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GROWTH ANALYSIS OF COMPOSITE ENTIRE FUNCTIONS FROM THE VIEW POINT OF RELATIVE (p, q)-TH ORDER

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Abstract. In this paper we study some comparative growth properties of composite entire functions on the basis of relative (p, q)-th order and relative (p, q)-th lower order of entire function with respect to another entire function where p and q are any two positive integers.

1. Introduction, Definitions and Notations

Throughout this paper, we assume that the reader is familiar with the fundamental results and the standard notations of the theory of entire functions which are available in [17]. For any entire function f defined in the open complex plane $\mathbb{C}$, the maximum modulus function $M_f(r)$ is defined as $M_f(r) = \max_{|z|=r}|f(z)|$. Since $M_f(r)$ is strictly increasing and continuous, therefore there exists its inverse function $M_f^{-1} : (|f(0)|, \infty) \to (0, \infty)$ with $\lim_{s\to\infty} M_f^{-1}(s) = \infty$. However, for another entire function g, $M_g(r)$ is defined and the ratio $\frac{M_f(r)}{M_g(r)}$ as $r \to \infty$ is called the growth of f with respect to g in terms of their maximum moduli. The maximum term $\mu_f(r)$ of f can be defined in the following

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way:

$$\mu_f(r) = \max_{n\geq 0}\left(|a_n| r^n\right).$$

In fact $\mu_f(r)$ is much weaker than $M_f(r)$ in some sense. So from another angle of view $\frac{\mu_f(r)}{\mu_g(r)}$ as $r \to \infty$ is also called the growth of f with respect to g where $\mu_g(r)$ denotes the maximum term of entire g.

However, the order ρ_f (resp. lower order λ_f) of an entire function f which is generally used in computational purpose is defined in terms of the growth of f with respect to the $\exp z$ function as

$$\rho_f = \overline{\lim_{r\to\infty}} \frac{\log\log M_f(r)}{\log\log M_{\exp z}(r)} = \overline{\lim_{r\to\infty}} \frac{\log\log M_f(r)}{\log r}$$

$$\left(\text{resp. } \lambda_f = \underline{\lim_{r\to\infty}} \frac{\log\log M_f(r)}{\log\log M_{\exp z}(r)} = \underline{\lim_{r\to\infty}} \frac{\log\log M_f(r)}{\log r}\right).$$

Extending this notion, Juneja et. al. [10] defined the (p,q)-th order (resp. (p,q)-th lower order) of an entire function f for any two positive integers p, q with $p \geq q$ which is as follows:

$$\rho_f(p,q) = \overline{\lim_{r\to\infty}} \frac{\log^{[p]} M_f(r)}{\log^{[q]} r} \quad \left(\text{resp. } \lambda_f(p,q) = \underline{\lim_{r\to\infty}} \frac{\log^{[p]} M_f(r)}{\log^{[q]} r}\right),$$

where

$$\log^{[k]} x = \log\left(\log^{[k-1]} x\right) \text{ for } k = 1, 2, 3, \cdots \text{ and}$$
$$\log^{[0]} x = x;$$

and

$$\exp^{[k]} x = \exp\left(\exp^{[k-1]} x\right) \text{ for } k = 1, 2, 3, \cdots \text{ and}$$
$$\exp^{[0]} x = x.$$

These definitions extend the generalized order $\rho_f^{[l]}$ and generalized lower order $\lambda_f^{[l]}$ of an entire function f considered in [13] for each integer $l \geq 2$ since these correspond to the particular case $\rho_f^{[l]} = \rho_f(l,1)$ and $\lambda_f^{[l]} = \lambda_f(l,1)$. Clearly, $\rho_f(2,1) = \rho_f$ and $\lambda_f(2,1) = \lambda_f$.

In this connection, let us recall that if $0 < \rho_f(p,q) < \infty$, then the following properties hold

$$\rho_f(p-n,q) = \infty \text{ for } n < p,\ \rho_f(p,q-n) = 0 \text{ for } n < q, \text{ and}$$

$$\rho_f(p+n,q+n) = 1 \text{ for } n = 1,2,...$$

Similarly for $0 < \lambda_f(p,q) < \infty$, one can easily verify that

$$\lambda_f(p-n,q) = \infty \text{ for } n < p,\ \lambda_f(p,q-n) = 0 \text{ for } n < q, \text{ and}$$

$$\lambda_f(p+n,q+n) = 1 \text{ for } n = 1,2,.... .$$

Recalling that for any pair of integer numbers m,n the Kroenecker function is defined by $\delta_{m,n} = 1$ for $m = n$ and $\delta_{m,n} = 0$ for $m \neq n$, the aforementioned properties provide the following definition.

Definition 1. [10] An entire function f is said to have index-pair $(1,1)$ if $0 < \rho_f(1,1) < \infty$. Otherwise, f is said to have index-pair $(p,q) \neq (1,1)$, $p \geq q \geq 1$, if $\delta_{p-q,0} < \rho_f(p,q) < \infty$ and $\rho_f(p-1,q-1) \notin \mathbb{R}^+$.

Definition 2. [10] An entire function f is said to have lower index-pair $(1,1)$ if $0 < \lambda_f(1,1) < \infty$. Otherwise, f is said to have lower index-pair $(p,q) \neq (1,1)$, $p \geq q \geq 1$, if $\delta_{p-q,0} < \lambda_f(p,q) < \infty$ and $\lambda_f(p-1,q-1) \notin \mathbb{R}^+$.

An entire function f of index-pair (p,q) is said to be of regular (p,q)-growth if its (p,q)-th order coincides with its (p,q)-th lower order, otherwise f is said to be of irregular (p,q)-growth.

Since for $0 \leq r < R$,

$$\mu_f(r) \leq M_f(r) \leq \frac{R}{R-r}\mu_f(R) \quad \{cf.\ [15]\ \}$$

it is easy to see that

$$\rho_f(p,q) = \overline{\lim_{r\to\infty}} \frac{\log^{[p]} \mu_f(r)}{\log^{[q]} r} \quad \left(\text{respectively } \lambda_f(p,q) = \underline{\lim}_{r\to\infty} \frac{\log^{[p]} \mu_f(r)}{\log^{[q]} r}\right).$$

L. Bernal [1,2] introduced the relative order between two entire functions to avoid comparing growth just with $\exp z$. In the case of relative order, it was then natural for Lahiri and Banerjee [11] to define the relative (p,q)-th order of entire functions as follows.

