Korean J. Math. **26** (2018), No. 3, pp. 387–403
`https://doi.org/10.11568/kjm.2018.26.3.387`

SHIFTING AND MODULATION FOR THE CONVOLUTION PRODUCT OF FUNCTIONALS IN A GENERALIZED FRESNEL CLASS

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Abstract. Shifting, scaling and modulation proprerties for the convolution product of the Fourier-Feynman transform of functionals in a generalized Fresnel class $\mathcal{F}_{A_1,A_2}$ are given. These properties help us to obtain convolution product of new functionals from the convolution product of old functionals which we know their convolution product.

1. Introduction

Let (H, B, ν) be an abstract Wiener space and let $\{e_j\}$ be a complete orthonormal system in H such that the e_j's are in B^*, the dual of B. For each $h \in H$ and $x \in B$, we define a stochastic inner product $(h, x)^\sim$ as follows:

$$(h,x)^\sim = \begin{cases} \lim\limits_{n\to\infty} \sum\limits_{j=1}^{n} \langle h, e_j\rangle (x, e_j), & \text{if the limit exists} \\ 0, & \text{otherwise,} \end{cases} \tag{1.1}$$

Received March 6, 2018. Revised July 16, 2018. Accepted July 20, 2018.

2010 Mathematics Subject Classification: 28C20, 60J25, 60J65.

Key words and phrases: analytic Feynman integral, Fourier-Feynman transform, convolution product, generalized Fresnel class, time shifting, frequency shifting, modulation.

† This study was supported by the Research Program funded by the Seoul National University of Science and Technology.

where $(\cdot,\cdot)$ denotes the natural dual pairing between B and B^*. It is well known [8,9] that for each $h(\neq 0)$ in H, $(h,\cdot)^\sim$ is a Gaussian random variable on B with mean zero and variance $|h|^2$, that is,

$$\int_B \exp\{i(h,x)^\sim\}\,d\nu(x) = \exp\Big\{-\frac{1}{2}|h|^2\Big\}. \tag{1.2}$$

A subset E of a product abstract Wiener space B^2 is said to be scale-invariant measurable provided $\{(\alpha x_1,\beta x_2) : (x_1,x_2)\in E\}$ is abstract Wiener measurable for every $\alpha > 0$ and $\beta > 0$, and a scale-invariant measurable set N is said to be scale-invariant null provided $(\nu\times\nu)(\{(\alpha x_1,\beta x_2) : (x_1,x_2)\in N\}) = 0$ for every $\alpha > 0$ and $\beta > 0$. A property that holds except on a scale-invariant null set is said to hold scale-invariant almost everywhere (s-a.e.) [7].

Let $\mathbb{C}$ denote the set of complex numbers and let

$$\Omega = \{\vec{\lambda} = (\lambda_1,\lambda_2)\in\mathbb{C}^2 : \operatorname{Re}\lambda_k > 0 \text{ for } k = 1,2\} \tag{1.3}$$

and

$$\tilde{\Omega} = \{\vec{\lambda} = (\lambda_1,\lambda_2)\in\mathbb{C}^2 : (\lambda_1,\lambda_2)\neq(0,0), \operatorname{Re}\lambda_k \geq 0 \text{ for } k = 1,2\}. \tag{1.4}$$

Let F be a complex-valued function on B^2 such that the integral

$$J_F(\lambda_1,\lambda_2) = \int_{B^2} F(\lambda_1^{-1/2}x_1, \lambda_2^{-1/2}x_2)\,d(\nu\times\nu)(x_1,x_2) \tag{1.5}$$

exists as a finite number for all real numbers $\lambda_1 > 0$ and $\lambda_2 > 0$. If there exists a function $J_F^*(\lambda_1,\lambda_2)$ analytic on Ω such that $J_F^*(\lambda_1,\lambda_2) = J_F(\lambda_1,\lambda_2)$ for all $\lambda_1 > 0$ and $\lambda_2 > 0$, then $J_F^*(\lambda_1,\lambda_2)$ is defined to be the analytic Wiener integral of F over B^2 with parameter $\vec{\lambda} = (\lambda_1,\lambda_2)$, and for $\vec{\lambda}\in\Omega$ we write

$$\int_{B^2}^{\mathrm{anw}_{\vec{\lambda}}} F(x_1,x_2)\,d(\nu\times\nu)(x_1,x_2) = J_F^*(\lambda_1,\lambda_2). \tag{1.6}$$

Let q_1 and q_2 be nonzero real numbers and F be a functional on B^2 such that $\int_{B^2}^{\mathrm{anw}_{\vec{\lambda}}} F(x_1,x_2)\,d(\nu\times\nu)(x_1,x_2)$ exists for all $\vec{\lambda}\in\Omega$. If the following limit exists, then we call it the analytic Feynman integral of F over B^2 with parameter $\vec{q} = (q_1,q_2)$ and we write

$$\int_{B^2}^{\mathrm{anf}_{\vec{q}}} F(x_1,x_2)\,d(\nu\times\nu)(x_1,x_2) = \lim_{\vec{\lambda}\to -i\vec{q}} \int_{B^2}^{\mathrm{anw}_{\vec{\lambda}}} F(x_1,x_2)\,d(\nu\times\nu)(x_1,x_2), \tag{1.7}$$

where $\vec{\lambda} = (\lambda_1, \lambda_2)$ approaches $(-iq_1, -iq_2)$ through Ω.

