L}(f_1) = \overline{\sigma}_{g_i}^{(p,q,t)L}(f_1) \mid i = 1, 2$ under the conditions specified in the theorem.

Likewise, if we consider $\rho_{g_1}^{(p,q,t)L}(f_1) > \rho_{g_2}^{(p,q,t)L}(f_1)$ and $g_1 \cdot g_2$, g_2 are satisfy the Property (A) with at least f_1 is of regular relative (p,q,t) growth with respect to g_1, then one can verify that $\overline{\sigma}_{g_1 \cdot g_2}^{(p,q,t)L}(f_1) = \overline{\sigma}_{g_2}^{(p,q,t)L}(f_1)$ provided $p > 1$.

Therefore the second part of theorem follows from Case III and Case IV.

Proof of the third part of the Theorem is omitted as it can be carried out in view of Theorem 13 and Theorem 15 and the above cases. □

THEOREM 24. *Let f_1, f_2, g_1 and g_2 be any four entire functions. Also let $\lambda_{g_1}^{(p,q,t)L}(f_1)$, $\lambda_{g_1}^{(p,q,t)L}(f_2)$, $\lambda_{g_2}^{(p,q,t)L}(f_1)$ and $\lambda_{g_2}^{(p,q,t)L}(f_2)$ are all non zero and finite where $p, q \in \mathbb{N}$ and $t \in \mathbb{N} \cup \{-1, 0\}$.*
(A) *Assume the functions f_1, f_2 and g_1 satisfy the following conditions:*
(i) $\lambda_{g_1}^{(p,q,t)L}(f_i) > \lambda_{g_1}^{(p,q,t)L}(f_j)$ with at least f_j is of regular relative (p,q,t) growth with respect to g_1 for $i = j = 1, 2$ and $i \neq j$;
(ii) g_1 satisfy the Property (A) and $q > 1$, then

$$\begin{aligned}\tau_{g_1}^{(p,q,t)L}(f_1 \cdot f_2) &= \tau_{g_1}^{(p,q,t)L}(f_i) \mid i = 1,2 \text{ and}\\ \overline{\tau}_{g_1}^{(p,q,t)L}(f_1 \cdot f_2) &= \overline{\tau}_{g_1}^{(p,q,t)L}(f_i) \mid i = 1,2 .\end{aligned}$$

Similarly,

$$\begin{aligned}\tau_{g_1}^{(p,q,t)L}\left(\frac{f_1}{f_2}\right) &= \tau_{g_1}^{(p,q,t)L}(f_i) \mid i = 1,2 \text{ and}\\ \overline{\tau}_{g_1}^{(p,q,t)L}\left(\frac{f_1}{f_2}\right) &= \overline{\tau}_{g_1}^{(p,q,t)L}(f_i) \mid i = 1,2\end{aligned}$$

holds provided $\frac{f_1}{f_2}$ *is entire, at least* f_2 *is of regular relative* (p,q,t) *growth with respect to* g_1, g_1 *satisfy the Property (A) and* $q>1$ *and* $\lambda_{g_1}^{(p,q,t)L}(f_i) > \lambda_{g_1}^{(p,q,t)L}(f_j) \mid i=1,2;\ j=1,2;\ i\neq j$.

(B) *Assume the functions* g_1, g_2 *and* f_1 *satisfy the following conditions:*
(i) $\lambda_{g_i}^{(p,q,t)L}(f_1) < \lambda_{g_j}^{(p,q,t)L}(f_1)$ *for* $i=j=1,2$ *and* $i\neq j$, *and* g_i *satisfy the Property (A)*
(ii) $g_1\cdot g_2$ *satisfy the Property (A) and* $p>1$, *then*

$$\begin{aligned}\tau_{g_1\cdot g_2}^{(p,q,t)L}(f_1) &= \tau_{g_i}^{(p,q,t)L}(f_1) \mid i=1,2 \text{ and}\\ \overline{\tau}_{g_1\cdot g_2}^{(p,q,t)L}(f_1) &= \overline{\tau}_{g_i}^{(p,q,t)L}(f_1) \mid i=1,2\ .\end{aligned}$$

Similarly,

$$\begin{aligned}\tau_{\frac{g_1}{g_2}}^{(p,q,t)L}(f_1) &= \tau_{g_i}^{(p,q,t)L}(f_1) \mid i=1,2 \text{ and}\\ \overline{\tau}_{\frac{g_1}{g_2}}^{(p,q,t)L}(f_1) &= \overline{\tau}_{gi}^{(p,q,t)L}(f_1) \mid i=1,2\end{aligned}$$

holds provided $\frac{g_1}{g_2}$ *is entire and satisfy the Property (A),* g_1 *satisfy the Property (A) and* $\lambda_{g_i}^{(p,q,t)L}(f_1) < \lambda_{g_j}^{(p,q,t)L}(f_1) \mid i=1,2;\ j=1,2;\ i\neq j$.

(C) *Assume the functions* f_1, f_2, g_1 *and* g_2 *satisfy the following conditions:*
(i) $g_1\cdot g_2$, g_1 *and* g_2 *are satisfy the Property (A),* $p>1$ *and* $q>1$;
(ii) $\lambda_{g_1}^{(p,q,t)L}(f_i) > \lambda_{g_1}^{(p,q,t)L}(f_j)$ *with at least* f_j *is of regular relative* (p,q,t) *growth with respect to* g_1 *for* $i=1,\ 2,\ j=1,2$ *and* $i\neq j$;
(iii) $\lambda_{g_2}^{(p,q,t)L}(f_i) > \lambda_{g_2}^{(p,q,t)L}(f_j)$ *with at least* f_j *is of regular relative* (p,q,t) *growth with respect to* g_2 *for* $i=1,\ 2,\ j=1,2$ *and* $i\neq j$;
(iv) $\lambda_{g_i}^{(p,q,t)L}(f_1) < \lambda_{g_j}^{(p,q,t)L}(f_1)$ *and* $\lambda_{g_i}^{(p,q,t)L}(f_2) < \lambda_{g_j}^{(p,q,t)L}(f_2)$ *holds simultaneously for* $i=1,2;\ j=1,2$ *and* $i\neq j$;
(v) $\lambda_{g_m}^{(p,q,t)L}(f_l) =$
$\min\left[\max\left\{\lambda_{g_1}^{(p,q,t)L}(f_1), \lambda_{g_1}^{(p,q,t)L}(f_2)\right\}, \max\left\{\lambda_{g_2}^{(p,q,t)L}(f_1), \lambda_{g_2}^{(p,q,t)L}(f_2)\right\}\right] \mid$
$l=m=1,2$; *then*

$$\begin{aligned}\tau_{g_1\cdot g_2}^{(p,q,t)L}(f_1\cdot f_2) = \tau_{g_m}^{(p,q,t)L}(f_l) \mid l=m=1,2 \text{ and}\\ \overline{\tau}_{g_1\cdot g_2}^{(p,q,t)L}(f_1\cdot f_2) = \overline{\tau}_{g_m}^{(p,q,t)L}(f_l) \mid l=m=1,2\ .\end{aligned}$$

