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A GENERALIZATION OF OSTROWSKI-TYPE INEQUALITY

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Abstract. A generalization of Ostrowski-type inequality involving functions of two independent variables is given.

1. Introduction

In 1938, Ostrowski established the following inequality which can be used to estimate the absolute deviation of a function form its integral mean.

Theorem 1.1. *Let $f : I \to R$ be a differentiable mapping in the interior $intI$ of I, where $I \subset R$ is an interval, and let $a, b \in intI$ with $a < b$. If $\mid f'(t) \mid \leq M, t \in [a, b]$, then we have*

$$\mid f(x) - \frac{1}{b-a}\int_a^b f(t)dt \mid \leq [\frac{1}{4} + \frac{(x - \frac{(a+b)}{2})^2}{(b-a)^2}](b-a)^2 M, x \in [a, b].$$

In [1], Ujević has showed the following Ostrowski-type inequality:

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THEOREM 1.2. *Let $f : [a,b] \to R$ be a differentiable mapping. If $\gamma \le f'(t) \le \Gamma, \forall t \in [a,b]$, then*

$$
\begin{aligned}
&| (b-a)[(1-\lambda)f(x) + \lambda f(\frac{a+b}{2}) - \frac{\Gamma+\gamma}{2}(1-\lambda)(x-\frac{a+b}{2})] \\
&- \int_a^b f(t)dt \,| \le \frac{\Gamma-\gamma}{2}[(\lambda^2+(1-\lambda)^2)\frac{(b-a)^2}{4} + (x-\frac{a+b}{2})^2],
\end{aligned}
$$

where $\lambda \in [0,1]$ and $a + \frac{b-a}{2}\lambda \le x \le b - \frac{b-a}{2}\lambda$.

Recently, many authors have studied Ostrowski-type inequality and obtained some good results [2–6]. Among them, Sarikaya [2] proved the following Ostrowski-type inequality involving functions of two independent variables.

THEOREM 1.3. *Let $f : [a,b] \times [c,d] \to R$ be an absolutely continuous function such that the partial derivatives of order 2 exists and $\gamma \le \frac{\partial^2 f(t,s)}{\partial t \partial s} \le \Gamma, \forall (t,s) \in [a,b] \times [c,d]$, then*

(1)
$$
\begin{aligned}
&|\frac{1}{4}f(x,y) + \frac{1}{4}G(x,y) - \frac{1}{2(b-a)}\int_a^b f(t,y)dt - \frac{1}{2(d-c)}\int_c^d f(x,s)ds \\
&\quad - \frac{1}{2(b-a)(d-c)}\int_a^b [(y-c)f(t,c) + (d-y)f(t,d)]dt \\
&\quad - \frac{1}{2(b-a)(d-c)}\int_c^d [(x-a)f(a,s) + (b-x)f(b,s)]dt \\
&\quad + \frac{1}{2(b-a)(d-c)}\int_a^b \int_c^d f(t,s)dsdt \\
&\le \frac{[(x-a)^2+(b-x)^2][(y-c)^2+(d-y)^2]}{32(b-a)(d-c)}(\Gamma-\gamma)
\end{aligned}
$$

for all $(x, y) \in [a, b] \times [c, d]$, *where*

$$\begin{aligned} G(x,y) =& \frac{(x-a)[(y-c)f(a,c)+(d-y)f(a,d)]}{(b-a)(d-c)} \\ &+ \frac{(b-x)[(y-c)f(b,c)+(d-y)f(b,d)]}{(b-a)(d-c)} \\ &+ \frac{(x-a)f(a,y)+(b-x)f(b,y)}{b-a} \\ &+ \frac{(y-c)f(x,c)+(d-y)f(x,d)}{d-c}. \end{aligned}$$

In this paper, our main purpose is to establish a generalization of the inequality (1).

2. Main Result

THEOREM 2.1. *Let* $f : [a, b] \times [c, d] \to R$ *be an absolutely continuous function such that the partial derivative of order 2 exists and* $\gamma \le \frac{\partial^2 f(t,s)}{\partial t \partial s} \le \Gamma, \forall (t, s) \in [a, b] \times [c, d]$, *then*

$$\begin{aligned} (2) \qquad & |(1-\frac{\lambda}{2})^2 f(x,y) + \tilde{G}(x,y) \\ & - (1-\frac{\lambda}{2})[\frac{1}{(b-a)}\int_a^b f(t,y)dt + \frac{1}{(d-c)}\int_c^d f(x,s)ds] \\ & - \frac{\lambda}{2(b-a)(d-c)}\int_a^b [(y-c)f(t,c)+(d-y)f(t,d)]dt \\ & - \frac{\lambda}{2(b-a)(d-c)}\int_c^d [(x-a)f(a,s)+(b-x)f(b,s)]dt \\ & + \frac{1}{2(b-a)(d-c)}\int_a^b \int_c^d f(t,s)dsdt \\ & - \frac{\Gamma+\gamma}{8}(1-\lambda)^2(a+b-2x)(c+d-2y)| \\ \le & \frac{\Gamma-\gamma}{8}(1-\lambda+\frac{\lambda^2}{2})^2 \frac{[(x-a)^2+(b-x)^2][(y-c)^2+(d-y)^2]}{(b-a)(d-c)} \end{aligned}$$

