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SURFACES GENERATED VIA THE EVOLUTION OF SPHERICAL IMAGE OF A SPACE CURVE

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Abstract. In this paper, we linked the motion of spherical images with the motion of their curves. Surfaces generated by the evolution of spherical image of a space curve are constructed. Also geometric proprieties of these surfaces are obtained.

1. Introduction

The authors [1–5] studied the motion of curves specified by

$$\frac{\partial \vec{\mathbf{x}}}{\partial t} = \alpha\vec{\mathbf{T}} + \beta\vec{\mathbf{N}} + \gamma\vec{\mathbf{B}}, \tag{1}$$

where α , β and γ are depending on the local values of curvatures and $\vec{\mathbf{T}}$, $\vec{\mathbf{N}}$ and $\vec{\mathbf{B}}$ are the unit tangent, principal normal and binormal vectors along the curve. The above authors obtained the evolution equations of the curvatures and constructed the evolving curve from its curvatures. Takeya Tsurumi et al. [6] studied the motions of curves specified by accelerations

$$\frac{\partial^2 \vec{\mathbf{x}}}{\partial t^2} = E\vec{\mathbf{T}} + F\vec{\mathbf{N}} + G\vec{\mathbf{B}}, \tag{2}$$

where E, F and G are the tangential, normal and binormal accelerations. They obtained six partial differential equations governing the motion of

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curves and showed that for a given (E, F, G) the motion of the curve is determined from these equations.

D. Y. Kwon and F. C. Park [7, 8] studied the evolutions of an inextensible flow of plane and space curves. They obtained the partial differential equation that governing the flow of that curves.

Curves associated to a space curve in three dimensional space have been discussed by many authors in recent years. Among these curves, the most studied ones are Mannheim partner curves, Bertrand curve couples, involute-evolute curve couples and spherical image of a space curve [9–17].

Talat Körpinar and Essin Turhan [18, 19] studied the surfaces generated via the binormal spherical image of a space curve. They obtained the time evolution equations for the orthogonal triad of binormal spherical image as a curve evolving on the sphere and constructed the first fundamental form, seconde fundamental forms, Gussian curvature and mean curvature of these surfaces.

If the curve moves with time, then spherical image of that curve which generated by the triad $(\vec{\mathbf{T}}, \vec{\mathbf{N}}, \vec{\mathbf{B}})$ evolves on a sphere. In this paper, we linked the motion of curves with the motion of its spherical image using a method different from the one proposed by Talat [18]. Time evolution equations for the curvature and torsion are obtained for the space curve. Surfaces generated by the evolution of the spherical image of the space curve are constructed.

This paper is organized as follows: In Section 2, we introduce the differential geometry of spherical images of a space curve . In section 3, we explain the proposed method for the evolving curves by two sets of Frenet frame. In section 4, surfaces generated by the evolution of spherical image of a space curve are constructed. Also geometric proprieties of these surfaces are obtained.

2. Differential geometry of spherical images of a space curve

In this section, we present the representation of the Frenet frame $(\vec{\mathbf{T}}, \vec{\mathbf{N}}, \vec{\mathbf{B}})$, curvature and torsion for spherical images of the curve in terms of the quantities associated with the curve. let $\vec{\mathbf{r}} = \vec{\mathbf{r}}(s)$ be a regular space curve in E^3 parameterized by s. The relations [20]

$$\frac{d}{ds}\begin{pmatrix}\vec{\mathbf{T}}\\ \vec{\mathbf{N}}\\ \vec{\mathbf{B}}\end{pmatrix}=\begin{pmatrix}0 & \kappa & 0\\ -\kappa & 0 & \tau\\ 0 & -\tau & 0\end{pmatrix}\begin{pmatrix}\vec{\mathbf{T}}\\ \vec{\mathbf{N}}\\ \vec{\mathbf{B}}\end{pmatrix} \tag{3}$$

are known as the Frenet-Serret formula, where $\vec{\mathbf{T}}$, $\vec{\mathbf{N}}$, $\vec{\mathbf{B}}$, κ and τ represent the tangent, the principal normal, the binormal, the curvature and the torsion of the curve, respectively.

Definition 2.1. The following space curves lies on a unit sphere [21]

$$\begin{aligned}\alpha_1 &= \vec{\mathbf{r}}_1(s_1)=\vec{\mathbf{T}},\\ \alpha_2 &= \vec{\mathbf{r}}_2(s_2)=\vec{\mathbf{N}},\\ \alpha_2 &= \vec{\mathbf{r}}_3(s_3)=\vec{\mathbf{B}},\end{aligned} \tag{4}$$

and called the spherical image of the tangent, the normal and the binormal to the curve.

Visualization of spherical images of the space curve are plotted in figure 1.