Similarly,

$$\tau^{(p,q,t)L}_{\frac{g_1}{g_2}}\left(\frac{f_1}{f_2}\right) = \tau^{(p,q,t)L}_{g_m}(f_l) \mid l = m = 1,2 \text{ and}$$
$$\overline{\tau}^{(p,q,t)L}_{\frac{g_1}{g_2}}\left(\frac{f_1}{f_2}\right) = \overline{\tau}^{(p,q,t)L}_{g_m}(f_l) \mid l = m = 1,2\ .$$

holds provided $\frac{f_1}{f_2}$ *and* $\frac{g_1}{g_2}$ *are entire functions which satisfy the following conditions:*
(i) $\frac{g_1}{g_2}$, g_1 *and* g_2 *satisfy the Property (A),* $p > 1$ *and* $q > 1$;
(ii) At least f_2 *is of regular relative* (p,q,t) *growth with respect to* g_1 *and* $\lambda^{(p,q,t)L}_{g_1}(f_1) \neq \lambda^{(p,q,t)L}_{g_1}(f_2)$;
(iii) At least f_2 *is of regular relative* (p,q,t) *growth with respect to* g_2 *and* $\lambda^{(p,q,t)L}_{g_2}(f_1) \neq \lambda^{(p,q,t)L}_{g_2}(f_2)$;
(iv) $\lambda^{(p,q,t)L}_{g_i}(f_1) < \lambda^{(p,q,t)L}_{g_j}(f_1)$ *and* $\lambda^{(p,q,t)L}_{g_i}(f_2) < \lambda^{(p,q,t)L}_{g_j}(f_2)$ *holds simultaneously for* $i = 1,2$; $j = 1,2$ *and* $i \neq j$;
(v) $\lambda^{(p,q,t)L}_{g_m}(f_l) = \min\left[\max\left\{\lambda^{(p,q,t)L}_{g_1}(f_1), \lambda^{(p,q,t)L}_{g_1}(f_2)\right\}, \max\left\{\lambda^{(p,q,t)L}_{g_2}(f_1), \lambda^{(p,q,t)L}_{g_2}(f_2)\right\}\right] \mid l = m = 1,2.$

Proof. Let us consider that $\lambda^{(p,q,t)L}_{g_1}(f)$, $\lambda^{(p,q,t)L}_{g_1}(f_2)$, $\lambda^{(p,q,t)L}_{g_2}(f_1)$ and $\lambda^{(p,q,t)L}_{g_2}(f_2)$ are all non zero and finite.

Case I. Suppose $\lambda^{(p,q,t)L}_{g_1}(f_1) > \lambda^{(p,q,t)L}_{g_1}(f_2)$ with at least f_2 is of regular relative (p,q,t) growth with respect to g_1 and g_1 satisfy the Property (A). Now for any arbitrary $\varepsilon > 0$, we obtain from (25) and (27) for a sequence values of r tending to infinity that

$$M_{f_1 \cdot f_2}(r) \leq$$
$$M_{g_1}\left[\exp^{[p-1]}\left\{\left(\tau^{(p,q,t)L}_{g_1}(f_1) + \frac{\varepsilon}{2}\right)\left[\log^{[q-1]} r \cdot \exp^{[t+1]} L(r)\right]^{\lambda^{(p,q,t)L}_{g_1}(f_1)}\right\}\right]$$
$$\times M_{g_1}\left[\exp^{[p-1]}\left\{\left(\overline{\tau}^{(p,q,t)L}_{g_1}(f_2) + \frac{\varepsilon}{2}\right)\left[\log^{[q-1]} r \cdot \exp^{[t+1]} L(r)\right]^{\lambda^{(p,q,t)L}_{g_1}(f_2)}\right\}\right]\ .$$

Now in view of $\lambda^{(p,q,t)L}_{g_1}(f_1) > \lambda^{(p,q,t)L}_{g_1}(f_2)$, we get that

$$\lim_{r\to\infty} \frac{\left(\tau^{(p,q,t)L}_{g_1}(f_1) + \frac{\varepsilon}{2}\right)\left[\log^{[q-1]} r \cdot \exp^{[t+1]} L(r)\right]^{\lambda^{(p,q,t)L}_{g_1}(f_1)}}{\left(\overline{\tau}^{(p,q,t)L}_{g_1}(f_2) + \frac{\varepsilon}{2}\right)\left[\log^{[q-1]} r \cdot \exp^{[t+1]} L(r)\right]^{\lambda^{(p,q,t)L}_{g_1}(f_2)}} = \infty\ .$$

Therefore we get for all sufficiently large values of r that

$$M_{g_1}\left[\exp^{[p-1]}\left\{\left(\tau_{g_1}^{(p,q,t)L}(f_1)+\frac{\varepsilon}{2}\right)\left[\log^{[q-1]} r\cdot\exp^{[t+1]} L(r)\right]^{\lambda_{g_1}^{(p,q,t)L}(f_1)}\right\}\right]$$

$$> M_{g_1}\left[\exp^{[p-1]}\left\{\left(\overline{\tau}_{g_1}^{(p,q,t)L}(f_2)+\frac{\varepsilon}{2}\right)\left[\log^{[q-1]} r\cdot\exp^{[t+1]} L(r)\right]^{\lambda_{g_1}^{(p,q,t)L}(f_2)}\right\}\right]$$

holds and therefore from the above arguments it follows for a sequence of values of r tending to infinity that

(43) $M_{f_1\cdot f_2}(r) <$

$$\left[M_{g_1}\left[\exp^{[p-1]}\left\{\left(\tau_{g_1}^{(p,q,t)L}(f_1)+\frac{\varepsilon}{2}\right)\left[\log^{[q-1]} r\cdot\exp^{[t+1]} L(r)\right]^{\lambda_{g_1}^{(p,q,t)L}(f_1)}\right\}\right]\right]^2 .$$

Now using the similar technique as explored in the proof of Case I of Theorem 23 we obtain from (43) that

$$\tau_{g_1}^{(p,q,t)L}(f_1\cdot f_2)\leq\tau_{g_1}^{(p,q,t)L}(f_1)\ . \tag{44}$$

In order to establish the equality of (44), let us restrict ourselves on the functions g_1 and $f_i \mid i=1,2$ such that $q>1$. Now let h, h_1, h_2 and k be any four entire functions such that $h=\frac{h_2}{h_1}$, k satisfy the Property (A) and h_1 is of regular relative (p,q,t) growth with respect to k. Now we know that $T_h(r)=T_{\frac{h_2}{h_1}}(r)\leq T_{h_2}(r)+T_{h_1}(r)$. Therefore in view of Lemma 3 and in the line of the construction of the proof as above it follows that

$$\tau_k^{(p,q,t)L}(h)=\tau_k^{(p,q,t)L}\left(\frac{h_2}{h_1}\right)\leq\tau_k^{(p,q,t)L}(h_2),$$

when $\lambda_k^{(p,q,t)L}(h_1)<\lambda_k^{(p,q,t)L}(h_2)$ with $q>1$ and

$$\tau_k^{(p,q,t)L}(h)=\tau_k^{(p,q,t)L}\left(\frac{h_2}{h_1}\right)\leq\tau_k^{(p,q,t)L}(h_1), \tag{45}$$

when $\lambda_k^{(p,q,t)L}(h_1)>\lambda_k^{(p,q,t)L}(h_2)$ with $q>1$.