Definition 3. [11] Let p and q be any two positive integers with $p > q$. The relative (p,q)-th order of f with respect to g is defined by

$$\rho_g^{(p,q)}(f) = \overline{\lim_{r\to\infty}} \frac{\log^{[p-1]} M_g^{-1} M_f(r)}{\log^{[q]} r}.$$

Then $\rho_{\exp z}^{(p,q)}(f) = \rho_f(p,q)$ and $\rho_g^{(k+1,1)}(f) = \rho_g^k(f)$ for any $k \geq 1$.

Sánchez Ruiz et al. [12] gave a more natural definition of relative (p,q) th order of an entire function in the light of index-pair which is as follows:

Definition 4. [12] Let f and g be any two entire functions with index-pairs (m,q) and (m,p) respectively where p, q, m are all positive integers such that $m \geq p$ and $m \geq q$. Then the relative (p,q) th order of f with respect to g is defined as

$$\rho_g^{(p,q)}(f) = \overline{\lim_{r\to\infty}} \frac{\log^{[p]} M_g^{-1} M_f(r)}{\log^{[q]} r}$$

Similarly one can define the relative (p,q) th lower order of an entire function f with respect to another entire function g denoted by $\lambda_g^{(p,q)}(f)$ where p and q are any two positive integers in the following way:

$$\lambda_g^{(p,q)}(f) = \underline{\lim_{r\to\infty}} \frac{\log^{[p]} M_g^{-1} M_f(r)}{\log^{[q]} r}.$$

In fact Definition 4 improves Definition 3 ignoring the restriction $p \geq q$.

If f and g have got index-pair $(m,1)$ and (m,k), respectively, then Definition 4 reduces to generalized relative order of f with respect to g. If the entire functions f and g have the same index-pair $(p,1)$ where p is any positive integer, we get the definition of relative order introduced by Bernal [1,2] and if $g = \exp^{[m-1]} z$, then $\rho_g(f) = \rho_f^{[m]}$ and $\rho_g^{(p,q)}(f) = \rho_f(m,q)$. Further if f is an entire function with index-pair $(2,1)$ and $g = \exp z$, then Definition 4 becomes the classical one given in [16].

An entire function f for which *relative* (p,q)*-th order* and *relative* (p,q)*-th lower order* with respect to another entire function g are the same is called a function of *regular relative* (p,q) *growth* with respect to g. Otherwise, f is said to be *irregular relative* (p,q) *growth* with respect to g.

In terms of maximum terms of entire functions, Definition 4 can be reformulated as:

Definition 5. For any positive integer p and q, the growth indicators $\rho_g^{(p,q)}(f)$ and $\lambda_g^{(p,q)}(f)$ of an entire function f with respect to another entire function g are defined as:

$$\rho_g^{(p,q)}(f) = \overline{\lim_{r\to\infty}} \frac{\log^{[p]} \mu_g^{-1}\mu_f(r)}{\log^{[q]} r} \text{ and } \lambda_g^{(p,q)}(f) = \underline{\lim_{r\to\infty}} \frac{\log^{[p]} \mu_g^{-1}\mu_f(r)}{\log^{[q]} r} .$$

In fact, Lemma 6 states the equivalence of Definition 4 and Definition 5.

For entire functions, the notions of their growth indicators such as order is classical in complex analysis and during the past decades, several researchers have already been exploring their studies in the area of comparative growth properties of composite entire functions in different directions using the classical growth indicators. But at that time, the concepts of relative orders of entire functions and as well as their technical advantages of not comparing with the growths of $\exp z$ are not at all known to the researchers of this area. Therefore the studies of the growths of composite entire functions in the light of their relative orders are the prime concern of this paper. In fact some light has already been thrown on such type of works by Datta et al. in [4], [5], [6], [7], [8] and [9]. Actually in this paper we establish some newly developed results related to the growth rates of composite entire functions on the basis of relative (p,q) th order and relative (p,q) th lower order improving some earlier results where p, q are any two positive integers.

2. Lemmas

In this section we present some lemmas which will be needed in the sequel.

Lemma 1. [3] *If f and g are any two entire functions with $g(0) = 0$. Let β satisfy $0 < \beta < 1$ and $c(\beta) = \frac{(1-\beta)^2}{4\beta}$. Then for all sufficiently large values of r,*

$$M_f(c(\beta) M_g(\beta r)) \leq M_{f\circ g}(r) \leq M_f(M_g(r)) .$$

In addition if $\beta = \frac{1}{2}$, then for all sufficiently large values of r,

$$M_{f\circ g}(r) \geq M_f\left(\frac{1}{8}M_g\left(\frac{r}{2}\right)\right) .$$

Lemma 2. [14] *Let f and g be any two entire functions. Then for every $\alpha > 1$ and $0 < r < R$,*

$$\mu_{f\circ g}(r) \leq \frac{\alpha}{\alpha - 1}\mu_f\left(\frac{\alpha R}{R - r}\mu_g(R)\right) .$$

Lemma 3. [14] *If f and g are any two entire functions with $g(0) = 0$, then for all sufficiently large values of r,*

$$\mu_{f\circ g}(r) \geq \frac{1}{2}\mu_f\left(\frac{1}{8}\mu_g\left(\frac{r}{4}\right) - |g(0)|\right) .$$

Lemma 4. [2] *Suppose f is an entire function and $\alpha > 1$, $0 < \beta < \alpha$. Then for all sufficiently large r,*

$$M_f(\alpha r) \geq \beta M_f(r).$$

Lemma 5. [6] *If f be an entire and $\alpha > 1$, $0 < \beta < \alpha$, then for all sufficiently large r,*

$$\mu_f(\alpha r) \geq \beta\mu_f(r) .$$

Lemma 6. *Definition 4 and Definition 5 are equivalent.*

Proof. Taking $R = \alpha r$ in the inequalities $\mu_g(r) \leq M_g(r) \leq \frac{R}{R-r}\mu_g(R)$ $\{cf.$ [15] $\}$, for $0 \leq r < R$ we obtain that

$$M_g^{-1}(r) \leq \mu_g^{-1}(r)$$

and

$$\mu_g^{-1}(r) \leq \alpha M_g^{-1}\left(\frac{\alpha r}{(\alpha - 1)}\right).$$

Since $M_g^{-1}(r)$ and $\mu_g^{-1}(r)$ are increasing functions of r, then for any $\alpha > 1$ it follows from the above and the inequalities $\mu_f(r) \leq M_f(r) \leq \frac{\alpha}{\alpha-1}\mu_f(\alpha r)$ $\{cf.$ [15] $\}$ that

$$M_g^{-1}M_f(r) \leq \mu_g^{-1}\left[\frac{\alpha}{(\alpha - 1)}\mu_f(\alpha r)\right] \tag{1}$$

and

$$\mu_g^{-1}\mu_f(r) \leq \alpha M_g^{-1}\left[\frac{\alpha}{(\alpha - 1)}M_f(r)\right]. \tag{2}$$