2.1. Spherical image of $\vec{\mathbf{T}}$. The Frenet frame $(\vec{\mathbf{T}}_1,\vec{\mathbf{N}}_1,\vec{\mathbf{B}}_1)$ of the curve α_1

$$\vec{\mathbf{r}}_1(s_1)=\vec{\mathbf{T}}, \tag{5}$$

is calculated and given by

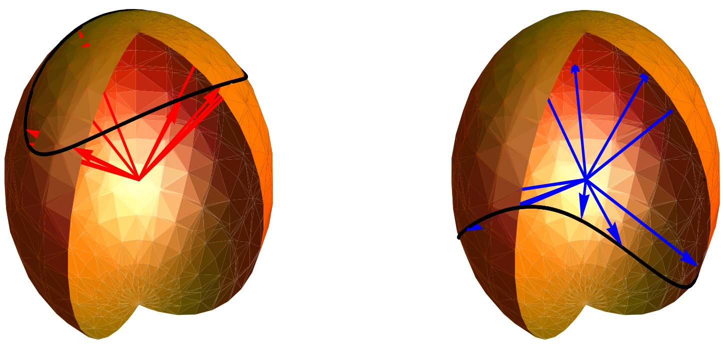
$$\begin{pmatrix}\vec{\mathbf{T}}_1\\ \vec{\mathbf{N}}_1\\ \vec{\mathbf{B}}_1\end{pmatrix}=\begin{pmatrix}0 & 1 & 0\\ \zeta_1 & 0 & \zeta_2\\ \zeta_2 & 0 & \zeta_1\end{pmatrix}\begin{pmatrix}\vec{\mathbf{T}}\\ \vec{\mathbf{N}}\\ \vec{\mathbf{B}}\end{pmatrix}, \tag{6}$$

where

$$\begin{aligned}\zeta_1 &= \frac{\kappa}{\sqrt{\kappa^2+\tau^2}},\\ \zeta_2 &= \frac{\tau}{\sqrt{\kappa^2+\tau^2}}.\end{aligned} \tag{7}$$

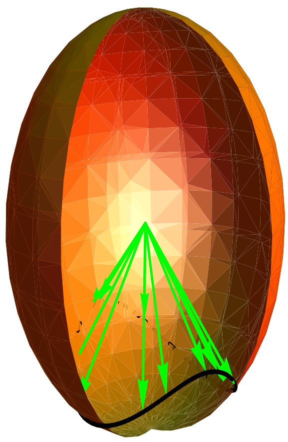
The curvature κ_1 and the torsion τ_1 are calculated and given by

$$\begin{aligned}\kappa_1 &= \frac{\sqrt{\kappa^2+\tau^2}}{\kappa},\\ \tau_1 &= \frac{-\kappa\tau'+\kappa'\tau}{\kappa\big(\kappa^2+\tau^2\big)},\end{aligned} \tag{8}$$



(a) Tangent spherical image

(b) Normal spherical image



(c) Binormal spherical image

FIGURE 1. Spherical images of the triad $(\vec{\mathbf{T}}, \vec{\mathbf{N}}, \vec{\mathbf{B}})$ for the curve $\vec{\mathbf{r}} = (\sin s, \cos s + \sin s, \cos s)$

where $'$ denoted to the derivative with respect to s.

2.2. Spherical image of $\vec{\mathbf{N}}$. The Frenet frame $(\vec{\mathbf{T}}_2, \vec{\mathbf{N}}_2, \vec{\mathbf{B}}_2)$ of the curve α_2

$$\vec{\mathbf{r}}_2(s_2) = \vec{\mathbf{N}}, \tag{9}$$

is calculated and given by

$$\begin{pmatrix} \vec{\mathbf{T}}_2 \\ \vec{\mathbf{N}}_2 \\ \vec{\mathbf{B}}_2 \end{pmatrix} = \begin{pmatrix} \zeta_1 & 0 & \zeta_2 \\ \zeta_3 & \zeta_4 & \zeta_5 \\ \zeta_6 & \zeta_7 & \zeta_8 \end{pmatrix} \begin{pmatrix} \vec{\mathbf{T}} \\ \vec{\mathbf{N}} \\ \vec{\mathbf{B}} \end{pmatrix}, \tag{10}$$