Further without loss of any generality, let $f=f_1\cdot f_2$ and $\lambda_{g_1}^{(p,q,t)L}(f_2)$ $<\lambda_{g_1}^{(p,q,t)L}(f_1)=\lambda_{g_1}^{(p,q,t)L}(f)$. Then in view of (44), we obtain that $\tau_{g_1}^{(p,q,t)L}(f)=\tau_{g_1}^{(p,q,t)L}(f_1\cdot f_2)\leq\tau_{g_1}^{(p,q,t)L}(f_1)$. Also $f_1=\frac{f}{f_2}$ and in this case we obtain from the above arguments that $\tau_{g_1}^{(p,q,t)L}(f_1)\leq\tau_{g_1}^{(p,q,t)L}(f)$

$= \tau_{g_1}^{(p,q,t)L}(f_1 \cdot f_2)$. Hence $\tau_{g_1}^{(p,q,t)L}(f) = \tau_{g_1}^{(p,q,t)L}(f_1) \Rightarrow \tau_{g_1}^{(p,q,t)L}(f_1 \cdot f_2) = \tau_{g_1}^{(p,q,t)L}(f_1)$ provided $q > 1$.

Similarly, if we consider $\lambda_{g_1}^{(p,q,t)L}(f_1) < \lambda_{g_1}^{(p,q,t)L}(f_2)$ with at least f_1 is of regular relative (p, q, t) growth with respect to g_1, then one can easily verify that $\tau_{g_1}^{(p,q,t)L}(f_1 \cdot f_2) = \tau_{g_1}^{(p,q,t)L}(f_2)$ provided $q > 1$.

Next we may suppose that $f = \frac{f_1}{f_2}$ with f_1, f_2 and f are all entire functions satisfying the conditions specified in the theorem.

Sub Case $\mathbf{I_A}$. Let $\lambda_{g_1}^{(p,q,t)L}(f_2) < \lambda_{g_1}^{(p,q,t)L}(f_1)$. Therefore in view of Theorem 8, $\lambda_{g_1}^{(p,q,t)L}(f_2) < \lambda_{g_1}^{(p,q,t)L}(f_1) = \lambda_{g_1}^{(p,q,t)L}(f)$. We have $f_1 = f \cdot f_2$. So $\tau_{g_1}^{(p,q,t)L}(f_1) = \tau_{g_1}^{(p,q,t)L}(f) = \tau_{g_1}^{(p,q,t)L}\left(\frac{f_1}{f_2}\right)$ provided $q > 1$.

Sub Case $\mathbf{I_B}$. Let $\lambda_{g_1}^{(p,q,t)L}(f_2) > \lambda_{g_1}^{(p,q,t)L}(f_1)$. Therefore in view of Theorem 8, $\lambda_{g_1}^{(p,q,t)L}(f_1) < \lambda_{g_1}^{(p,q,t)L}(f_2) = \lambda_{g_1}^{(p,q,t)L}(f)$. Now in view of (45), we get that $\tau_{g_1}^{(p,q,t)L}\left(\frac{f_1}{f_2}\right) \leq \tau_{g_1}^{(p,q,t)L}(f_2)$. Further we have $f_2 = \frac{f_1}{f}$ and in this case $\tau_{g_1}^{(p,q,t)L}(f_2) \leq \tau_{g_1}^{(p,q,t)L}\left(\frac{f_1}{f_2}\right)$. So $\tau_{g_1}^{(p,q,t)L}\left(\frac{f_1}{f_2}\right) = \tau_{g_1}^{(p,q,t)L}(f_2)$ provided $q > 1$.

Case II. Let $\lambda_{g_1}^{(p,q,t)L}(f_1) > \lambda_{g_1}^{(p,q,t)L}(f_2)$ with at least f_2 is of regular relative (p, q, t) growth with respect to g_1 and g_1 satisfy the Property (A). Now for any arbitrary $\varepsilon > 0$, we get from (25) for all sufficiently large values of r that

$$M_{f_1 \cdot f_2}(r) \leq$$

$$M_{g_1}\left[\exp^{[p-1]}\left\{\left(\overline{\tau}_{g_1}^{(p,q,t)L}(f_1) + \frac{\varepsilon}{2}\right)\left[\log^{[q-1]} r \cdot \exp^{[t+1]} L(r)\right]^{\lambda_{g_1}^{(p,q,t)L}(f_1)}\right\}\right]$$

$$\times M_{g_1}\left[\exp^{[p-1]}\left\{\left(\overline{\tau}_{g_1}^{(p,q,t)L}(f_2) + \frac{\varepsilon}{2}\right)\left[\log^{[q-1]} r \cdot \exp^{[t+1]} L(r)\right]^{\lambda_{g_1}^{(p,q,t)L}(f_2)}\right\}\right].$$

Now in view of $\lambda_{g_1}^{(p,q,t)L}(f_1) > \lambda_{g_1}^{(p,q,t)L}(f_2)$, we get that

$$\lim_{r\to\infty} \frac{\left(\overline{\tau}_{g_1}^{(p,q,t)L}(f_1) + \frac{\varepsilon}{2}\right)\left[\log^{[q-1]} r \cdot \exp^{[t+1]} L(r)\right]^{\lambda_{g_1}^{(p,q,t)L}(f_1)}}{\left(\overline{\tau}_{g_1}^{(p,q,t)L}(f_2) + \frac{\varepsilon}{2}\right)\left[\log^{[q-1]} r \cdot \exp^{[t+1]} L(r)\right]^{\lambda_{g_1}^{(p,q,t)L}(f_2)}} = \infty .$$