Therefore in view of Lemma 5, we have from (1) that

$$M_g^{-1} M_f(r) \le \mu_g^{-1} \mu_f \left[\frac{(2\alpha - 1)\,\alpha}{(\alpha - 1)} \cdot r \right] .$$

Thus from above we get that

$$\begin{aligned} \frac{\log^{[p]} M_g^{-1} M_f(r)}{\log^{[q]} r} &\le \frac{\log^{[p]} \mu_g^{-1} \mu_f \left[\frac{(2\alpha-1)\alpha}{(\alpha-1)} \cdot r \right]}{\log^{[q]} r} \\ i.e.,\ \frac{\log^{[p]} M_g^{-1} M_f(r)}{\log^{[q]} r} &\le \frac{\log^{[p]} \mu_g^{-1} \mu_f \left[\frac{(2\alpha-1)\alpha}{(\alpha-1)} \cdot r \right]}{\log^{[q]} \left[\frac{(2\alpha-1)\alpha}{(\alpha-1)} \cdot r \right] + O(1)} \end{aligned}$$

$$\text{(3)} \qquad i.e.,\ \rho_g^{(p,q)}(f) \le \varlimsup_{r \to \infty} \frac{\log^{[p]} \mu_g^{-1} \mu_f(r)}{\log^{[q]} r}$$

and accordingly

$$\text{(4)} \qquad \lambda_g^{(p,q)}(f) \le \varliminf_{r \to \infty} \frac{\log^{[p]} \mu_g^{-1} \mu_f(r)}{\log^{[q]} r} .$$

Similarly, in view of Lemma 4 it follows from (2) that

$$\mu_g^{-1} \mu_f(r) \le \alpha M_g^{-1} M_f \left[\left(\frac{2\alpha - 1}{\alpha - 1} \right) \cdot r \right]$$

and from above we obtain that

$$\begin{aligned} \frac{\log^{[p]} \mu_g^{-1} \mu_f(r)}{\log^{[q]} r} &\le \frac{\log^{[p]} \alpha M_g^{-1} M_f \left[\left(\frac{2\alpha-1}{\alpha-1} \right) \cdot r \right]}{\log^{[q]} r} \\ i.e.,\ \frac{\log^{[p]} \mu_g^{-1} \mu_f(r)}{\log^{[q]} r} &\le \frac{\log^{[p]} M_g^{-1} M_f \left[\left(\frac{2\alpha-1}{\alpha-1} \right) \cdot r \right] + O(1)}{\log^{[q]} \left[\left(\frac{2\alpha-1}{\alpha-1} \right) \cdot r \right] + O(1)} \end{aligned}$$

$$\text{(5)} \qquad i.e.,\ \rho_g^{(p,q)}(f) \ge \varlimsup_{r \to \infty} \frac{\log^{[p]} \mu_g^{-1} \mu_f(r)}{\log^{[q]} r}$$

and consequently

$$\text{(6)} \qquad \lambda_g^{(p,q)}(f) \ge \varliminf_{r \to \infty} \frac{\log^{[p]} \mu_g^{-1} \mu_f(r)}{\log^{[q]} r} .$$

Combining (3), (5) and (4), (6) we obtain that

$$\rho_g^{(p,q)}(f) = \overline{\lim_{r\to\infty}} \frac{\log^{[p]} \mu_g^{-1}\mu_f(r)}{\log^{[q]} r} \text{ and } \lambda_g^{(p,q)}(f) = \underline{\lim_{r\to\infty}} \frac{\log^{[p]} \mu_g^{-1}\mu_f(r)}{\log^{[q]} r} .$$

This proves the lemma. □

3. Main Results

In this section we present the main results of the paper.

THEOREM 1. *Let f and h be any two entire functions such that $0 < \lambda_h^{(p,q)}(f) \le \rho_h^{(p,q)}(f) < \infty$. Also let g be an entire function with finite (m,n)-th lower order where p,q,m,n are all positive integers with $m > n$. Then for every positive constant A and every real number α,*

$$(i) \lim_{r\to\infty} \frac{\log^{[p]} \mu_h^{-1}\mu_{f\circ g}(\exp^{[n-1]} r)}{\left\{\log^{[p]} \mu_h^{-1}\mu_f(r^A)\right\}^{1+\alpha}} = \infty \text{ if } q = 1,$$

$$(ii) \lim_{r\to\infty} \frac{\log^{[p]} \mu_h^{-1}\mu_{f\circ g}(\exp^{[n-1]} r)}{\left\{\log^{[p]} \mu_h^{-1}\mu_f(\exp^{[q-1]} r^A)\right\}^{1+\alpha}} = \infty \text{ if } q > 1 \text{ and } q < m$$

and

$$(iii) \lim_{r\to\infty} \frac{\log^{[p-1]} \mu_h^{-1}\mu_{f\circ g}(\exp^{[n-1]} r)}{\left\{\log^{[p]} \mu_h^{-1}\mu_f(r^A)\right\}^{1+\alpha}} = \infty \text{ if } q > 1 \text{ and } q \ge m.$$

Proof. If α be such that $1+\alpha \le 0$ then the theorem is trivial. So we suppose that $1+\alpha > 0$. Now from the definition of $\rho_h^{(p,q)}(f)$ in terms of maximum terms, it follows for all sufficiently large values of r that

$$(7) \qquad \log^{[p]} \mu_h^{-1}\mu_f(\exp^{[q-1]} r^A) \le \left(\rho_h^{(p,q)}(f) + \varepsilon\right) \log^{[q]} \exp^{[q-1]} r^A.$$

Case I. Let $q = 1$. Then we have from (7) for all sufficiently large values of r that

$$\log^{[p]} \mu_h^{-1}\mu_f(r^A) \le \left(\rho_h^{(p,q)}(f) + \varepsilon\right) A \log r$$

$$(8) \text{ i.e., } \left\{\log^{[p]} \mu_h^{-1}\mu_f(r^A)\right\}^{1+\alpha} \le \left(\rho_h^{[p]}(f) + \varepsilon\right)^{1+\alpha} A^{1+\alpha} (\log r)^{1+\alpha} .$$

Case II. Let $q > 1$. Then we obtain from (7) for all sufficiently large values of r that

$$\log^{[p]} \mu_h^{-1}\mu_f(\exp^{[q-1]} r^A) \leq \left(\rho_h^{(p,q)}(f) + \varepsilon\right) A \log r$$

$$(9) \qquad i.e., \left\{\log^{[p]} \mu_h^{-1}\mu_f(\exp^{[q-1]} r^A)\right\}^{1+\alpha} \leq \left(\rho_h^{(p,q)}(f) + \varepsilon\right)^{1+\alpha} A^{1+\alpha} (\log r)^{1+\alpha} .$$