Let $M(H)$ denote the space of complex-valued countably additive Borel measures on H. Under the total variation norm $\|\cdot\|$ and with convolution as multiplication, $M(H)$ is a commutative Banach algebra with identity [2].

Now we will introduce the class of functionals that we work with in this paper. Let A_1 and A_2 be bounded, non-negative self-adjoint operators on H. The generalized Fresnel class $\mathcal{F}_{A_1,A_2}$, which was introduced by Kallianpur and Bromley [8], is the space of all s-equivalence classes of functionals F on B^2 which have the form

$$F(x_1, x_2) = \int_H \exp\Big\{ i \sum_{j=1}^{2} (A_j^{1/2} h, x_j)^{\sim} \Big\} \, d\sigma(h) \tag{1.8}$$

for some complex-valued countably additive Borel measure σ on H.

As is customary, we will identify a functional with its s-equivalence class and think of $\mathcal{F}_{A_1,A_2}$ as a collection of functionals on B^2 rather than as a collection of equivalence classes. Moreover the map $\sigma \mapsto [F]$ defined by (1.8) sets up an algebra isomorphism between $M(H)$ and $\mathcal{F}_{A_1,A_2}$ if the range of $A_1 + A_2$ is dense in H. In this case, $\mathcal{F}_{A_1,A_2}$ becomes a Banach algebra under the norm $\|F\| = \|\sigma\|$ [8].

Remark 1.1. Let $\mathcal{F}(B)$ denote the Fresnel class of functions F on B of the form

$$F(x) = \int_H \exp\{ i (h, x)^{\sim} \} \, d\sigma(h) \tag{1.9}$$

for some $\sigma \in M(H)$. If A_1 is the identity operator on H and $A_2 = O$, the zero operator, then $\mathcal{F}_{A_1,A_2}$ is essentially the Fresnel class $\mathcal{F}(B)$.

Recently in [10], the first author studied shifting, scaling, modulation and variational properties for analytic Fourier-Feynman transform and convolution product of functionals in a Banach algebra $\mathcal{S}$ introduced by Cameron and Storvick [3] on Wiener space. Moreover in [11], some of the results in [10] are extended for functionals in a generalized Fresnel class $\mathcal{F}_{A_1,A_2}$.

In this paper we develop time shifting, frequency shifting and modulation properties for the convolution product of functionals in a generalized Fresnel class. As we commented in Remark 1.1, since $\mathcal{F}_{A_1,A_2}$ is a generalization of the Fresnel class $\mathcal{F}(B)$ which is an abstract Wiener space version of the Banach algebra $\mathcal{S}$, the results in Section 4 of [10] can be obtained as corollaries of our results.

2. Shifting for the convolution

In this section we develop some properties relavant to shifting (translating) and computational rules for the convolution product of functionals in a generalized Fresnel class $\mathcal{F}_{A_1,A_2}$. Let us begin with the definition of convolution product of functionals on abstract Wiener space.

Let $\vec{q} = (q_1, q_2)$, where q_1 and q_2 are nonzero real numbers throughout this paper.

Definition 2.1. Let F and G be functionals on B^2. For $\vec{\lambda} = (\lambda_1, \lambda_2) \in \Omega$ and $(y_1, y_2) \in B^2$, we define the convolution product by

$$(F * G)_{\vec{\lambda}}(y_1, y_2) = \int_{B^2}^{\mathrm{anw}_{\vec{\lambda}}} F\Big(\frac{y_1 + x_1}{\sqrt{2}}, \frac{y_2 + x_2}{\sqrt{2}}\Big) G\Big(\frac{y_1 - x_1}{\sqrt{2}}, \frac{y_2 - x_2}{\sqrt{2}}\Big) d(\nu \times \nu)(x_1, x_2) \tag{2.1}$$

and

$$(F * G)_{\vec{q}}(y_1, y_2) = \int_{B^2}^{\mathrm{anf}_{\vec{q}}} F\Big(\frac{y_1 + x_1}{\sqrt{2}}, \frac{y_2 + x_2}{\sqrt{2}}\Big) G\Big(\frac{y_1 - x_1}{\sqrt{2}}, \frac{y_2 - x_2}{\sqrt{2}}\Big) d(\nu \times \nu)(x_1, x_2) \tag{2.2}$$

if it exists [4–6, 12, 14, 15].

Obviously the convolution product is bilinear in the sense that

$$\begin{aligned} &[(F_1 + F_2) * (G_1 + G_2)]_{\vec{q}}(y_1, y_2) \\ &= (F_1 * G_1)_{\vec{q}}(y_1, y_2) + (F_1 * G_2)_{\vec{q}}(y_1, y_2) \\ &\quad + (F_2 * G_1)_{\vec{q}}(y_1, y_2) + (F_2 * G_2)_{\vec{q}}(y_1, y_2) \end{aligned} \tag{2.3}$$

for all functionals F_j, G_j on B^2 for $j = 1, 2$, whenever each convolution products exist.

Huffman, Park and Skoug [6] established the existence of convolution product on $C_0[0, T]$ for functionals in $\mathcal{S}$. And Chang, Kim and Yoo [4] extended the results for functionals in $\mathcal{F}_{A_1,A_2}$.