Therefore it follows for all sufficiently large values of r that

$$M_{g_1}\left[\exp^{[p-1]}\left\{\left(\overline{\tau}_{g_1}^{(p,q,t)L}(f_1)+\frac{\varepsilon}{2}\right)\left[\log^{[q-1]} r\cdot\exp^{[t+1]} L(r)\right]^{\lambda_{g_1}^{(p,q,t)L}(f_1)}\right\}\right]$$
$$> M_{g_1}\left[\exp^{[p-1]}\left\{\left(\overline{\tau}_{g_1}^{(p,q,t)L}(f_2)+\frac{\varepsilon}{2}\right)\left[\log^{[q-1]} r\cdot\exp^{[t+1]} L(r)\right]^{\lambda_{g_1}^{(p,q,t)L}(f_2)}\right\}\right]$$

holds and therefore from the above arguments we get for all sufficiently large values of r that

(46) $M_{f_1\cdot f_2}(r) <$

$$\left[M_{g_1}\left[\exp^{[p-1]}\left\{\left(\overline{\tau}_{g_1}^{(p,q,t)L}(f_1)+\frac{\varepsilon}{2}\right)\left[\log^{[q-1]} r\cdot\exp^{[t+1]} L(r)\right]^{\lambda_{g_1}^{(p,q,t)L}(f_1)}\right\}\right]\right]^2 .$$

Now using the similar technique as explored in the proof of Case I of Theorem 24 we obtain from (46) that $\overline{\tau}_{g_1}^{(p,q,t)L}(f_1\cdot f_2)=\overline{\tau}_{g_1}^{(p,q,t)L}(f_1)$ and $\overline{\tau}_{\frac{g_1}{g_2}}^{(p,q,t)L}(f_1)=\overline{\tau}_{gi}^{(p,q,t)L}(f_1)\mid i=1,2$ under the conditions specified in the theorem.

Likewise, if we consider $\lambda_{g_1}^{(p,q,t)L}(f_1)<\lambda_{g_1}^{(p,q,t)L}(f_2)$ with at least f_1 is of regular relative (p,q,t) growth with respect to g_1, then one can easily verify that $\overline{\tau}_{g_1}^{(p,q,t)L}(f_1\cdot f_2)=\overline{\tau}_{g_1}^{(p,q,t)L}(f_2)$ provided $q>1$.

Therefore the first part of theorem follows Case I and Case II.

Case III. Let $\lambda_{g_1}^{(p,q,t)L}(f_1)<\lambda_{g_2}^{(p,q,t)L}(f_1)$, $g_1\cdot g_2$ and g_1 are satisfy the Property (A). Now for all sufficiently large values of r and $\lambda_{g_1}^{(p,q,t)L}(f_1)<\lambda_{g_2}^{(p,q,t)L}(f_1)$, we get that

$$\exp^{[p-1]}\left\{\left(\tau_{g_2}^{(p,q,t)L}(f_1)-\varepsilon\right)\left[\log^{[q-1]} r\cdot\exp^{[t+1]} L(r)\right]^{\lambda_{g_2}^{(p,q,t)L}(f_1)}\right\} >$$
$$\exp^{[p-1]}\left\{\left(\tau_{g_1}^{(p,q,t)L}(f_1)-\varepsilon\right)\left[\log^{[q-1]} r\cdot\exp^{[t+1]} L(r)\right]^{\lambda_{g_1}^{(p,q,t)L}(f_1)}\right\}$$

holds. Therefore

$$M_{g_2}\left[\exp^{[p-1]}\left\{\left(\tau_{g_2}^{(p,q,t)L}(f_1)-\varepsilon\right)\left[\log^{[q-1]} r\cdot\exp^{[t+1]} L(r)\right]^{\lambda_{g_2}^{(p,q,t)L}(f_1)}\right\}\right] >$$
$$M_{g_2}\left[\exp^{[p-1]}\left\{\left(\tau_{g_1}^{(p,q,t)L}(f_1)-\varepsilon\right)\left[\log^{[q-1]} r\cdot\exp^{[t+1]} L(r)\right]^{\lambda_{g_1}^{(p,q,t)L}(f_1)}\right\}\right] .$$

also holds.

Therefore in view of (26) we obtain for all sufficiently large values of r that

(47)

$$M_{g_1\cdot g_2}\left[\exp^{[p-1]}\left\{\left(\tau_{g_1}^{(p,q,t)L}(f_1)-\varepsilon\right)\left[\log^{[q-1]}r\cdot\exp^{[t+1]}L(r)\right]^{\lambda_{g_1}^{(p,q,t)L}(f_1)}\right\}\right] \leq \left[M_{f_1}(r)\right]^2 .$$

Now using the similar technique as explored in the proof of Case III of Theorem 23 we obtain from (47) that

$$\tau_{g_1\cdot g_2}^{(p,q,t)L}(f_1)\leq\tau_{g_1}^{(p,q,t)L}(f_1) . \tag{48}$$

In order to establish the equality of (48), let us restrict ourselves on the functions $g_i \mid i=1,2$ and f_1 such that $p>1$. Now let h, h_1, h_2 and k be any four entire functions such that $h=\frac{h_1}{h_2}$, h and h_1 are satisfy the Property (A). Now we know that $T_h(r)=T_{\frac{h_2}{h_1}}(r)\leq T_{h_2}(r)+T_{h_1}(r)$. Therefore in view of Lemma 3 and in the line of the construction of the proof as above it follows that

$$\tau_h^{(p,q,t)L}(k)=\tau_{\frac{h_1}{h_2}}^{(p,q,t)L}(k)\geq\tau_{h_1}^{(p,q,t)L}(k),$$

when $\lambda_{h_1}^{(p,q,t)L}(k)<\lambda_{h_2}^{(p,q,t)L}(k)$ with $p>1$ and

$$\tau_h^{(p,q,t)L}(k)=\tau_{\frac{h_1}{h_2}}^{(p,q,t)L}(k)\geq\tau_{h_2}^{(p,q,t)L}(k), \tag{49}$$

when $\lambda_{h_1}^{(p,q,t)L}(k)>\lambda_{h_2}^{(p,q,t)L}(k)$ with $p>1$.

Further without loss of any generality, let $g=g_1\cdot g_2$ and $\lambda_g^{(p,q,t)L}(f_1)$ $=\lambda_{g_1}^{(p,q,t)L}(f_1)<\lambda_{g_2}^{(p,q,t)L}(f_1)$. Then in view of (48), we obtain that $\tau_g^{(p,q,t)L}(f_1)=\tau_{g_1\cdot g_2}^{(p,q,t)L}(f_1)\geq\tau_{g_1}^{(p,q,t)L}(f_1)$. Also $g_1=\frac{g}{g_2}$ and in this case we obtain from above arguments that $\tau_{g_1}^{(p,q,t)L}(f_1)\geq\tau_g^{(p,q,t)L}(f_1)=\tau_{g_1\cdot g_2}^{(p,q,t)L}(f_1)$. Hence $\tau_g^{(p,q,t)L}(f_1)=\tau_{g_1}^{(p,q,t)L}(f_1)\Rightarrow\tau_{g_1\cdot g_2}^{(p,q,t)L}(f_1)=\tau_{g_1}^{(p,q,t)L}(f_1)$ provided $p>1$.