for all $(x,y) \in [a,b] \times [c,d]$ *and* $\lambda \in [0,1]$, *where*

$$\begin{aligned}\tilde{G}(x,y) &= \frac{\lambda}{2}(1-\frac{\lambda}{2})\frac{(x-a)[(y-c)f(a,c)+(d-y)f(a,d)]}{(b-a)(d-c)} \\ &+ \frac{\lambda}{2}(1-\frac{\lambda}{2})\frac{(b-x)[(y-c)f(b,c)+(d-y)f(b,d)]}{(b-a)(d-c)} \\ &+ (\frac{\lambda}{2})^2\frac{(x-a)f(a,y)+(b-x)f(b,y)}{b-a} \\ &+ (\frac{\lambda}{2})^2\frac{(y-c)f(x,c)+(d-y)f(x,d)}{d-c}.\end{aligned}$$

Proof. We define the mappings $p : [a,b] \times [a,b] \to R$ and $q : [c,d] \times [c,d] \to R$ given by

$$p(x,t) = \begin{cases} t-(a+\lambda\frac{x-a}{2}), & t \in [a,x], \\ t-(b-\lambda\frac{b-x}{2}), & t \in (x,b] \end{cases}$$

and

$$q(y,s) = \begin{cases} s-(c+\lambda\frac{y-c}{2}), & t \in [c,y], \\ s-(d-\lambda\frac{d-y}{2}), & t \in (y,d]. \end{cases}$$

Then, we have

$$\begin{aligned}&\int_a^b\int_c^d p(x,t)q(y,s)\frac{\partial^2 f(t,s)}{\partial t\partial s}dsdt \qquad (3)\\ &= \int_a^x\int_c^y [t-(a+\lambda\frac{x-a}{2})][s-(c+\lambda\frac{y-c}{2})]\frac{\partial^2 f(t,s)}{\partial t\partial s}dsdt \\ &+ \int_a^x\int_y^d [t-(a+\lambda\frac{x-a}{2})][s-(d-\lambda\frac{d-y}{2})]\frac{\partial^2 f(t,s)}{\partial t\partial s}dsdt \\ &+ \int_x^b\int_c^y [t-(b-\lambda\frac{b-x}{2})][s-(c+\lambda\frac{y-c}{2})]\frac{\partial^2 f(t,s)}{\partial t\partial s}dsdt \\ &+ \int_x^b\int_y^d [t-(b-\lambda\frac{b-x}{2})][s-(d-\lambda\frac{d-y}{2})]\frac{\partial^2 f(t,s)}{\partial t\partial s}dsdt.\end{aligned}$$

Integrating by parts twice, we have

(4)
$$
\begin{aligned}
&\int_a^x \int_c^y [t-(a+\lambda\frac{x-a}{2})][s-(c+\lambda\frac{y-c}{2})]\frac{\partial^2 f(t,s)}{\partial t\partial s}dsdt\\
=&(1-\frac{\lambda}{2})(y-c)[(1-\frac{\lambda}{2})(x-a)f(x,y)+\frac{\lambda}{2}(x-a)f(a,y)-\int_a^x f(t,y)dt]\\
&+\frac{\lambda}{2}(y-c)[(1-\frac{\lambda}{2})(x-a)f(x,c)+\frac{\lambda}{2}(x-a)f(a,c)-\int_a^x f(t,c)dt]\\
&-(x-a)[(1-\frac{\lambda}{2})\int_c^y f(x,s)ds+\frac{\lambda}{2}\int_c^y f(a,s)ds]+\int_a^x\int_c^y f(t,s)dsdt.
\end{aligned}
$$

(5)
$$
\begin{aligned}
&\int_a^x \int_y^d [t-(a+\lambda\frac{x-a}{2})][s-(d-\lambda\frac{d-y}{2})]\frac{\partial^2 f(t,s)}{\partial t\partial s}dsdt\\
=&\frac{\lambda}{2}(d-y)[(1-\frac{\lambda}{2})(x-a)f(x,d)+\frac{\lambda}{2}(x-a)f(a,d)-\int_a^x f(t,d)dt]\\
&+(1-\frac{\lambda}{2})(d-y)[(1-\frac{\lambda}{2})(x-a)f(x,y)+\frac{\lambda}{2}(x-a)f(a,y)-\int_a^x f(t,y)dt]\\
&-(x-a)[(1-\frac{\lambda}{2})\int_y^d f(x,s)ds-\frac{\lambda}{2}\int_y^d f(a,s)ds]+\int_a^x\int_y^d f(t,s)dsdt.
\end{aligned}
$$