Case III. Again let $q > 1$. Then it follows from (7) for all sufficiently large values of r that

$$\log^{[p]} \mu_h^{-1}\mu_f(r^A) \leq \left(\rho_h^{(p,q)}(f) + \varepsilon\right) \log^{[q]} r + O(1)$$

$$\log^{[p]} \mu_h^{-1}\mu_f(r^A) \leq \left(\rho_h^{(p,q)}(f) + \varepsilon\right) \log^{[q]} r \left(1 + \frac{O(1)}{\left(\rho_h^{(p,q)}(f) + \varepsilon\right) \log^{[q]} r}\right)$$

$$(10) \quad i.e., \left\{\log^{[p]} \mu_h^{-1}\mu_f(r^A)\right\}^{1+\alpha} \leq \left(\rho_h^{(p,q)}(f) + \varepsilon\right)^{1+\alpha} \left(\log^{[q]} r\right)^{1+\alpha} \left(1 + \frac{O(1)}{\left(\rho_h^{(p,q)}(f) + \varepsilon\right) \log^{[q]} r}\right)^{[1+\alpha]} .$$

Let us choose $0 < \varepsilon < \min(\lambda_h^{(p,q)}(f), \lambda_{(m,n)}(g))$.

Now from Lemma 3 we get for all sufficiently large values of r that

$$\mu_{f\circ g}(r) \geq \frac{1}{2}\mu_f\left(\frac{1}{16}\mu_g\left(\frac{r}{4}\right)\right) \quad \{cf.\ [15]\ \} \ .$$

Therefore in view of Lemma 5, we obtain from abvove for all sufficiently large values of r that

$$(11) \qquad \mu_{f\circ g}(r) \geq \mu_f\left(\frac{1}{48}\mu_g\left(\frac{r}{4}\right)\right) \ .$$

Since $\mu_h^{-1}(r)$ is an increasing function of r, it follows from (11) for all sufficiently large values r that

$$\mu_h^{-1}\mu_{f\circ g}(\exp^{[n-1]} r) \geq \mu_h^{-1}\mu_f\left(\frac{1}{48}\mu_g\left(\frac{\exp^{[n-1]} r}{4}\right)\right)$$

$$i.e., \ \log^{[p]} \mu_h^{-1}\mu_{f\circ g}(\exp^{[n-1]} r) \geq$$

$$\left(\lambda_h^{(p,q)}(f)-\varepsilon\right)\log^{[q]}\left\{\frac{1}{48}\mu_g\left(\frac{\exp^{[n-1]} r}{4}\right)\right\}$$

$$\text{(12)}\quad i.e.,\ \log^{[p]}\mu_h^{-1}\mu_{f\circ g}(\exp^{[n-1]} r)\geq \left(\lambda_h^{(p,q)}(f)-\varepsilon\right)\log^{[q]}\mu_g\left(\frac{\exp^{[n-1]} r}{4}\right)+O(1).$$

Case IV. Let $q < m$. Then from (12) it follows for all sufficiently large values of r that

$$\text{(13)}\quad \log^{[p]}\mu_h^{-1}\mu_{f\circ g}(\exp^{[n-1]} r)\geq \left(\lambda_h^{(p,q)}(f)-\varepsilon\right)\exp^{[m-q-1]}\log^{[m-1]}\mu_g\left(\frac{\exp^{[n-1]} r}{4}\right)+O(1).$$

Now from the definition of (m,n)-th lower order of g in terms of maximum terms we obtain for arbitrary positive $\varepsilon\,(>0)$ and for all sufficiently large values of r that

$$\log^{[m]}\mu_g\left(\frac{\exp^{[n-1]} r}{4}\right)\geqslant(\lambda_g(m,n)-\varepsilon)\log^{[n]}\left(\frac{\exp^{[n-1]} r}{4}\right)$$

$$i.e.,\quad \log^{[m]}\mu_g\left(\frac{\exp^{[n-1]} r}{4}\right)\geqslant(\lambda_g(m,n)-\varepsilon)\log r+O(1)$$

$$i.e.,\quad \log^{[m]}\mu_g\left(\frac{\exp^{[n-1]} r}{4}\right)\geqslant\log r^{(\lambda_g(m,n)-\varepsilon)}+O(1)\ .$$

Also for all sufficiently large values of r we get from (**??**) that

$$\text{(14)}\qquad \log^{[m-1]}\mu_g\left(\frac{\exp^{[n-1]} r}{4}\right)\geqslant r^{(\lambda_g(m,n)-\varepsilon)}+O(1)\ .$$

Now from (13) and (14) it follows for all sufficiently large values of r that

$$\text{(15)}\quad \log^{[p]}\mu_h^{-1}\mu_{f\circ g}(\exp^{[n-1]} r)\geq \left(\lambda_h^{(p,q)}(f)-\varepsilon\right)\exp^{[m-q-1]} r^{(\lambda_g(m,n)-\varepsilon)}+O(1)\ .$$

Case V. Let $q\geqslant m$. Then from (12) we obtain for all sufficiently large values of r that

$$\text{(16)}\quad \log^{[p]}\mu_h^{-1}\mu_{f\circ g}(\exp^{[n-1]} r)\geqslant\left(\lambda_h^{(p,q)}(f)-\varepsilon\right)\log^{[q-m]}\log^{[m]}\mu_g\left(\frac{\exp^{[n-1]} r}{4}\right)+O(1).$$

Now from (??) and (16) we have for all sufficiently large values of r that

$$\log^{[p]}\mu_h^{-1}\mu_{f\circ g}(\exp^{[n-1]} r)$$
$$\geqslant \left(\lambda_h^{(p,q)}(f)-\varepsilon\right)\log^{[q-m]}\log r^{(\lambda_g(m,n)-\varepsilon)}+O(1)$$

$$i.e.,\ \log^{[p]}\mu_h^{-1}\mu_{f\circ g}(\exp^{[n-1]} r)\geqslant \left(\lambda_h^{(p,q)}(f)-\varepsilon\right)\log^{[q-m+1]} r+O(1)$$

$$i.e.,\ \log^{[p]}\mu_h^{-1}\mu_{f\circ g}(\exp^{[n-1]} r)\geqslant \log\left(\log^{[q-m]} r\right)^{\left(\lambda_h^{(p,q)}(f)-\varepsilon\right)}+O(1)$$

$$i.e.,\ \log^{[p-1]}\mu_h^{-1}\mu_{f\circ g}(\exp^{[n-1]} r)\geqslant \left(\log^{[q-m]} r\right)^{\left(\lambda_h^{(p,q)}(f)-\varepsilon\right)}+O(1). \tag{17}$$