THEOREM 2.2 (Theorem 3.3 of [4]). *Let F and G be elements of $\mathcal{F}_{A_1,A_2}$ with corresponding finite Borel measures σ and ρ in $M(H)$, respectively. Then their convolution product $(F*G)_{\vec{q}}$ exists and is given by the formula*

$$\begin{aligned}(F*G)_{\vec{q}}(y_1,y_2) = \int_{H^2} \exp\Big\{ i\sum_{j=1}^{2}\Big[\frac{1}{\sqrt{2}}(A_j^{1/2}(h+k),y_j)^\sim \\ - \frac{1}{4q_j}|A_j^{1/2}(h-k)|^2\Big]\Big\}\, d\sigma(h)\, d\rho(k)\end{aligned} \tag{2.4}$$

for s-a.e. $(y_1,y_2)\in B^2$.

In the classical Fourier analysis, the Fourier transform $\mathcal{F}$ turns a function f into a new function $\mathcal{F}[f]$. Because the transform is used in signal analysis, we usually use t for time as the variable of the function f, and ω as the variable of the transformed function $\mathcal{F}[f]$, that is,

$$\mathcal{F}[f](\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}\, dt.$$

Engineers refer to the variable ω in the transformed function as the frequency of the signal f [13].

We will use the same convention in this paper, that is, for a convolution product $(F*G)_{\vec{q}}(y_1,y_2)$ of $F(x_1,x_2)$ and $G(x_1,x_2)$, we call the variable (x_1,x_2) as a time and the variable (y_1,y_2) as a freqeuency.

Our first result in this section is a relationship between time shifting and frequency shifting of convolution product on Wiener space.

THEOREM 2.3. *Let F and G be functionals on B^2 and let $(w_1,w_2)\in B^2$. Then we have*

$$\begin{aligned}&[F(\cdot - w_1, \cdot - w_2) * G(\cdot - w_1, \cdot - w_2)]_{\vec{q}}(y_1,y_2) \\ &= (F*G)_{\vec{q}}(y_1-\sqrt{2}w_1, y_2-\sqrt{2}w_2)\end{aligned} \tag{2.5}$$

if each sides exist.

Proof. For all $\lambda_1, \lambda_2 > 0$ and for s-a.e. $(y_1, y_2) \in B^2$, we have

$$\begin{aligned}
&[F(\cdot - w_1, \cdot - w_2) * G(\cdot - w_1, \cdot - w_2)]_{\vec{\lambda}}(y_1, y_2) \\
&= \int_{B^2} F\Big(\frac{y_1 + \lambda_1^{-1/2} x_1}{\sqrt{2}} - w_1, \frac{y_2 + \lambda_2^{-1/2} x_2}{\sqrt{2}} - w_2\Big) \\
&\quad G\Big(\frac{y_1 - \lambda_1^{-1/2} x_1}{\sqrt{2}} - w_1, \frac{y_2 - \lambda_2^{-1/2} x_2}{\sqrt{2}} - w_2\Big)\, d(\nu \times \nu)(x_1, x_2) \\
&= (F * G)_{\vec{\lambda}}(y_1 - \sqrt{2} w_1, y_2 - \sqrt{2} w_2)
\end{aligned}$$

if the abstract Wiener integral exists. Extending analytically each sides and taking limits as $\vec{\lambda} \to -i\vec{q}$, we have the result. □

The following theorem is reminiscent of the time shifting theorem for the convolution product of the classical Fourier transform. Hence we call the following theorem as time shifting formula for the convolution product of the Fourier-Feynman transform on a product abstract Wiener space. It says that if we shift back (w_1, w_2) for F and shift front (w_1, w_2) for G, then the convolution product of this shifted functions is equal to the convolution product of

$$F(x_1, x_2) \exp\Big\{ i \sum_{j=1}^{2} q_j (w_j, x_j)^{\sim} \Big\}$$

and

$$G(x_1, x_2) \exp\Big\{ -i \sum_{j=1}^{2} q_j (w_j, x_j)^{\sim} \Big\}$$

multiplied by an exponential factor.

Theorem 2.4 (time shifting). *Let F and G be given as in Theorem 2.2 and let $(w_1, w_2) \in H^2$. Then we have*

$$\begin{aligned}
&[F(\cdot - w_1, \cdot - w_2) * G(\cdot + w_1, \cdot + w_2)]_{\vec{q}}(y_1, y_2) \\
&= \exp\Big\{ i \sum_{j=1}^{2} q_j |w_j|^2 \Big\} \Big[F(\cdot, \cdot) \exp\Big\{ i \sum_{j=1}^{2} q_j (w_j, \cdot)^{\sim} \Big\} \qquad (2.6) \\
&\quad * G(\cdot, \cdot) \exp\Big\{ -i \sum_{j=1}^{2} q_j (w_j, \cdot)^{\sim} \Big\} \Big]_{\vec{q}}(y_1, y_2)
\end{aligned}$$

for s-a.e. $(y_1, y_2) \in B^2$.