(6)
$$
\begin{aligned}
&\int_x^b \int_c^y [t-(b-\lambda\frac{b-x}{2})][s-(c+\lambda\frac{y-c}{2})]\frac{\partial^2 f(t,s)}{\partial t\partial s}dsdt\\
=&(1-\frac{\lambda}{2})(y-c)[\frac{\lambda}{2}(b-x)f(b,y)+(1-\frac{\lambda}{2})(b-x)f(x,y)-\int_x^b f(t,y)dt]\\
&+\frac{\lambda}{2}(y-c)[\frac{\lambda}{2}(b-x)f(x,c)+(1-\frac{\lambda}{2})(b-x)f(x,c)-\int_x^b f(t,c)dt]\\
&-(b-x)[\frac{\lambda}{2}\int_c^y f(b,s)ds+(1-\frac{\lambda}{2})\int_c^y f(x,s)ds]+\int_x^b\int_c^y f(t,s)dsdt.
\end{aligned}
$$

(7)

$$
\begin{aligned}
&\int_x^b \int_y^d [t-(b-\lambda\frac{b-x}{2})][s-(d-\lambda\frac{d-y}{2})]\frac{\partial^2 f(t,s)}{\partial t\partial s}dsdt \\
=&\frac{\lambda}{2}(d-y)[\frac{\lambda}{2}(b-x)f(b,d)+(1-\frac{\lambda}{2})(b-x)f(x,d)-\int_x^b f(t,d)dt] \\
&+(1-\frac{\lambda}{2})(d-y)[\frac{\lambda}{2}(b-x)f(b,y)+(1-\frac{\lambda}{2})(b-x)f(x,y)-\int_x^b f(t,y)dt] \\
&-(b-x)[\frac{\lambda}{2}\int_y^d f(b,s)ds-(1-\frac{\lambda}{2})\int_y^d f(x,s)ds]+\int_x^b\int_y^d f(t,s)dsdt.
\end{aligned}
$$

Summing up (4)-(7), we get

$$
\begin{aligned}
(8)\qquad &\int_a^b\int_c^d p(x,t)q(y,s)\frac{\partial^2 f(t,s)}{\partial t\partial s}dsdt \\
=&(1-\frac{\lambda}{2})^2(b-a)(d-c)f(x,y) \\
&+\frac{\lambda}{2}(1-\frac{\lambda}{2})[(x-a)f(a,y)+(b-x)f(b,y)](d-c) \\
&+\frac{\lambda}{2}(1-\frac{\lambda}{2})[(y-c)f(x,c)+(d-y)f(x,d)](b-a) \\
&+(\frac{\lambda}{2})^2[(y-c)f(a,c)+(d-y)f(a,d)](x-a) \\
&+(\frac{\lambda}{2})^2[(y-c)f(b,c)+(d-y)f(b,d)](b-x) \\
&-(1-\frac{\lambda}{2})(d-c)\int_a^b f(t,y)dt-(1-\frac{\lambda}{2})(b-a)\int_c^d f(x,s)ds \\
&-\frac{\lambda}{2}\int_a^b[(y-c)f(t,c)+(d-y)f(t,d)]dt \\
&-\frac{\lambda}{2}\int_c^d[(x-a)f(a,s)ds+(b-x)f(b,s)]ds \\
&+\int_a^b\int_c^d f(t,s)dsdt.
\end{aligned}
$$

We also have

$$\int_a^b \int_c^d p(x,t)q(y,s)dsdt = \frac{1}{4}(1-\lambda)^2(b-a)(d-c)(a+b-2x)(c+d-2y). \tag{9}$$

Let $C = \frac{\Gamma+\gamma}{2}$. Form (9), we get

$$\begin{aligned} &\int_a^b \int_c^d p(x,t)q(y,s)(\frac{\partial^2 f(t,s)}{\partial t \partial s} - C)dsdt \\ =&\int_a^b \int_c^d p(x,t)q(y,s)\frac{\partial^2 f(t,s)}{\partial t \partial s}dsdt \\ &- \frac{\Gamma+\gamma}{8}(1-\lambda)^2(b-a)(d-c)(a+b-2x)(c+d-2y). \end{aligned} \tag{10}$$