Now combining (8) of Case I and (15) of Case IV it follows for all sufficiently large values of r that

$$\frac{\log^{[p]}\mu_h^{-1}\mu_{f\circ g}(\exp^{[n-1]} r)}{\left\{\log^{[p]}\mu_h^{-1}\mu_f(r^A)\right\}^{1+\alpha}}\geq\frac{\left(\lambda_h^{[p]}(f)-\varepsilon\right)\exp^{[m-2]} r^{(\lambda_g(m,n)-\varepsilon)}+O(1)}{\left(\rho_h^{[p]}(f)+\varepsilon\right)^{1+\alpha}A^{1+\alpha}(\log r)^{1+\alpha}}.$$

Since $\frac{\exp^{[m-2]} r^{(\lambda_g(m,n)-\varepsilon)}}{(\log r)^{1+\alpha}}\to\infty$ as $r\to\infty$, then from above it follows that

$$\varliminf_{r\to\infty}\frac{\log^{[p]}\mu_h^{-1}\mu_{f\circ g}(\exp^{[n-1]} r)}{\left\{\log^{[p]}\mu_h^{-1}\mu_f(r^A)\right\}^{1+\alpha}}=\infty,$$

from which the first part of the theorem follows.

Again combining (9) of Case II and (15) of Case IV we obtain for all sufficiently large values of r that

$$\frac{\log^{[p]}\mu_h^{-1}\mu_{f\circ g}(\exp^{[n-1]} r)}{\left\{\log^{[p]}\mu_h^{-1}\mu_f(\exp^{[q-1]} r^A)\right\}^{1+\alpha}}\geq\frac{\left(\lambda_h^{(p,q)}(f)-\varepsilon\right)\exp^{[m-q-1]} r^{(\lambda_g(m,n)-\varepsilon)}+O(1)}{\left(\rho_h^{(p,q)}(f)+\varepsilon\right)^{1+\alpha}A^{1+\alpha}(\log r)^{1+\alpha}}.$$

As $\frac{\exp^{[m-q-1]} r^{(\lambda_g(m,n)-\varepsilon)}}{(\log r)^{1+\alpha}}\to\infty$ as $r\to\infty$, then we obtain from above that

$$\lim_{r\to\infty}\frac{\log^{[p]}\mu_h^{-1}\mu_{f\circ g}(\exp^{[n-1]} r)}{\left\{\log^{[p]}\mu_h^{-1}\mu_f(\exp^{[q-1]} r^A)\right\}^{1+\alpha}}=\infty\ .$$

This establishes the second part of the theorem .

Again combining (10) of Case III and (17) of Case V it follows for all sufficiently large values of r that

$$\frac{\log^{[p-1]} \mu_h^{-1} \mu_{f\circ g}(\exp^{[n-1]} r)}{\left\{\log^{[p]} \mu_h^{-1} \mu_f(r^A)\right\}^{1+\alpha}}$$

$$\geq \frac{\left(\log^{[q-m]} r\right)^{\left(\lambda_h^{(p,q)}(f)-\varepsilon\right)} + O(1)}{\left(\rho_h^{(p,q)}(f)+\varepsilon\right)^{1+\alpha} \left(\log^{[q]} r\right)^{1+\alpha} \left(1 + \frac{O(1)}{\left(\rho_h^{(p,q)}(f)+\varepsilon\right) \log^{[q]} r}\right)^{1+\alpha}}.$$

Since $q - m < q$, so $\frac{\left(\log^{[q-m]} r\right)^{\left(\lambda_h^{(p,q)}(f)-\varepsilon\right)}}{\left(\log^{[q]} r\right)^{1+\alpha}} \to \infty$ as $r \to \infty$. Thus it follows from above that

$$\lim_{r\to\infty} \frac{\log^{[p-1]} \mu_h^{-1} \mu_{f\circ g}(\exp^{[n-1]} r)}{\left\{\log^{[p]} \mu_h^{-1} \mu_f(r^A)\right\}^{1+\alpha}} = \infty \ .$$

This proves the third part of the theorem.

Thus the theorem follows . □

Remark 1. Theorem 1 is still valid with "limit superior" instead of "limit" if we replace the condition " $0 < \lambda_h^{(p,q)}(f) \leq \rho_h^{(p,q)}(f) < \infty$" by " $0 < \lambda_h^{(p,q)}(f) < \infty$".

In the line of Theorem 1 one may state the following theorem without proof:

Theorem 2. *Let f, g and h be any three entire functions such that g is of finite (m,n)-th lower order, $\lambda_h^{(p,q)}(f) > 0$ and $\rho_h^{(p,n)}(g) < \infty$ where p, q, m, n are all positive integers with $m > \min\{p,q,n\}$. Then for every positive constant A and every real number α,*

$$\lim_{r\to\infty} \frac{\log^{[p]} \mu_h^{-1} \mu_{f\circ g}(\exp^{[n-1]} r)}{\left\{\log^{[p]} \mu_h^{-1} \mu_g(\exp^{[n-1]} r^A)\right\}^{1+\alpha}} = \infty.$$

Remark 2. In Theorem 2 if we take the condition $\lambda_h^{(p,n)}(g) < \infty$ instead of $\rho_h^{(p,n)}(g) < \infty$, then also Theorem 2 remains true with "limit superior" in place of " limit ".

Theorem 3. *Let f and h be any two entire functions such that $\lambda_h^{(p,q)}(f) > 0$ and g be an entire function with finite (m,n)-th order where p,q,m,n are all positive integers with $m > n$. Then for each $\alpha \in (-\infty, \infty)$,*

$$(i)\ \lim_{r\to\infty} \frac{\left\{\log^{[p]} \mu_h^{-1}\mu_{f\circ g}(r)\right\}^{1+\alpha}}{\log^{[p]} \mu_h^{-1}\mu_f\left(\exp^{[q]}(r^A)\right)} = 0 \text{ if } q \geqslant m$$

and

$$(ii)\ \lim_{r\to\infty} \frac{\left\{\log^{[p+m-q-1]} \mu_h^{-1}\mu_{f\circ g}(r)\right\}^{1+\alpha}}{\log^{[p]} \mu_h^{-1}\mu_f\left(\exp^{[q]}(r^A)\right)} = 0 \text{ if } q < m,$$

where $A > (1+\alpha)\rho_g(m,n)$.