If $\lambda_{g_1}^{(p,q,t)L}(f_1)>\lambda_{g_2}^{(p,q,t)L}(f_1)$, then one can easily verify that $\tau_{g_1\cdot g_2}^{(p,q,t)L}(f_1)$ $=\tau_{g_2}^{(p,q,t)L}(f_1)$ provided $p>1$.

Next we may suppose that $g=\frac{g_1}{g_2}$ with g_1, g_2, g are all entire functions satisfying the conditions specified in the theorem.

Sub Case III$_\mathbf{A}$. Let $\lambda_{g_1}^{(p,q,t)L}(f_1) < \lambda_{g_2}^{(p,q,t)L}(f_1)$. Therefore in view of Theorem 10, $\lambda_{g}^{(p,q,t)L}(f_1) = \lambda_{g_1}^{(p,q,t)L}(f_1) < \lambda_{g_2}^{(p,q,t)L}(f_1)$. We have $g_1 = g \cdot g_2$. So $\tau_{g_1}^{(p,q,t)L}(f_1) = \tau_{g}^{(p,q,t)L}(f_1) = \tau_{\frac{g_1}{g_2}}^{(p,q,t)L}(f_1)$ provided $p > 1$.

Sub Case III$_\mathbf{B}$. Let $\lambda_{g_1}^{(p,q,t)L}(f_1) > \lambda_{g_2}^{(p,q,t)L}(f_1)$. Therefore in view of Theorem 10, $\lambda_{g}^{(p,q,t)L}(f_1) = \lambda_{g_2}^{(p,q,t)L}(f_1) < \lambda_{g_1}^{(p,q,t)L}(f_1)$. Now in view of (49), we get that $\tau_{\frac{g_1}{g_2}}^{(p,q,t)L}(f_1) \leq \tau_{g_2}^{(p,q,t)L}(f_1)$. Further we have $g_2 = \frac{g_1}{g}$ and in this case $\tau_{g_2}^{(p,q,t)L}(f_1) \leq \tau_{\frac{g_1}{g_2}}^{(p,q,t)L}(f_1)$. So $\tau_{\frac{g_1}{g_2}}^{(p,q,t)L}(f_1) = \tau_{g_2}^{(p,q,t)L}(f_1)$ provided $p > 1$.

Case IV. Suppose $\lambda_{g_1}^{(p,q,t)L}(f_1) < \lambda_{g_2}^{(p,q,t)L}(f_1)$, $g_1 \cdot g_2$ and g_1 are satisfy the Property (A). Therefore for all sufficiently large values of r we obtain that

$$\exp^{[p-1]}\left\{\left(\tau_{g_2}^{(p,q,t)L}(f_1) - \varepsilon\right)\left[\log^{[q-1]} r_n \cdot \exp^{[t+1]} L(r_n)\right]^{\lambda_{g_2}^{(p,q,t)L}(f_1)}\right\} >$$

$$\exp^{[p-1]}\left\{\left(\overline{\tau}_{g_1}^{(p,q,t)L}(f_1) - \varepsilon\right)\left[\log^{[q-1]} r_n \cdot \exp^{[t+1]} L(r_n)\right]^{\lambda_{g_1}^{(p,q,t)L}(f_1)}\right\}$$

holds. Naturally,

$$M_{g_2}\left[\exp^{[p-1]}\left\{\left(\tau_{g_2}^{(p,q,t)L}(f_1) - \varepsilon\right)\left[\log^{[q-1]} r_n \cdot \exp^{[t+1]} L(r_n)\right]^{\lambda_{g_2}^{(p,q,t)L}(f_1)}\right\}\right] >$$

$$M_{g_2}\left[\exp^{[p-1]}\left\{\left(\overline{\tau}_{g_1}^{(p,q,t)L}(f_1) - \varepsilon\right)\left[\log^{[q-1]} r_n \cdot \exp^{[t+1]} L(r_n)\right]^{\lambda_{g_1}^{(p,q,t)L}(f_1)}\right\}\right].$$

also holds.

Therefore in view of (26) and (27) we obtain for a sequence of values of r tending to infinity that

(50)

$$M_{g_1 \cdot g_2}\left[\exp^{[p-1]}\left\{\left(\overline{\tau}_{g_1}^{(p,q,t)L}(f_1) - \varepsilon\right)\left[\log^{[q-1]} r_n \cdot \exp^{[t+1]} L(r_n)\right]^{\lambda_{g_1}^{(p,q,t)L}(f_1)}\right\}\right]$$

$$\leq \left[M_{f_1}(r)\right]^2 .$$

Now using the similar technique as explored in the proof of Case III of Theorem 24, we obtain from (50) that $\overline{\tau}_{g_1\cdot g_2}^{(p,q,t)L}(f_1) = \overline{\tau}_{g_1}^{(p,q,t)L}(f_1)$ and $\overline{\tau}_{\frac{g_1}{g_2}}^{(p,q,t)L}(f_1) = \overline{\tau}_{gi}^{(p,q,t)L}(f_1) \mid i = 1, 2$.

Similarly if we consider that $\lambda_{g_1}^{(p,q,t)L}(f_1) > \lambda_{g_2}^{(p,q,t)L}(f_1)$, then one can easily verify that $\overline{\tau}_{g_1\cdot g_2}^{(p,q,t)L}(f_1) = \overline{\tau}_{g_2}^{(p,q,t)L}(f_1)$ provided $p > 1$.

Therefore the second part of the theorem follows from Case III and Case IV.

Proof of the third part of the Theorem is omitted as it can be carried out in view of Theorem 14 , Theorem 16 and the above cases. □

Theorem 25. *Let f_1, f_2, g_1 and g_2 be any four entire functions. Also let $p, q \in \mathbb{N}$ and $t \in \mathbb{N} \cup \{-1, 0\}$.*
(A) *The following condition is assumed to be satisfied:*
(i) Either $\sigma_{g_1}^{(p,q,t)L}(f_1) \neq \sigma_{g_1}^{(p,q,t)L}(f_2)$ or $\overline{\sigma}_{g_1}^{(p,q,t)L}(f_1) \neq \overline{\sigma}_{g_1}^{(p,q,t)L}(f_2)$ holds and $q > 1$;
(ii) g_1 satisfies the Property (A), then

$$\rho_{g_1}^{(p,q,t)L}(f_1 \cdot f_2) = \rho_{g_1}^{(p,q,t)L}(f_1) = \rho_{g_1}^{(p,q,t)L}(f_2) \ .$$

(B) *The following conditions are assumed to be satisfied:*
(i) Either $\sigma_{g_1}^{(p,q,t)L}(f_1) \neq \sigma_{g_2}^{(p,q,t)L}(f_1)$ or $\overline{\sigma}_{g_1}^{(p,q,t)L}(f_1) \neq \overline{\sigma}_{g_2}^{(p,q,t)L}(f_1)$ holds and $p > 1$;
(ii) f_1 is of regular relative (p, q, t) growth with respect to at least any one of g_1 or g_2. Also $g_1 \cdot g_2$ satisfy the Property (A). Then we have

$$\rho_{g_1\cdot g_2}^{(p,q,t)L}(f_1) = \rho_{g_1}^{(p,q,t)L}(f_1) = \rho_{g_2}^{(p,q,t)L}(f_1) \ .$$

Proof. Let f_1, f_2, g_1 and g_2 be any four entire functions satisfying the conditions of the theorem.