where

$$\begin{aligned}
\zeta_1 &= \frac{\kappa}{\sqrt{\kappa^2+\tau^2}}, \\
\zeta_2 &= \frac{\tau}{\sqrt{\kappa^2+\tau^2}}, \\
\zeta_3 &= \frac{\tau(-\tau\kappa'+\kappa\tau')}{\sqrt{(\kappa^2+\tau^2)^4+(-\tau\kappa'+\kappa\tau')(\kappa^2+\tau^2)}}, \\
\zeta_4 &= -\frac{(\kappa^2+\tau^2)^{3/2}}{\sqrt{(\kappa^2+\tau^2)^3+(\tau\kappa'-\kappa\tau')^2}}, \\
\zeta_5 &= \frac{\kappa(-\tau\kappa'+\kappa\tau')}{\sqrt{(\kappa^2+\tau^2)^3+(\tau\kappa'-\kappa\tau')^2)}}, \\
\zeta_6 &= \frac{\tau(\kappa^2+\tau^2)}{\sqrt{(\kappa^2+\tau^2)^3+(\tau\kappa'-\kappa\tau')^2}}, \\
\zeta_7 &= \frac{-\tau\kappa'+\kappa\tau'}{\sqrt{(\kappa^2+\tau^2)^3+(\tau\kappa'-\kappa\tau')^2}}, \\
\zeta_8 &= \frac{\kappa(\kappa^2+\tau^2)}{\sqrt{(\kappa^2+\tau^2)^3+(\tau\kappa'-\kappa\tau')^2}}.
\end{aligned} \tag{11}$$

The curvature κ_2 and the torsion τ_2 are calculated and given by

(12)

$$\begin{aligned}
\kappa_2 &= \frac{\sqrt{(\kappa^3+\kappa\tau^2)^2+(\kappa^2\tau+\tau^3)^2+(\tau\kappa'-\kappa\tau')^2}}{(\kappa^2+\tau^2)^{3/2}}, \\
\tau_2 &= \frac{\tau^2(3\kappa'\tau'-\tau\kappa'')-\kappa^2(3\kappa'\tau'+\tau\kappa'')+\kappa^3\tau''+\kappa\tau(3\kappa'^2-3\tau'^2+\tau\tau')}{(\kappa^3+\kappa\tau^2)^2+(\kappa^2\tau+\tau^3)^2+(\tau\kappa'-\kappa\tau')^2}.
\end{aligned}$$

2.3. Spherical image of $\vec{\mathbf{B}}$. The Frenet frame $(\vec{\mathbf{T}}_3, \vec{\mathbf{N}}_3, \vec{\mathbf{B}}_3)$ of the curve α_3

$$\vec{\mathbf{r}}_3(s_3) = \vec{\mathbf{B}}, \tag{13}$$

is calculated and given by

$$\begin{pmatrix} \vec{\mathbf{T}}_3 \\ \vec{\mathbf{N}}_3 \\ \vec{\mathbf{B}}_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ \zeta_1 & 0 & -\zeta_2 \\ \zeta_2 & 0 & \zeta_1 \end{pmatrix} \begin{pmatrix} \vec{\mathbf{T}} \\ \vec{\mathbf{N}} \\ \vec{\mathbf{B}} \end{pmatrix}, \tag{14}$$

where

$$\begin{aligned} \zeta_1 &= \frac{\kappa}{\sqrt{\kappa^2 + \tau^2}}, \\ \zeta_2 &= \frac{\tau}{\sqrt{\kappa^2 + \tau^2}}. \end{aligned} \tag{15}$$

The curvature κ_3 and the torsion τ_3 are calculated and given by

$$\begin{aligned} \kappa_3 &= \frac{\sqrt{\kappa^2 + \tau^2}}{\tau}, \\ \tau_3 &= \frac{-\kappa\tau' + \kappa'\tau}{\tau(\kappa^2 + \tau^2)}. \end{aligned} \tag{16}$$

3. Time evolution equation of a space curve

In this section, we derive the time evolution equations that the intrinsic quantities of curves satisfy.

If the curve evolves in time t, then $\vec{\mathbf{r}} = \vec{\mathbf{r}}(s,t)$, where s is the arc-length and t represent time evolution in the space.

The moving frame $(\vec{\mathbf{T}}, \vec{\mathbf{N}}, \vec{\mathbf{B}})$ vary along the curve according to Serret-Frenet equations

$$\frac{\partial}{\partial s} \begin{pmatrix} \vec{\mathbf{T}} \\ \vec{\mathbf{N}} \\ \vec{\mathbf{B}} \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} \vec{\mathbf{T}} \\ \vec{\mathbf{N}} \\ \vec{\mathbf{B}} \end{pmatrix}. \tag{17}$$

If the curve moves withe time t in the space, then the frame $(\vec{\mathbf{T}}, \vec{\mathbf{N}}, \vec{\mathbf{B}})$ evolves over the curve according to [22]

$$\frac{\partial}{\partial t}\begin{pmatrix}\vec{\mathbf{T}}\\ \vec{\mathbf{N}}\\ \vec{\mathbf{B}}\end{pmatrix} = \begin{pmatrix}0 & \rho_1 & \rho_2\\ -\rho_1 & 0 & \rho_3\\ -\rho_2 & -\rho_3 & 0\end{pmatrix}\begin{pmatrix}\vec{\mathbf{T}}\\ \vec{\mathbf{N}}\\ \vec{\mathbf{B}}\end{pmatrix}, \tag{18}$$

where the parameters ρ_1, ρ_2 and ρ_3 are function of s and t.