Proof. If $1+\alpha \leq 0$, then the theorem is obvious. We consider $1+\alpha > 0$. Let us we choose ε such that

$$0 < \varepsilon < \min\left\{\lambda_h^{(p,q)}(f), \frac{A}{1+\alpha} - \rho_g(m,n)\right\}. \tag{18}$$

Since $\mu_h^{-1}(r)$ is an increasing function of r, taking $R = \beta r$ in Lemma 2 and in view of Lemma 5 it follows for all sufficiently large values of r that

$$\log^{[p]} \mu_h^{-1}\mu_{f\circ g}(r) \leq \log^{[p]} \mu_h^{-1}\mu_f\left(\frac{(2\alpha-1)\,\alpha\beta}{(\alpha-1)(\beta-1)}\mu_g(\beta r)\right)$$

$$i.e.,\ \log^{[p]} \mu_h^{-1}\mu_{f\circ g}(r) \leq \left(\rho_h^{(p,q)}(f) + \varepsilon\right)\log^{[q]} \mu_g(\beta r) + O(1)\ . \tag{19}$$

Now the following cases may arise :

Case I. Let $q \geqslant m$. Then we have from (19) for all sufficiently large values of r that

$$\log^{[p]} \mu_h^{-1}\mu_{f\circ g}(r) \leq \left(\rho_h^{(p,q)}(f) + \varepsilon\right)\log^{[m-1]} \mu_g(\beta r) + O(1)\ . \tag{20}$$

Now from the definition of (m,n)-th order of g in terms of maximum terms, we get for arbitrary positive ε and for all sufficiently large values of r that

$$\log^{[m]} \mu_g(\beta r) \leqslant (\rho_g(m,n) + \varepsilon)\log^{[n]} r + O(1)$$

$$i.e.,\ \log^{[m]} \mu_g(\beta r) \leqslant (\rho_g(m,n) + \varepsilon)\log r + O(1)\ . \tag{21}$$

Also for all sufficiently large values of r it follows from (21) that

$$\log^{[m-1]} \mu_g(\beta r) \leqslant r^{(\rho_g(m,n)+\varepsilon)} + O(1)\ . \tag{22}$$

So from (20) and (22) it follows for all sufficiently large values of r that

$$\log^{[p]} \mu_h^{-1} \mu_{f \circ g} (r) \leq \left(\rho_h^{(p,q)} (f) + \varepsilon\right) r^{(\rho_g(m,n)+\varepsilon)} + O(1) \ . \tag{23}$$

Case II. Let $q < m$. Then we get from(19) for all sufficiently large values of r that

$$\log^{[p]} \mu_h^{-1} \mu_{f \circ g} (r) \leq \left(\rho_h^{(p,q)} (f) + \varepsilon\right) \exp^{[m-q]} \log^{[m]} \mu_g (\beta r) + O(1). \tag{24}$$

Since $\frac{(\rho_g(m,n)+\varepsilon)\log r}{O(1)} \to \infty$ as $r \to \infty$, we obtain from (21) for all sufficiently large values of r, that

$$\exp^{[m-q]} \log^{[m]} \mu_g (\beta r) \leqslant \exp^{[m-q]} \log r^{(2\rho_g(m,n)+\varepsilon)}$$

$$i.e., \ \exp^{[m-q]} \log^{[m]} \mu_g (\beta r) \leqslant \exp^{[m-q-1]} r^{(2\rho_g(m,n)+\varepsilon)}. \tag{25}$$

Now from (24) and (25) we obtain for all sufficiently large values of r that

$$\log^{[p]} \mu_h^{-1} \mu_{f \circ g} (r) \leqslant \left(\rho_h^{(p,q)} (f) + \varepsilon\right) \exp^{[m-q-1]} r^{(2\rho_g(m,n)+\varepsilon)} + O(1)$$

$$i.e., \ \log^{[p+m-q-1]} \mu_h^{-1} \mu_{f \circ g} (r) \leqslant r^{(2\rho_g(m,n)+\varepsilon)} + O(1) \ . \tag{26}$$

Again for all sufficiently large values of r we get that

$$\log^{[p]} \mu_h^{-1} \mu_f \left(\exp^{[q]} \left(r^A\right)\right) \ \geq \ \left(\lambda_h^{(p,q)} (f) - \varepsilon\right) \log^{[q]} \exp^{[q]} \left(r^A\right)$$

$$i.e., \ \log^{[p]} \mu_h^{-1} \mu_f \left(\exp^{[q]} \left(r^A\right)\right) \ \geq \ \left(\lambda_h^{(p,q)} (f) - \varepsilon\right) r^A \ . \tag{27}$$

Now if $q \geqslant m$, we get from $(23), (27)$ and in view of (18) for all sufficiently large values of r that

$$\frac{\left\{\log^{[p]} \mu_h^{-1} \mu_{f \circ g} (r)\right\}^{1+\alpha}}{\log^{[p]} \mu_h^{-1} \mu_f \left(\exp^{[q]} (r^A)\right)} \leqslant \frac{\left(\rho_h^{(p,q)} (f) + \varepsilon\right)^{1+\alpha} r^{(\rho_g(m,n)+\varepsilon)(1+\alpha)} + O(1)}{\left(\lambda_h^{(p,q)} (f) - \varepsilon\right) r^A}$$

$$i.e., \ \lim_{r \to \infty} \frac{\left\{\log^{[p]} \mu_h^{-1} \mu_{f \circ g} (r)\right\}^{1+\alpha}}{\log^{[p]} \mu_h^{-1} \mu_f \left(\exp^{[q]} (r^A)\right)} = 0 \ ,$$

which proves the first part of the theorem.

Again when $q < m$, we obtain from (26), (27) and (18) for all sufficiently large values of r that

$$\frac{\left\{\log^{[p+m-q-1]}\mu_h^{-1}\mu_{f\circ g}(r)\right\}^{1+\alpha}}{\log^{[p]}\mu_h^{-1}\mu_f\left(\exp^{[q]}\left(r^A\right)\right)} \leqslant \frac{r^{(2\rho_g(m,n)+\varepsilon)(1+\alpha)}\left(1+\frac{O(1)}{r^{(2\rho_g(m,n)+\varepsilon)}}\right)^{1+\alpha}}{\left(\lambda_h^{(p,q)}(f)-\varepsilon\right)r^A}$$

$$i.e., \ \varlimsup_{r\to\infty}\frac{\left\{\log^{[p+m-q-1]}\mu_h^{-1}\mu_{f\circ g}(r)\right\}^{1+\alpha}}{\log^{[p]}\mu_h^{-1}\mu_f\left(\exp^{[q]}\left(r^A\right)\right)} = 0 \ .$$

This proves the second part of the theorem. □

Remark 3. In Theorem 3 if we take the condition $\rho_h^{(p,q)}(f) > 0$ instead of $\lambda_h^{(p,q)}(f) > 0$, the theorem remains true with " limit inferior" in place of "limit ".