Case I. Suppose that $\rho_{g_1}^{(p,q,t)L}(f_1) = \rho_{g_1}^{(p,q,t)L}(f_2)$ $(0 < \rho_{g_1}^{(p,q,t)L}(f_1), \rho_{g_1}^{(p,q,t)L}(f_2) < \infty)$ and g_1 satisfy the Property (A). Now in view of Theorem 9, it is easy to see that $\rho_{g_1}^{(p,q,t)L}(f_1 \cdot f_2) \leq \rho_{g_1}^{(p,q,t)L}(f_1) = \rho_{g_1}^{(p,q,t)L}(f_2)$. If possible let

$$\rho_{g_1}^{(p,q,t)L}(f_1 \cdot f_2) < \rho_{g_1}^{(p,q,t)L}(f_1) = \rho_{g_1}^{(p,q,t)L}(f_2) \ . \tag{51}$$

Let $\sigma_{g_1}^{(p,q,t)L}(f_1) \neq \sigma_{g_1}^{(p,q,t)L}(f_2)$. Now in view of the first part of Theorem 23 and (51) we obtain that $\sigma_{g_1}^{(p,q,t)L}(f_1) = \sigma_{g_1}^{(p,q,t)L}\left(\frac{f_1\cdot f_2}{f_2}\right) = \sigma_{g_1}^{(p,q,t)L}(f_2)$ which is a contradiction. Hence $\rho_{g_1}^{(p,q,t)L}(f_1 \cdot f_2) = \rho_{g_1}^{(p,q,t)L}(f_1)$

$= \rho_{g_1}^{(p,q,t)L}(f_2)$. Similarly with the help of the first part of Theorem 23, one can obtain the same conclusion under the hypothesis $\overline{\sigma}_{g_1}^{(p,q,t)L}(f_1) \neq \overline{\sigma}_{g_1}^{(p,q,t)L}(f_2)$. This prove the first part of the theorem.

Case II. Let us consider that $\rho_{g_1}^{(p,q,t)L}(f_1) = \rho_{g_2}^{(p,q,t)L}(f_1)$ $(0 < \rho_{g_1}^{(p,q,t)L}(f_1), \rho_{g_2}^{(p,q,t)L}(f_1) < \infty)$, f_1 is of regular relative (p,q,t) growth with respect to at least any one of g_1 or g_2. Also $g_1 \cdot g_2$ satisfy the Property (A). Therefore in view of Theorem 11, it follows that $\rho_{g_1\cdot g_2}^{(p,q,t)L}(f_1) \geq \rho_{g_1}^{(p,q,t)L}(f_1) = \rho_{g_2}^{(p,q,t)L}(f_1)$ and if possible let

$$\rho_{g_1\cdot g_2}^{(p,q,t)L}(f_1) > \rho_{g_1}^{(p,q,t)L}(f_1) = \rho_{g_2}^{(p,q,t)L}(f_1) \quad . \tag{52}$$

Further suppose that $\sigma_{g_1}^{(p,q,t)L}(f_1) \neq \sigma_{g_2}^{(p,q,t)L}(f_1)$. Therefore in view of the proof of the second part of Theorem 23 and (52), we obtain that $\sigma_{g_1}^{(p,q,t)L}(f_1) = \sigma_{\frac{g_1\cdot g_2}{g_2}}^{(p,q,t)L}(f_1) = \sigma_{g_2}^{(p,q,t)L}(f_1)$ which is a contradiction. Hence $\rho_{g_1\cdot g_2}^{(p,q,t)L}(f_1) = \rho_{g_1}^{(p,q,t)L}(f_1) = \rho_{g_2}^{(p,q,t)L}(f_1)$. Likewise in view of the proof of second part of Theorem 23, one can obtain the same conclusion under the hypothesis $\overline{\sigma}_{g_1}^{(p,q,t)L}(f_1) \neq \overline{\sigma}_{g_2}^{(p,q,t)L}(f_1)$. This proves the second part of the theorem. □

THEOREM 26. *Let f_1, f_2, g_1 and g_2 be any four entire functions. Also let $p, q \in \mathbb{N}$ and $t \in \mathbb{N} \cup \{-1, 0\}$.*
(A) *The following conditions are assumed to be satisfied:*
(i) $(f_1 \cdot f_2)$ is of regular relative (p,q,t) growth with respect to at least any one g_1 or g_2;
(ii) $(g_1 \cdot g_2)$, g_1 and g_2 all satisfy the Property (A);
*(iii) Either $\sigma_{g_1}^{(p,q,t)L}(f_1 \cdot f_2) \neq \sigma_{g_2}^{(p,q,t)L}(f_1 \cdot f_2)$ or $\overline{\sigma}_{g_1}^{(p,q,t)L}(f_1 \cdot f_2)$
$\neq \overline{\sigma}_{g_2}^{(p,q,t)L}(f_1 \cdot f_2)$;*
(iv) Either $\sigma_{g_1}^{(p,q,t)L}(f_1) \neq \sigma_{g_1}^{(p,q,t)L}(f_2)$ or $\overline{\sigma}_{g_1}^{(p,q,t)L}(f_1) \neq \overline{\sigma}_{g_1}^{(p,q,t)L}(f_2)$;
(v) Either $\sigma_{g_2}^{(p,q,t)L}(f_1) \neq \sigma_{g_2}^{(p,q,t)L}(f_2)$ or $\overline{\sigma}_{g_2}^{(p,q,t)L}(f_1) \neq \overline{\sigma}_{g_2}^{(p,q,t)L}(f_2)$;
(vi) $\min\{p,q\} > 1$; then

$$\rho_{g_1\cdot g_2}^{(p,q,t)L}(f_1 \cdot f_2) = \rho_{g_1}^{(p,q,t)L}(f_1) = \rho_{g_1}^{(p,q,t)L}(f_2) = \rho_{g_2}^{(p,q,t)L}(f_1) = \rho_{g_2}^{(p,q,t)L}(f_2) \ .$$