Proof. Let $F_0(x_1,x_2)=F(x_1-w_1,x_2-w_2)$ and $G_0(x_1,x_2)=G(x_1+w_1,x_2+w_2)$. By (1.8) we have

$$F_0(x_1,x_2)=\int_H \exp\Big\{i\sum_{j=1}^{2}(A_j^{1/2}h,x_j)^{\sim}\Big\}\,d\sigma_0(h)$$

and

$$G_0(x_1,x_2)=\int_H \exp\Big\{i\sum_{j=1}^{2}(A_j^{1/2}k,x_j)^{\sim}\Big\}\,d\rho_0(h)$$

belong to $\mathcal{F}_{A_1,A_2}$, where $\sigma_0(E)=\int_E\exp\{-i\sum_{j=1}^{2}(A_j^{1/2}h,w_j)^{\sim}\}\,d\sigma(h)$ and $\rho_0(E)=\int_E\exp\{i\sum_{j=1}^{2}(A_j^{1/2}k,w_j)^{\sim}\}\,d\rho(k)$ for a Borel subset E of H. Then by Theorem 2.2, the left hand side of (2.6) is given by

$$\begin{aligned}(F_0*G_0)_{\vec{q}}(y_1,y_2)=\int_{H^2}\exp\Big\{i\sum_{j=1}^{2}\Big[&\frac{1}{\sqrt{2}}(A_j^{1/2}(h+k),y_j)^{\sim}\\&-\frac{1}{4q_j}|A_j^{1/2}(h-k)|^2\Big]\Big\}\,d\sigma_0(h)\,d\rho_0(k).\end{aligned}$$

Now rewrite the above expression using the definitions of the measures σ_0 and ρ_0. Further we use the fact that the stochastic inner product $(A_j^{1/2}(h-k),w_j)^{\sim}$ is equal to the inner product $\langle A_j^{1/2}(h-k),w_j\rangle$, since $A_j^{1/2}(h-k)$ and w_j belong to H for $j=1,2$. Hence we obtain

$$\begin{aligned}(F_0*G_0)_{\vec{q}}(y_1,y_2)=\int_{H^2}\exp\Big\{i\sum_{j=1}^{2}\Big[&\frac{1}{\sqrt{2}}(A_j^{1/2}(h+k),y_j)^{\sim}\\&-\frac{1}{4q_j}|A_j^{1/2}(h-k)|^2-\langle A_j^{1/2}(h-k),w_j\rangle\Big]\Big\}\\&d\sigma(h)\,d\rho(k)\end{aligned}$$

for s-a.e. $(y_1,y_2)\in B^2$. To consider the right hand side of (2.6), let

$$\begin{aligned}F_1(x_1,x_2)&=F(x_1,x_2)\exp\Big\{i\sum_{j=1}^{2}q_j(w_j,x_j)^{\sim}\Big\}\\&=\int_H\exp\Big\{i\sum_{j=1}^{2}(A_j^{1/2}h+q_jw_j,x_j)^{\sim}\Big\}\,d\sigma(h)\end{aligned}$$

and

$$\begin{aligned} G_1(x_1,x_2) &= G(x_1,x_2)\exp\Big\{-i\sum_{j=1}^{2} q_j(w_j,x_j)^{\sim}\Big\} \\ &= \int_H \exp\Big\{i\sum_{j=1}^{2}(A_j^{1/2}k - q_jw_j, x_j)^{\sim}\Big\}\, d\rho(k). \end{aligned}$$

For all $\lambda_1, \lambda_2 > 0$ and s-a.e. $(y_1, y_2) \in B^2$, by the Definition 2.1 of the convolution product, we have

$$\begin{aligned} (F_1 * G_1)_{\vec{\lambda}}(y_1,y_2) = \int_{B^2}\int_{H^2} & \exp\Big\{i\sum_{j=1}^{2}\Big[\frac{1}{\sqrt{2}}(A_j^{1/2}(h+k), y_j)^{\sim} \\ & + \frac{1}{\sqrt{2\lambda_j}}(A_j^{1/2}(h-k) + 2q_jw_j, x_j)^{\sim}\Big]\Big\} \\ & d\sigma(h)\, d\rho(k)\, d(\nu\times\nu)(x_1,x_2). \end{aligned}$$

Using the Fubini theorem and (1.2), we obtain

$$\begin{aligned} (F_1 * G_1)_{\vec{\lambda}}(y_1,y_2) = \int_{H^2} & \exp\Big\{\sum_{j=1}^{2}\Big[\frac{i}{\sqrt{2}}(A_j^{1/2}(h+k), y_j)^{\sim} \\ & - \frac{1}{4\lambda_j}|A_j^{1/2}(h-k) + 2q_jw_j|^2\Big]\Big\}\, d\sigma(h)\, d\rho(k). \end{aligned}$$

Extending analytically and using the dominated convergence theorem we have that

$$\begin{aligned} (F_1 * G_1)_{\vec{q}}(y_1,y_2) = \int_{H^2} & \exp\Big\{i\sum_{j=1}^{2}\Big[\frac{1}{\sqrt{2}}(A_j^{1/2}(h+k), y_j)^{\sim} \\ & - \frac{1}{4q_j}|A_j^{1/2}(h-k) + 2q_jw_j|^2\Big]\Big\}\, d\sigma(h)\, d\rho(k) \end{aligned}$$

for s-a.e. $(y_1, y_2) \in B^2$. Since $|A_j^{1/2}(h-k) + 2q_j w_j|^2 = |A_j^{1/2}(h-k)|^2 + 4q_j^2|w_j|^2 + 4q_j\langle A_j^{1/2}(h-k), w_j\rangle$, we have