On the other hand, we have

$$\begin{aligned} &|\int_a^b \int_c^d p(x,t)q(y,s)(\frac{\partial^2 f(t,s)}{\partial t \partial s} - C)dsdt| \\ \leq &\max_{(t,s)\in[a,b]\times[c,d]} |\frac{\partial^2 f(t,s)}{\partial t \partial s} - C| \int_a^b \int_c^d |p(x,t)q(y,s)|dsdt. \end{aligned} \tag{11}$$

We also have

$$\max_{(t,s)\in[a,b]\times[c,d]} |\frac{\partial^2 f(t,s)}{\partial t \partial s} - C| \leq \frac{\Gamma-\gamma}{2}. \tag{12}$$

and

$$\begin{aligned} &\int_a^b \int_c^d |p(x,t)q(y,s)|dsdt \\ =&\frac{1}{4}(1-\lambda+\frac{\lambda^2}{2})^2[(x-a)^2+(b-x)^2][(y-c)^2+(d-y)^2]. \end{aligned} \tag{13}$$

By (11) to (13),we have

$$\begin{aligned} &|\int_a^b \int_c^d p(x,t)q(y,s)(\frac{\partial^2 f(t,s)}{\partial t \partial s} - C)dsdt| \\ \leq&\frac{\Gamma-\gamma}{8}(1-\lambda+\frac{\lambda^2}{2})^2[(x-a)^2+(b-x)^2][(y-c)^2+(d-y)^2]. \end{aligned} \tag{14}$$

Form (10) and (14), we get (2). □

Remark 2.2. We choose $\lambda = 0$ and $x = \frac{a+b}{2}, y = \frac{c+d}{2}$ in (2) to obtain [3, Corollary 1] .

Remark 2.3. We choose $\lambda = 1$ in (2) to obtain the inequality (1).

Corollary 2.4. *Let the assumptions of Theorem 2.1 hold. We have*

$$
\begin{aligned}
(15)\qquad & |9f(x,y) + H(x,y) \\
& - [\frac{12}{(b-a)} \int_a^b f(t,y)dt + \frac{12}{(d-c)} \int_c^d f(x,s)ds] \\
& - \frac{4}{(b-a)(d-c)} \int_a^b [(y-c)f(t,c) + (d-y)f(t,d)]dt \\
& - \frac{4}{(b-a)(d-c)} \int_c^d [(x-a)f(a,s) + (b-x)f(b,s)]dt \\
& + \frac{8}{(b-a)(d-c)} \int_a^b \int_c^d f(t,s)]dsdt \\
& - \frac{\Gamma+\gamma}{2}(a+b-2x)(c+d-2y)| \\
\leq & \frac{25(\Gamma-\gamma)}{32} \frac{[(x-a)^2 + (b-x)^2][(y-c)^2 + (d-y)^2]}{(b-a)(d-c)}
\end{aligned}
$$

for all $(x,y) \in [a,b] \times [c,d]$ *where*

$$
\begin{aligned}
H(x,y) = & \frac{3(x-a)[(y-c)f(a,c) + (d-y)f(a,d)]}{(b-a)(d-c)} \\
& + \frac{3(b-x)[(y-c)f(b,c) + (d-y)f(b,d)]}{(b-a)(d-c)} \\
& + \frac{(x-a)f(a,y) + (b-x)f(b,y)}{b-a} \\
& + \frac{(y-c)f(x,c) + (d-y)f(x,d)}{d-c}.
\end{aligned}
$$

and in particular

$$
\begin{aligned}
(16) \qquad & |36f(\frac{a+b}{2},\frac{c+d}{2}) + I(\frac{a+b}{2},\frac{c+d}{2}) \\
& - 48[\frac{1}{(b-a)}\int_a^b f(t,\frac{c+d}{2})dt + \frac{1}{(d-c)}\int_c^d f(\frac{a+b}{2},s)ds] \\
& - \frac{8}{(b-a)}\int_a^b [f(t,c)+f(t,d)]dt \\
& - \frac{8}{(d-c)}\int_c^d [f(a,s)+f(b,s)]dt \\
& + \frac{32}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)]dsdt| \\
\leq & \frac{25(\Gamma-\gamma)}{32}(b-a)(d-c)
\end{aligned}
$$

for all $(x, y) \in [a, b] \times [c, d]$ *where*

$$
\begin{aligned}
I(\frac{a+b}{2},\frac{c+d}{2}) &= 3[f(a,c)+f(a,d)] + 3[f(b,c)+f(b,d)] \\
&+ 2[f(a,\frac{c+d}{2}) + f(b,\frac{c+d}{2})] + 2[f(\frac{a+b}{2},c) + f(\frac{a+b}{2},d)].
\end{aligned}
$$

Proof. We choose $\lambda = \frac{1}{2}$ in (2) to obtain (15) and $x = \frac{a+b}{2}, y = \frac{c+d}{2}$ in (15) to obtain (16). □

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