(B) *The following conditions are assumed to be satisfied:*
(i) $(g_1 \cdot g_2)$ satisfy the Property (A);
(ii) f_1 and f_2 are of regular relative (p,q,t) growth with respect to at least any one g_1 or g_2;
(iii) Either $\sigma_{g_1\cdot g_2}^{(p,q,t)L}(f_1) \neq \sigma_{g_1\cdot g_2}^{(p,q,t)L}(f_2)$ or $\overline{\sigma}_{g_1\cdot g_2}^{(p,q,t)L}(f_1) \neq \overline{\sigma}_{g_1\cdot g_2}^{(p,q,t)L}(f_2)$;

(iv) *Either* $\sigma_{g_1}^{(p,q,t)L}(f_1) \neq \sigma_{g_2}^{(p,q,t)L}(f_1)$ *or* $\overline{\sigma}_{g_1}^{(p,q,t)L}(f_1) \neq \overline{\sigma}_{g_2}^{(p,q,t)L}(f_1)$*;*
(v) *Either* $\sigma_{g_1}^{(p,q,t)L}(f_2) \neq \sigma_{g_2}^{(p,q,t)L}(f_2)$ *or* $\overline{\sigma}_{g_1}^{(p,q,t)L}(f_2) \neq \overline{\sigma}_{g_2}^{(p,q,t)L}(f_2)$*;*
(vi) $\min\{p,q\} > 1$*; then*

$$\rho_{g_1 \cdot g_2}^{(p,q,t)L}(f_1 \cdot f_2) = \rho_{g_1}^{(p,q,t)L}(f_1) = \rho_{g_1}^{(p,q,t)L}(f_2) = \rho_{g_2}^{(p,q,t)L}(f_1) = \rho_{g_2}^{(p,q,t)L}(f_2)\ .$$

We omit the proof of Theorem 26 as it is a natural consequence of Theorem 25.

Theorem 27. *Let* f_1*,* f_2*,* g_1 *and* g_2 *be any four entire functions.*
(A) *The following conditions are assumed to be satisfied:*
(i) *At least any one of* f_1 *or* f_2 *are of regular relative* (p,q,t) *growth with respect to* g_1 *where* $p,q \in \mathbb{N}$ *and* $t \in \mathbb{N} \cup \{-1,0\}$*;*
(ii) *Either* $\tau_{g_1}^{(p,q,t)L}(f_1) \neq \tau_{g_1}^{(p,q,t)L}(f_2)$ *or* $\overline{\tau}_{g_1}^{(p,q,t)L}(f_1) \neq \overline{\tau}_{g_1}^{(p,q,t)L}(f_2)$ *holds and* $q > 1$*.*
(iii) g_1 *satisfies the Property (A), then*

$$\lambda_{g_1}^{(p,q,t)L}(f_1 \cdot f_2) = \lambda_{g_1}^{(p,q,t)L}(f_1) = \lambda_{g_1}^{(p,q,t)L}(f_2)\ .$$

(B) *The following conditions are assumed to be satisfied:*
(i) f_1*,* g_1 *and* g_2 *be any three entire functions such that* $\lambda_{g_1}^{(p,q,t)L}(f_1)$ *and* $\lambda_{g_2}^{(p,q,t)L}(f_1)$ *exist where* $p,q \in \mathbb{N}$*,* $t \in \mathbb{N} \cup \{-1,0\}$ *and* $g_1 \cdot g_2$ *satisfies the Property (A);*
(ii) *Either* $\tau_{g_1}^{(p,q,t)L}(f_1) \neq \tau_{g_2}^{(p,q,t)L}(f_1)$ *or* $\overline{\tau}_{g_1}^{(p,q,t)L}(f_1) \neq \overline{\tau}_{g_2}^{(p,q,t)L}(f_1)$ *holds and* $p > 1$*, then*

$$\lambda_{g_1 \cdot g_2}^{(p,q,t)L}(f_1) = \lambda_{g_1}^{(p,q,t)L}(f_1) = \lambda_{g_2}^{(p,q,t)L}(f_1)\ .$$

Proof. Let f_1, f_2, g_1 and g_2 be any four entire functions satisfying the conditions of the theorem.

Case I. Let $\lambda_{g_1}^{(p,q,t)L}(f_1) = \lambda_{g_1}^{(p,q,t)L}(f_2)$ $\left(0 < \lambda_{g_1}^{(p,q,t)L}(f_1), \lambda_{g_1}^{(p,q,t)L}(f_2) < \infty\right)$, g_1 satisfy the Property (A) and at least f_1 or f_2 is of regular relative (p,q,t) growth with respect to g_1. Now in view of Theorem 7 it is easy to see that $\lambda_{g_1}^{(p,q,t)L}(f_1 \cdot f_2) \leq \lambda_{g_1}^{(p,q,t)L}(f_1) = \lambda_{g_1}^{(p,q,t)L}(f_2)$. If possible let

$$\lambda_{g_1}^{(p,q,t)L}(f_1 \cdot f_2) < \lambda_{g_1}^{(p,q,t)L}(f_1) = \lambda_{g_1}^{(p,q,t)L}(f_2)\ . \tag{53}$$

Also let $\tau_{g_1}^{(p,q,t)L}(f_1) \neq \tau_{g_1}^{(p,q,t)L}(f_2)$. Then in view of the proof of first part of Theorem 24 and (53), we obtain that $\tau_{g_1}^{(p,q,t)L}(f_1) = \tau_{g_1}^{(p,q,t)L}\left(\frac{f_1 \cdot f_2}{f_2}\right)$ $= \tau_{g_1}^{(p,q,t)L}(f_2)$ which is a contradiction. Hence $\lambda_{g_1}^{(p,q,t)L}(f_1 \cdot f_2) = \lambda_{g_1}^{(p,q,t)L}(f_1)$

$= \lambda_{g_1}^{(p,q,t)L}(f_2)$. Analogously, in view of the proof of first part of Theorem 24, one can derived the same conclusion under the hypothesis $\overline{\tau}_{g_1}^{(p,q,t)L}(f_1) \neq \overline{\tau}(p,q,t)L(f_2)$. Hence the first part of the theorem is established.