$$\begin{aligned}
&(F_1 * G_1)_{\vec{q}}(y_1, y_2)\\
&= \exp\Big\{-i\sum_{j=1}^{2} q_j|w_j|^2\Big\}\int_{H^2}\exp\Big\{i\sum_{j=1}^{2}\Big[\frac{1}{\sqrt{2}}(A_j^{1/2}(h+k), y_j)^{\sim}\\
&\quad - \frac{1}{4q_j}|A_j^{1/2}(h-k)|^2 - \langle A_j^{1/2}(h-k), w_j\rangle\Big]\Big\}\,d\sigma(h)\,d\rho(k)
\end{aligned}$$

for s-a.e. $(y_1, y_2) \in B^2$. Finally we have

$$(F_0 * G_0)_{\vec{q}}(y_1, y_2) = \exp\Big\{i\sum_{j=1}^{2} q_j|w_j|^2\Big\}(F_1 * G_1)_{\vec{q}}(y_1, y_2)$$

for s-a.e. $(y_1, y_2) \in B^2$ and this completes the proof. □

Modifying the second part of the proof of Theorem 2.4, we have the following result. In this theorem we shift back (w_1, w_2) for F and G.

THEOREM 2.5. *Let F and G be given as in Theorem 2.2 and let $(w_1, w_2) \in H^2$. Then we have*

(2.7)

$$\begin{aligned}
&\Big[F(\cdot,\cdot)\exp\Big\{i\sum_{j=1}^{2} q_j(w_j,\cdot)^{\sim}\Big\} * G(\cdot,\cdot)\exp\Big\{i\sum_{j=1}^{2} q_j(w_j,\cdot)^{\sim}\Big\}\Big]_{\vec{q}}(y_1, y_2)\\
&= \exp\Big\{\sqrt{2}i\sum_{j=1}^{2} q_j(w_j, y_j)^{\sim}\Big\}(F * G)_{\vec{q}}(y_1, y_2)
\end{aligned}$$

for s-a.e. $(y_1, y_2) \in B^2$.

Proof. Let

$$\begin{aligned}
F_1(x_1, x_2) &= F(x_1, x_2)\exp\Big\{i\sum_{j=1}^{2} q_j(w_j, x_j)^{\sim}\Big\}\\
&= \int_H \exp\Big\{i\sum_{j=1}^{2}(A_j^{1/2}h + q_j w_j, x_j)^{\sim}\Big\}\,d\sigma(h)
\end{aligned}$$

and

$$
\begin{aligned}
G_1(x_1,x_2) &= G(x_1,x_2)\exp\Big\{i\sum_{j=1}^{2} q_j(w_j,x_j)^{\sim}\Big\}\\
&= \int_H \exp\Big\{i\sum_{j=1}^{2}(A_j^{1/2}k+q_jw_j,x_j)^{\sim}\Big\}\,d\rho(k).
\end{aligned}
$$

For all $\lambda_1,\lambda_2>0$ and s-a.e. $(y_1,y_2)\in B^2$, by the Definition 2.1 of the convolution product, we have

$$
\begin{aligned}
(F_1*G_1)_{\vec{\lambda}}(y_1,y_2) = \int_{B^2}\int_{H^2}\exp\Big\{i\sum_{j=1}^{2}\Big[\frac{1}{\sqrt{2}}(A_j^{1/2}(h+k)+2q_jw_j,y_j)^{\sim}\\
+\frac{1}{\sqrt{2\lambda_j}}(A_j^{1/2}(h-k),x_j)^{\sim}\Big]\Big\}\\
d\sigma(h)\,d\rho(k)\,d(\nu\times\nu)(x_1,x_2).
\end{aligned}
$$

Using the Fubini theorem and (1.2), we obtain

$$
\begin{aligned}
(F_1*G_1)_{\vec{\lambda}}(y_1,y_2) = \int_{H^2}\exp\Big\{\sum_{j=1}^{2}\Big[\frac{i}{\sqrt{2}}(A_j^{1/2}(h+k)+2q_jw_j,y_j)^{\sim}\\
-\frac{1}{4\lambda_j}|A_j^{1/2}(h-k)|^2\Big]\Big\}\,d\sigma(h)\,d\rho(k).
\end{aligned}
$$

Extending analytically and using the dominated convergence theorem we have that

$$
\begin{aligned}
(F_1*G_1)_{\vec{q}}(y_1,y_2) = \exp\Big\{\sqrt{2}i\sum_{j=1}^{2} q_j(w_j,y_j)^{\sim}\Big\}\\
\int_{H^2}\exp\Big\{i\sum_{j=1}^{2}\Big[\frac{1}{\sqrt{2}}(A_j^{1/2}(h+k),y_j)^{\sim}\\
-\frac{1}{4q_j}|A_j^{1/2}(h-k)|^2\Big]\Big\}\,d\sigma(h)\,d\rho(k)
\end{aligned}
$$

for s-a.e. $(y_1,y_2)\in B^2$. Finally by Theorem 2.2 we have the result. □

3. Scaling and modulation for the convolution product

In this section, we study scaling and modulation properties for the convolution product.

The following theorem is called a scaling theorem because we want the convolution product not of $F(x_1, x_2)$ and $G(x_1, x_2)$, but of $F(a_1x_1, a_2x_2)$ and $G(a_1x_1, a_2x_2)$, in which a_1 and a_2 can be thought as scaling factors.