In view of Theorem 3 the following theorem can be carried out:

Theorem 4. *Let f, g and h be any three entire functions where g is of finite (m,n) th order, $\lambda_h^{(p,n)}(g) > 0$ and $\rho_h^{(p,q)}(f) < \infty$ where p,q,m,n are all positive integers with $m > \min\{p,q,n\}$. Then for each $\alpha \in (-\infty,\infty)$,*

$$\lim_{r\to\infty}\frac{\left\{\log^{[p+m-q-1]}\mu_h^{-1}\mu_{f\circ g}(r)\right\}^{1+\alpha}}{\log^{[p]}\mu_h^{-1}\mu_g\left(\exp^{[n]}\left(r^A\right)\right)} = 0 \text{ if } A > (1+\alpha)\rho_g(m,n)\,.$$

The proof is omitted.

Remark 4. In Theorem 4, if we take the condition $\rho_h^{(p,n)}(g) > 0$ instead of $\lambda_h^{(p,n)}(g) > 0$, the theorem remains true with " limit replaced by limit inferior".

Theorem 5. *Let f,g and h be any three entire functions such that $\rho_h^{(p,q)}(f) < \infty$ and $\lambda_h^{(p,q)}(f\circ g) = \infty$ where p and q are any positive integers with $q > 1$. Then for every $A\,(> 0)$,*

$$\lim_{r\to\infty}\frac{\log^{[p]}\mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[p]}\mu_h^{-1}\mu_f\left(r^A\right)} = \infty \ .$$

Proof. If possible, let there exists a constant β such that for a sequence of values of r tending to infinity we have

$$(28) \qquad \log^{[p]}\mu_h^{-1}\mu_{f\circ g}(r) \le \beta\cdot\log^{[p]}\mu_h^{-1}\mu_f\left(r^A\right) \ .$$

Again from the definition of $\rho_h^{(p,q)}(f)$ in terms of maximum terms, it follows for all sufficiently large values of r that

$$\log^{[p]}\mu_h^{-1}\mu_f\left(r^A\right) \le \left(\rho_h^{(p,q)}(f)+\varepsilon\right)\log^{[q]} r + O(1)\ . \tag{29}$$

Now combining (28) and (29) we obtain for a sequence of values of r tending to infinity that

$$\begin{aligned} \log^{[p]}\mu_h^{-1}\mu_{f\circ g}(r) &\le \beta\cdot\left(\rho_h^{(p,q)}(f)+\varepsilon\right)\log^{[q]} r + O(1) \\ \text{i.e., } \lambda_h^{(p,q)}(f\circ g) &\le \beta\cdot\left(\rho_h^{(p,q)}(f)+\varepsilon\right), \end{aligned}$$

which contradicts the condition $\lambda_h^{(p,q)}(f\circ g)=\infty$. So for any positive integer q and for all sufficiently large values of r we get that

$$\log^{[p]}\mu_h^{-1}\mu_{f\circ g}(r) \ge \beta\cdot\log^{[p]}\mu_h^{-1}\mu_f\left(r^A\right),$$

from which the theorem follows. □

In the line of Theorem 5, one can easily prove the following theorem and therefore its proof is omitted.

THEOREM 6. *Let f, g and h be any three entire functions such that $\rho_h^{(p,q)}(g) < \infty$ and $\lambda_h^{(p,q)}(f\circ g)=\infty$ where p and q are any positive integers with $q>1$. Then for every $A\,(>0)$,*

$$\lim_{r\to\infty}\frac{\log^{[p]}\mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[p]}\mu_h^{-1}\mu_g(r^A)}=\infty\ .$$

REMARK 5. Theorem 5 is also valid with "limit superior" instead of "limit" if $\lambda_h^{(p,q)}(f\circ g)=\infty$ is replaced by $\rho_h^{(p,q)}(f\circ g)=\infty$ and the other conditions remain the same.

REMARK 6. Theorem 6 is also valid with "limit superior" instead of "limit" if $\lambda_h^{(p,q)}(f\circ g)=\infty$ is replaced by $\rho_h^{(p,q)}(f\circ g)=\infty$ and the other conditions remain the same.

COROLLARY 1. *Under the assumptions of Theorem 5 and Remark 5,*

$$\lim_{r\to\infty}\frac{\log^{[p-1]}\mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[p-1]}\mu_h^{-1}\mu_f(r^A)}=\infty \text{ and } \overline{\lim_{r\to\infty}}\frac{\log^{[p-1]}\mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[p-1]}\mu_h^{-1}\mu_f(r^A)}=\infty$$

respectively.

Proof. By Theorem 5 we obtain for all sufficiently large values of r and for $K > 1$,

$$\begin{aligned} \log^{[p]} \mu_h^{-1}\mu_{f\circ g}(r) &\geq K \cdot \log^{[p]} \mu_h^{-1}\mu_f\left(r^A\right) \\ \textit{i.e.,}\ \log^{[p-1]} \mu_h^{-1}\mu_{f\circ g}(r) &\geq \left\{\log^{[p-1]} \mu_h^{-1}\mu_f\left(r^A\right)\right\}^K, \end{aligned}$$

from which the first part of the corollary follows.

Similarly using Remark 5, we obtain the second part of the corollary. □

Corollary 2. *Under the assumptions of Theorem 6 and Remark 6,*

$$\lim_{r\to\infty} \frac{\log^{[p-1]} \mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[p-1]} \mu_h^{-1}\mu_g\left(r^A\right)} = \infty \ \textit{and}\ \overline{\lim_{r\to\infty}} \frac{\log^{[p-1]} \mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[p-1]} \mu_h^{-1}\mu_g\left(r^A\right)} = \infty$$

respectively.

In the line of Corollary 1, one can easily verify Corollary 2 with the help of Theorem 6 and Remark 6 respectively and therefore its proof is omitted.

Theorem 7. *If f, g and h be any three entire functions such that $\rho_g(m,n) < \lambda_h^{(p,q)}(f) \leq \rho_h^{(p,q)}(f) < \infty$ where p, q, m, n are all positive integers with $m > n$. Then*

$$(i)\ \lim_{r\to\infty} \frac{\log^{[p]} \mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[p-1]} \mu_h^{-1}\mu_f\left(\exp^{[q-1]} r^A\right)} = 0 \ \textit{if}\ q \geqslant m$$

and

$$(ii)\ \lim_{r\to\infty} \frac{\log^{[p+m-q-1]} \mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[p-1]} \mu_h^{-1}\mu_f\left(\exp^{[q-1]} r^A\right)} = 0 \ \textit{if}\ q < m$$

where $A > 0$.