Case II. Let us consider that $\lambda_{g_1}^{(p,q,t)L}(f_1) = \lambda_{g_2}^{(p,q,t)L}(f_1)$ $(0 < \lambda_{g_1}^{(p,q,t)L}(f_1)$, $\lambda_{g_2}^{(p,q,t)L}(f_1) < \infty)$ and $g_1 \cdot g_2$ satisfy the Property (A). Therefore in view of Theorem 10, it follows that $\lambda_{g_1 \cdot g_2}^{(p,q,t)L}(f_1) \geq \lambda_{g_1}^{(p,q,t)L}(f_1) = \lambda_{g_2}^{(p,q,t)L}(f_1)$ and if possible let

$$\lambda_{g_1 \cdot g_2}^{(p,q,t)L}(f_1) > \lambda_{g_1}^{(p,q,t)L}(f_1) = \lambda_{g_2}^{(p,q,t)L}(f_1) \ . \tag{54}$$

Further let $\tau_{g_1}^{(p,q,t)L}(f_1) \neq \tau_{g_2}^{(p,q,t)L}(f_1)$. Then in view of second part of Theorem 24 and (54), we obtain that $\tau_{g_1}^{(p,q,t)L}(f_1) = \tau_{\frac{g_1 \cdot g_2}{g_2}}^{(p,q,t)L}(f_1) = \tau_{g_2}^{(p,q,t)L}(f_1)$ which is a contradiction. Hence $\lambda_{g_1 \cdot g_2}^{(p,q,t)L}(f_1) = \lambda_{g_1}^{(p,q,t)L}(f_1)$ $= \lambda_{g_2}^{(p,q,t)L}(f_1)$. Similarly by second part of Theorem 24, we get the same conclusion when $\overline{\tau}_{g_1}^{(p,q,t)L}(f_1) \neq \overline{\tau}_{g_2}^{(p,q,t)L}(f_1)$ and therefore the second part of the theorem follows. □

THEOREM 28. *Let f_1, f_2, g_1 and g_2 be any four entire functions.*
(A) *The following conditions are assumed to be satisfied:*
(i) $g_1 \cdot g_2$, g_1 and g_2 satisfy the Property (A);
(ii) At least any one of f_1 or f_2 are of regular relative (p,q,t) growth with respect to g_1 and g_2 where $p,q \in \mathbb{N}$ and $t \in \mathbb{N} \cup \{-1,0\}$;
(iii) Either $\tau_{g_1}^{(p,q,t)L}(f_1 \cdot f_2) \neq \tau_{g_2}^{(p,q,t)L}(f_1 \cdot f_2)$ or $\overline{\tau}_{g_1}^{(p,q,t)L}(f_1 \cdot f_2)$ $\neq \overline{\tau}_{g_2}^{(p,q,t)L}(f_1 \cdot f_2)$;
(iv) Either $\tau_{g_1}^{(p,q,t)L}(f_1) \neq \tau_{g_1}^{(p,q,t)L}(f_2)$ or $\overline{\tau}_{g_1}^{(p,q,t)L}(f_1) \neq \overline{\tau}_{g_1}^{(p,q,t)L}(f_2)$;
(v) Either $\tau_{g_2}^{(p,q,t)L}(f_1) \neq \tau_{g_2}^{(p,q,t)L}(f_2)$ or $\overline{\tau}_{g_2}^{(p,q,t)L}(f_1) \neq \overline{\tau}_{g_2}^{(p,q,t)L}(f_2)$; then

$$\lambda_{g_1 \cdot g_2}^{(p,q,t)L}(f_1 \cdot f_2) = \lambda_{g_1}^{(p,q,t)L}(f_1) = \lambda_{g_1}^{(p,q,t)L}(f_2) = \lambda_{g_2}^{(p,q,t)L}(f_1) = \lambda_{g_2}^{(p,q,t)L}(f_2) \ .$$

(B) *The following conditions are assumed to be satisfied:*
(i) $g_1 \cdot g_2$ satisfy the Property (A);
(ii) At least any one of f_1 or f_2 are of regular relative (p,q,t) growth with respect to $g_1 \cdot g_2$ where $p,q \in \mathbb{N}$ and $t \in \mathbb{N} \cup \{-1,0\}$;
(iii) Either $\tau_{g_1 \cdot g_2}^{(p,q,t)L}(f_1) \neq \tau_{g_1 \cdot g_2}^{(p,q,t)L}(f_2)$ or $\overline{\tau}_{g_1 \cdot g_2}^{(p,q,t)L}(f_1) \neq \overline{\tau}_{g_1 \cdot g_2}^{(p,q,t)L}(f_2)$ holds;
(iv) Either $\tau_{g_1}^{(p,q,t)L}(f_1) \neq \tau_{g_2}^{(p,q,t)L}(f_1)$ or $\overline{\tau}_{g_1}^{(p,q,t)L}(f_1) \neq \overline{\tau}_{g_2}^{(p,q,t)L}(f_1)$ holds;
(v) Either $\tau_{g_1}^{(p,q,t)L}(f_2) \neq \tau_{g_2}^{(p,q,t)L}(f_2)$ or $\overline{\tau}_{g_1}^{(p,q,t)L}(f_2) \neq \overline{\tau}_{g_2}^{(p,q,t)L}(f_2)$ holds,

then

$\lambda_{g_1 \cdot g_2}^{(p,q,t)L}(f_1 \cdot f_2) = \lambda_{g_1}^{(p,q,t)L}(f_1) = \lambda_{g_1}^{(p,q,t)L}(f_2) = \lambda_{g_2}^{(p,q,t)L}(f_1) = \lambda_{g_2}^{(p,q,t)L}(f_2)$.

We omit the proof of Theorem 28 as it is a natural consequence of Theorem 27.

Remark 1. If we take $\frac{f_1}{f_2}$ instead of $f_1 \cdot f_2$ and $\frac{g_1}{g_2}$ instead of $g_1 \cdot g_2$ where $\frac{f_1}{f_2}$ and $\frac{g_1}{g_2}$ are entire functions and the other conditions of Theorem 25, Theorem 26, Theorem 27 and Theorem 28 remain the same, then conclusion of Theorem 25, Theorem 26, Theorem 27 and Theorem 28 remains valid.

4. Concluding Remarks

In this paper, we study certain properties of relative $(p,q,t)\,L$-th order, relative $(p,q,t)\,L$-th type, and relative $(p,q,t)\,L$-th weak type of entire functions with respect to another entire function where $p,q \in \mathbb{N}$ and $t \in \mathbb{N} \cup \{-1,0\}$. Moreover, if we rewrite Definition 2 as

$$\begin{matrix} \rho_g^{(p,q,t)L}(f) \\ \lambda_g^{(p,q,t)L}(f) \end{matrix} = \lim_{r\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log^{[p]} M_g^{-1} M_f(r)}{\log^{[q]}\left[r \exp^{[t]} L(r)\right]},$$

and also alter Definition 3 and Definition 4 accordingly where $p,q \in \mathbb{N}$ and $t \in \mathbb{N} \cup \{0\}$, then substituting $\log^{[q]} r + \exp^{[t]} L(r)$ and $\log^{[q-1]} r \cdot \exp^{[t+1]} L(r)$ by $\log^{[q]}\left[r \exp^{[t]} L(r)\right]$ and $\log^{[q-1]}\left[r \exp^{[t]} L(r)\right]$ respectively, all the above results can be derived which gives another direction of growth measurement of entire functions.

Acknowledgment

The author is thankful to the referee for his/her valuable suggestions towards the improvement of the paper.

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