Theorem 3.1 (scaling). *Let F and G be given as in Theorem 2.2 and let a_1, a_2 be nonzero real numbers. Then we have*

$$[F(a_1\cdot, a_2\cdot) * G(a_1\cdot, a_2\cdot)]_{\vec{q}}(y_1, y_2) = (F * G)_{(q_1/a_1^2, q_2/a_2^2)}(a_1y_1, a_2y_2) \tag{3.1}$$

for s-a.e. $(y_1, y_2) \in B^2$.

Proof. For all $\lambda_1 > 0$ and $\lambda_2 > 0$, using (1.8), the Fubini theorem and (1.2), we have

$$\begin{aligned}&[F(a_1\cdot, a_2\cdot) * G(a_1\cdot, a_2\cdot)]_{\vec{\lambda}}(y_1, y_2)\\ &= \int_{H^2} \exp\Big\{\sum_{j=1}^{2}\Big[i\frac{a_j}{\sqrt{2}}(A_j^{1/2}(h+k), y_j)^{\sim} - \frac{a_j^2}{4\lambda_j}|A_j^{1/2}(h-k))|^2\Big]\Big\}\\ &d\sigma(h)\, d\rho(k)\end{aligned}$$

for s-a.e. $(y_1, y_2) \in B^2$. Extending analytically and using the dominated convergence theorem, we have

$$\begin{aligned}&[F(a_1\cdot, a_2\cdot) * G(a_1\cdot, a_2\cdot)]_{\vec{q}}(y_1, y_2)\\ &= \int_{H^2} \exp\Big\{i\sum_{j=1}^{2}\Big[\frac{a_j}{\sqrt{2}}(A_j^{1/2}(h+k), y_j)^{\sim} - \frac{a_j^2}{4q_j}|A_j^{1/2}(h-k))|^2\Big]\Big\}\\ &d\sigma(h)\, d\rho(k)\end{aligned}$$

for s-a.e. $(y_1, y_2) \in B^2$. Finally by Theorem 2.2, we see that the expression on the right hand side is equal to the right hand side of (3.1) and this completes the proof. □

Next corollary follows immediately from the scaling theorem above by putting $a_1 = a_2 = -1$. This result is called time reversal because we replace (x_1, x_2) by $(-x_1, -x_2)$ in $F(x_1, x_2)$ and $G(x_1, x_2)$ to get $F(-x_1, -x_2)$ and $G(-x_1, -x_2)$, respectively. The convolution product of these new functionals is obtained by simply replacing (y_1, y_2) by $(-y_1, -y_2)$ in the convolution product of $F(x_1, x_2)$ and $G(x_1, x_2)$.

COROLLARY 3.2 (time reversal). *Let F and G be given as in Theorem 2.2. Then we have*

$$[F(-\cdot,-\cdot)*G(-\cdot,-\cdot)]_{\vec{q}}(y_1,y_2)=(F*G)_{\vec{q}}(-y_1,-y_2) \tag{3.2}$$

for s-a.e. $(y_1,y_2)\in B^2$.

Our next theorem is useful to obtain the convolution product of new functionals from the convolution product of old functionals which we know their convolution product.

THEOREM 3.3 (modulation). *Let F and G be given as in Theorem 2.2 and let $(w_1,w_2)\in H^2$. Then*

$$\begin{aligned}&\Big[F(\cdot,\cdot)\cos\Big(\sum_{j=1}^{2}q_j(w_j,\cdot)^{\sim}\Big)*G(\cdot,\cdot)\cos\Big(\sum_{j=1}^{2}q_j(w_j,\cdot)^{\sim}\Big)\Big]_{\vec{q}}(y_1,y_2)\\&=\frac{1}{4}[Q_{1,0,0}(\vec{q};w_1,w_2;y_1,y_2)+Q_{0,1,1}(\vec{q};w_1,w_2;y_1,y_2)\\&\quad+Q_{0,1,-1}(\vec{q};w_1,w_2;y_1,y_2)+Q_{-1,0,0}(\vec{q};w_1,w_2;y_1,y_2)]\end{aligned} \tag{3.3}$$

and

$$\begin{aligned}&\Big[F(\cdot,\cdot)\cos\Big(\sum_{j=1}^{2}q_j(w_j,\cdot)^{\sim}\Big)*G(\cdot,\cdot)\sin\Big(\sum_{j=1}^{2}q_j(w_j,\cdot)^{\sim}\Big)\Big]_{\vec{q}}(y_1,y_2)\\&=\frac{1}{4i}[Q_{1,0,0}(\vec{q};w_1,w_2;y_1,y_2)-Q_{0,1,1}(\vec{q};w_1,w_2;y_1,y_2)\\&\quad+Q_{0,1,-1}(\vec{q};w_1,w_2;y_1,y_2)-Q_{-1,0,0}(\vec{q};w_1,w_2;y_1,y_2)]\end{aligned} \tag{3.4}$$