Proof. From the definition of relative (p,q)-th order in terms of maximum terms, we obtain for all sufficiently large values of r that

$$\log^{[p]} \mu_h^{-1}\mu_f\left(\exp^{[q-1]} r^A\right) \geqslant \left(\lambda_h^{(p,q)}(f) - \varepsilon\right) \log^{[q]} \exp^{[q-1]} r^A$$

$$\textit{i.e.,}\ \log^{[p-1]} \mu_h^{-1}\mu_f\left(\exp^{[q-1]} r^A\right) \geqslant r^{A\left(\lambda_h^{(p,q)}(f)-\varepsilon\right)}. \tag{30}$$

As $\rho_g(m,n) < \lambda_h^{(p,q)}(f)$, we can choose $\varepsilon\,(> 0)$ in such a way that

$$\rho_g(m,n) + \varepsilon < A\left(\lambda_h^{(p,q)}(f) - \varepsilon\right). \tag{31}$$

Now if $q \geqslant m$, combining (23), (30) and in view of (31) we have for all sufficiently large values of r that

$$\frac{\log^{[p]} \mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[p-1]} \mu_h^{-1}\mu_f\left(\exp^{[q-1]} r^A\right)} \leqslant \frac{\left(\rho_h^{(p,q)}(f)+\varepsilon\right) r^{(\rho_g(m,n)+\varepsilon)} + O(1)}{r^{A\left(\lambda_h^{(p,q)}(f)-\varepsilon\right)}}$$

$$i.e., \lim_{r\to\infty} \frac{\log^{[p]} \mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[p-1]} \mu_h^{-1}\mu_f\left(\exp^{[q-1]} r^A\right)} = 0\ .$$

This proves the first part of the theorem.

When $q < m$, combining (26) and (30) it follows for all sufficiently large values of r that

$$\frac{\log^{[p+m-q-1]} \mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[p-1]} \mu_h^{-1}\mu_f\left(\exp^{[q-1]} r^A\right)} \leqslant \frac{r^{(2\rho_g(m,n)+\varepsilon)}\left(1+\frac{O(1)}{r^{(2\rho_g(m,n)+\varepsilon)}}\right)}{r^{A\left(\lambda_h^{(p,q)}(f)-\varepsilon\right)}}\ .$$

Since $\rho_g(m,n) < \lambda_h^{(p,q)}(f)$ and $\varepsilon\,(>0)$ is arbitrary, we get from above

$$\lim_{r\to\infty} \frac{\log^{[p+m-q-1]} \mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[p-1]} \mu_h^{-1}\mu_f\left(\exp^{[q-1]} r^A\right)} = 0,$$

which is the second part of the theorem. □

Theorem 8. *If f, g and h be any three entire functions with $\lambda_g(m,n) < \lambda_h^{(p,q)}(f) \le \rho_h^{(p,q)}(f) < \infty$ where p, q, m, n are all positive integers such that $m > n$. Then*

$$(i)\ \ \underline{\lim}_{r\to\infty} \frac{\log^{[p]} \mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[p-1]} \mu_h^{-1}\mu_f\left(\exp^{[q-1]} r^A\right)} = 0 \text{ if } q \geqslant m$$

and

$$(ii)\ \ \underline{\lim}_{r\to\infty} \frac{\log^{[p+m-q-1]} \mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[p-1]} \mu_h^{-1}\mu_f\left(\exp^{[q-1]} r^A\right)} = 0 \text{ if } q < m$$

where $A > 0$.

Proof of Theorem 8 is omitted as it can be carried out in the line of Theorem 7.

Theorem 9. *Let h and f be any two entire functions such that $0 < \lambda_h^{(p,q)}(f) \le \rho_h^{(p,q)}(f) < \infty$. Then for any entire function g with finite (m,q) th order,*

$$\overline{\lim}_{r\to\infty} \frac{\log^{[p+m-q]} \mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[p]} \mu_h^{-1}\mu_f(r)} \le \frac{\rho_g(m,q)}{\lambda_h^{(p,q)}(f)},$$

where p, q and m are all positive integers with $m > q$.

Proof. Since $q < m$, we get from(24) for all sufficiently large values of r that

$$\log^{[p+m-q]} \mu_h^{-1} \mu_{f \circ g}(r) \leqslant \log^{[m]} \mu_g(\beta r) + O(1)$$

$$i.e., \ \frac{\log^{[p+m-q]} \mu_h^{-1} \mu_{f \circ g}(r)}{\log^{[p]} \mu_h^{-1} \mu_f(r)} \leq \frac{\log^{[m]} \mu_g(\beta r) + O(1)}{\log^{[q]} r} \cdot \frac{\log^{[q]} r}{\log^{[p]} \mu_h^{-1} \mu_f(r)}$$

$$i.e., \ \overline{\lim_{r \to \infty}} \frac{\log^{[p+m-q]} \mu_h^{-1} \mu_{f \circ g}(r)}{\log^{[p]} \mu_h^{-1} \mu_f(r)} \leq \overline{\lim_{r \to \infty}} \frac{\log^{[m]} \mu_g(\beta r) + O(1)}{\log^{[q]}(\beta r) + O(1)} \cdot \overline{\lim_{r \to \infty}} \frac{\log^{[q]} r}{\log^{[p]} \mu_h^{-1} \mu_f(r)}$$

$$i.e., \ \overline{\lim_{r \to \infty}} \frac{\log^{[p+m-q]} \mu_h^{-1} \mu_{f \circ g}(r)}{\log^{[p]} \mu_h^{-1} \mu_f(r)} \leq \rho_g(m, q) \cdot \frac{1}{\lambda_h^{(p,q)}(f)} = \frac{\rho_g(m, q)}{\lambda_h^{(p,q)}(f)} .$$

This proves the theorem. □

In the line of Theorem 9 we may state the following theorem without proof.

THEOREM 10. *Let f, g and h be any three entire functions satisfying (i) $\rho_h^{(p,q)}(f) < \infty$, (ii) $\lambda_h^{(p,n)}(g) > 0$ and (iii) $\rho_g(m, n) < \infty$.Then*

$$\overline{\lim_{r \to \infty}} \frac{\log^{[p+m-q]} \mu_h^{-1} \mu_{f \circ g}(r)}{\log^{[p]} \mu_h^{-1} \mu_g(r)} \leq \frac{\rho_g(m, n)}{\lambda_h^{(p,n)}(g)},$$

where p, q, m and n are all positive integers with $m > n$.

REMARK 7. The same results of above theorems, remarks and corollary in terms of maximum modulus of entire functions can also be deduced with the help of Lemma 1.

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