and

$$\begin{aligned}&\Big[F(\cdot,\cdot)\sin\Big(\sum_{j=1}^{2}q_j(w_j,\cdot)^{\sim}\Big)*G(\cdot,\cdot)\cos\Big(\sum_{j=1}^{2}q_j(w_j,\cdot)^{\sim}\Big)\Big]_{\vec{q}}(y_1,y_2)\\&=\frac{1}{4i}[Q_{1,0,0}(\vec{q};w_1,w_2;y_1,y_2)+Q_{0,1,1}(\vec{q};w_1,w_2;y_1,y_2)\\&\quad-Q_{0,1,-1}(\vec{q};w_1,w_2;y_1,y_2)-Q_{-1,0,0}(\vec{q};w_1,w_2;y_1,y_2)]\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}
&\Big[F(\cdot,\cdot)\sin\Big(\sum_{j=1}^{2}q_j(w_j,\cdot)^{\sim}\Big)*G(\cdot,\cdot)\sin\Big(\sum_{j=1}^{2}q_j(w_j,\cdot)^{\sim}\Big)\Big]_{\vec{q}}(y_1,y_2)\\
&=\frac{1}{4}[Q_{1,0,0}(\vec{q};w_1,w_2;y_1,y_2)-Q_{0,1,1}(\vec{q};w_1,w_2;y_1,y_2)\\
&\quad-Q_{0,1,-1}(\vec{q};w_1,w_2;y_1,y_2)+Q_{-1,0,0}(\vec{q};w_1,w_2;y_1,y_2)],
\end{aligned}\tag{3.6}$$

where

$$\begin{aligned}
&Q_{\alpha,\beta,\gamma}(\vec{q};w_1,w_2;y_1,y_2)\\
&=\exp\Big\{i\sum_{j=1}^{2}\Big[\sqrt{2}q_j(\alpha w_j,y_j)^{\sim}-q_j|\beta w_j|^2\Big]\Big\}\\
&\quad[F(\cdot-\gamma w_1,\cdot-\gamma w_2)*G(\cdot+\gamma w_1,\cdot+\gamma w_2)]_{\vec{q}}(y_1,y_2)
\end{aligned}\tag{3.7}$$

for s*-a.e.* $(y_1,y_2)\in B^2$.

Proof. Since

$$\begin{aligned}
&\cos\Big(\sum_{j=1}^{2}q_j(w_j,\cdot)^{\sim}\Big)\\
&=\frac{1}{2}\Big(\exp\Big\{i\sum_{j=1}^{2}q_j(w_j,\cdot)^{\sim}\Big\}+\exp\Big\{-i\sum_{j=1}^{2}q_j(w_j,\cdot)^{\sim}\Big\}\Big),
\end{aligned}$$

we use the bilinearity (2.3) of convolution product to get

$$\begin{aligned}
&\Big[F(\cdot,\cdot)\cos\Big(\sum_{j=1}^{2}q_j(w_j,\cdot)^{\sim}\Big)*G(\cdot,\cdot)\cos\Big(\sum_{j=1}^{2}q_j(w_j,\cdot)^{\sim}\Big)\Big]_{\vec{q}}(y_1,y_2)\\
&=\frac{1}{4}([F_1(\cdot,\cdot)*G_1(\cdot,\cdot)]_{\vec{q}}(y_1,y_2)+[F_1(\cdot,\cdot)*G_{-1}(\cdot,\cdot)]_{\vec{q}}(y_1,y_2)\\
&\quad+[F_{-1}(\cdot,\cdot)*G_1(\cdot,\cdot)]_{\vec{q}}(y_1,y_2)+[F_{-1}(\cdot,\cdot)*G_{-1}(\cdot,\cdot)]_{\vec{q}}(y_1,y_2)),
\end{aligned}$$

where

$$F_\alpha(\cdot,\cdot)=F(\cdot,\cdot)\exp\Big\{\alpha i\sum_{j=1}^{2}q_j(w_j,\cdot)^{\sim}\Big\}$$

and

$$G_\alpha(\cdot,\cdot)=G(\cdot,\cdot)\exp\Big\{\alpha i\sum_{j=1}^{2}q_j(w_j,\cdot)^{\sim}\Big\}$$

for $\alpha = 1, -1$. By (2.6) and (2.7) we obtain (3.3). Using the identity

$$\begin{aligned}&\sin\Big(\sum_{j=1}^{2} q_j(w_j,\cdot)^{\sim}\Big)\\ &= \frac{1}{2i}\Big(\exp\Big\{i\sum_{j=1}^{2} q_j(w_j,\cdot)^{\sim}\Big\} - \exp\Big\{-i\sum_{j=1}^{2} q_j(w_j,\cdot)^{\sim}\Big\}\Big),\end{aligned}$$

the other conclusions are proved similarly. □

Since the Dirac measure concentrated at $h = 0$ in H is a complex Borel measure, the constant function $F \equiv 1$ belongs to $\mathcal{F}_{A_1,A_2}$. Hence we have the following corollary.

Corollary 3.4. *Let $(w_1, w_2) \in H^2$. Then we have*

$$\begin{aligned}&\Big[\cos\Big(\sum_{j=1}^{2} q_j(w_j,\cdot)^{\sim}\Big) * \cos\Big(\sum_{j=1}^{2} q_j(w_j,\cdot)^{\sim}\Big)\Big]_{\vec{q}}(y_1,y_2)\\ &= \frac{1}{2}\Big[\cos\Big(\sqrt{2}\sum_{j=1}^{2} q_j(w_j,y_j)^{\sim}\Big) + \exp\Big\{-i\sum_{j=1}^{2} q_j|w_j|^2\Big\}\Big]\end{aligned} \tag{3.8}$$

and

$$\begin{aligned}&\Big[\cos\Big(\sum_{j=1}^{2} q_j(w_j,\cdot)^{\sim}\Big) * \sin\Big(\sum_{j=1}^{2} q_j(w_j,\cdot)^{\sim}\Big)\Big]_{\vec{q}}(y_1,y_2)\\ &= \Big[\sin\Big(\sum_{j=1}^{2} q_j(w_j,\cdot)^{\sim}\Big) * \cos\Big(\sum_{j=1}^{2} q_j(w_j,\cdot)^{\sim}\Big)\Big]_{\vec{q}}(y_1,y_2)\\ &= \frac{1}{2}\sin\Big(\sqrt{2}\sum_{j=1}^{2} q_j(w_j,y_j)^{\sim}\Big)\end{aligned} \tag{3.9}$$

and

$$\begin{aligned}&\Big[\sin\Big(\sum_{j=1}^{2} q_j(w_j,\cdot)^{\sim}\Big) * \sin\Big(\sum_{j=1}^{2} q_j(w_j,\cdot)^{\sim}\Big)\Big]_{\vec{q}}(y_1,y_2)\\ &= -\frac{1}{2}\Big[\cos\Big(\sqrt{2}\sum_{j=1}^{2} q_j(w_j,y)^{\sim}\Big) - \exp\Big\{-i\sum_{j=1}^{2} q_j|w_j|^2\Big\}\Big]\end{aligned} \tag{3.10}$$

for s-a.e. $(y_1, y_2) \in B^2$.

Proof. Since

$$[F(\cdot-\gamma w_1,\cdot-\gamma w_2)*G(\cdot+\gamma w_1,\cdot+\gamma w_2)]_{\vec{q}}(y_1,y_2)\equiv 1$$

for $F\equiv G\equiv 1$, by the modulation property Theorem 3.3 and Euler's formula, the results follows immediately. □

The generalized Fresnel class $\mathcal{F}_{A_1,A_2}$ becomes the Fresnel class $\mathcal{F}(B)$ if we take A_1 to be the identity operators on H and $A_2=O$, the zero operator. (See Remark 1.1.) Hence we have the following time shifting and modulation properties for the convolution product of functionals in $\mathcal{F}(B)$ as corollaries of our Theorems 2.4 and 3.3, respectively. Corollaries 3.5 and 3.6 below are the abstract Wiener space version of Theorems 20 and 23 in [10].

Corollary 3.5 (time shifting). *Let F and G be given as in* (1.9) *with corresponding finite Borel measures σ and ρ in $M(H)$, respectively. Then, for a nonzero real number q, we have*

$$\begin{aligned}&[F(\cdot-w)*G(\cdot+w)]_q(y)\\&=\exp\{iq|w|^2\}[F(\cdot)\exp\{iq(w,\cdot)^{\sim}\}*G(\cdot)\exp\{-iq(w,\cdot)^{\sim}\}]_q(y)\end{aligned}\tag{3.11}$$

for s-a.e. $y\in B$.

Corollary 3.6 (modulation). *Let F and G be given as in Corollary* 3.5 *and let $w\in H$. Then, for a nonzero real number q, we have*

$$\begin{aligned}&[F(\cdot)\cos(q(w,\cdot)^{\sim})*G(\cdot)\cos(q(w,\cdot)^{\sim})]_q(y)\\&=\frac{1}{4}[R_{1,0,0}(q,w,y)+R_{0,1,1}(q,w,y)+R_{0,1,-1}(q,w,y)+R_{-1,0,0}(q,w,y)]\end{aligned}\tag{3.12}$$

and

$$\begin{aligned}&[F(\cdot)\cos(q(w,\cdot)^{\sim})*G(\cdot)\sin(q(w,\cdot)^{\sim})]_q(y)\\&=\frac{1}{4i}[R_{1,0,0}(q,w,y)-R_{0,1,1}(q,w,y)+R_{0,1,-1}(q,w,y)-R_{-1,0,0}(q,w,y)]\end{aligned}\tag{3.13}$$

and

$$\begin{aligned}&[F(\cdot)\sin(q(w,\cdot)^{\sim})*G(\cdot)\cos(q(w,\cdot)^{\sim})]_q(y)\\&=\frac{1}{4i}[R_{1,0,0}(q,w,y)+R_{0,1,1}(q,w,y)-R_{0,1,-1}(q,w,y)-R_{-1,0,0}(q,w,y)]\end{aligned}\tag{3.14}$$

and

(3.15)
$$[F(\cdot)\sin(q(w,\cdot)^{\sim}) * G(\cdot)\sin(q(w,\cdot)^{\sim})]_q(y)$$
$$= \frac{1}{4}[R_{1,0,0}(q,w,y) - R_{0,1,1}(q,w,y) - R_{0,1,-1}(q,w,y) + R_{-1,0,0}(q,w,y)],$$

where

$$R_{\alpha,\beta,\gamma}(q,w,y) = \exp\{i[\sqrt{2}q(\alpha w,y)^{\sim} - q|\beta w|^2]\} \\ [F(\cdot - \gamma w) * G(\cdot + \gamma w)]_q(y) \tag{3.16}$$

for s*-a.e.* $y \in B$.

Of course we can also write down scaling and time reversal properties for the convolution product of functionals in $\mathcal{F}(B)$ as corollaries of our Theorem 3.1 and Corollary 3.2, respectively.

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