Applying the compatibility condition

$$\frac{\partial}{\partial s}\frac{\partial}{\partial t}\begin{pmatrix}\vec{\mathbf{T}}\\ \vec{\mathbf{N}}\\ \vec{\mathbf{B}}\end{pmatrix} = \frac{\partial}{\partial t}\frac{\partial}{\partial s}\begin{pmatrix}\vec{\mathbf{T}}\\ \vec{\mathbf{N}}\\ \vec{\mathbf{B}}\end{pmatrix}, \tag{19}$$

a short calculation using Eqs. (17) and (18) leads to

(20)
$$\begin{pmatrix}0 & (\frac{\partial\kappa}{\partial t} - \frac{\partial\rho_1}{\partial s} + \tau\rho_2) & (\frac{\partial\rho_2}{\partial s} - \kappa\rho_3 + \tau\rho_1)\\ -(\frac{\partial\kappa}{\partial t} - \frac{\partial\rho_1}{\partial s} + \tau\rho_2) & 0 & (\frac{\partial\tau}{\partial t} - \frac{\partial\rho_3}{\partial s} - \kappa\rho_2)\\ -(\frac{\partial\rho_2}{\partial s} - \kappa\rho_3 + \tau\rho_1) & -(\frac{\partial\tau}{\partial t} - \frac{\partial\rho_3}{\partial s} - \kappa\rho_2) & 0\end{pmatrix} = 0_{3\times 3}.$$

Thus the compatibility conditions become

$$\begin{aligned}\frac{\partial\kappa}{\partial t} &= \frac{\partial\rho_1}{\partial s} - \tau\rho_2,\\ \frac{\partial\tau}{\partial t} &= \frac{\partial\rho_3}{\partial s} + \kappa\rho_2,\\ \frac{\partial\rho_2}{\partial s} &= \kappa\rho_3 - \tau\rho_1.\end{aligned} \tag{21}$$

The curvature and torsion evolves according to

$$\begin{aligned}\frac{\partial\kappa}{\partial t} &= \frac{\partial\rho_1}{\partial s} - \rho_2\tau,\\ \frac{\partial\tau}{\partial t} &= \frac{\partial}{\partial s}\left(\frac{\frac{\partial\rho_2}{\partial s} + \tau\rho_1}{\kappa}\right) + \kappa\rho_2.\end{aligned} \tag{22}$$

4. Surfaces generated by the evolution of the spherical image of a space curve

In this section, we shall study the surfaces generated by the evolution of the spherical image of the tangent, spherical image of the normal and spherical image of the binormal to the curve.

4.1. Surfaces generated via spherical image of the tangent vector field. The equation of surfaces generated by the tangent spherical image is given by

$$\vec{\psi} = \vec{\mathbf{T}}(s,t). \tag{23}$$

The tangent space to the surface $\vec{\psi}$ is given by

$$\begin{aligned} \vec{\psi}_s &= \kappa \vec{\mathbf{N}}, \\ \vec{\psi}_t &= \alpha \vec{\mathbf{N}} + \beta \vec{\mathbf{B}}, \end{aligned} \tag{24}$$

where the subscripts denote partial differentiation.

The unit normal to $\vec{\psi}$ is given by

$$\vec{\mathbf{N}}_\psi = \frac{\vec{\psi}_s \wedge \vec{\psi}_t}{|\vec{\psi}_t \wedge \vec{\psi}_t|} = \vec{\mathbf{T}}. \tag{25}$$

Using equations (17), (18) and (24), the second derivative are calculated and given by

$$\begin{aligned} \vec{\psi}_{ss} &= -\kappa^2 \vec{\mathbf{T}} + \kappa\tau \vec{\mathbf{B}} + \kappa_s \vec{\mathbf{N}}, \\ \vec{\psi}_{tt} &= (\beta^2 - \alpha^2)\vec{\mathbf{T}} + (\alpha_t - \beta\gamma)\vec{\mathbf{N}} + (\alpha\gamma + \beta_t)\vec{\mathbf{B}}, \\ \vec{\psi}_{st} &= (\beta^2 - \alpha^2)\vec{\mathbf{T}} + (\alpha_t - \beta\gamma)\vec{\mathbf{N}} + (\alpha\gamma + \beta_t)\vec{\mathbf{B}}. \end{aligned} \tag{26}$$

If we compute components of the first fundamental form g_{ij}, we have

$$\begin{aligned} g_{11} &= \vec{\psi}_s \wedge \vec{\psi}_s = \kappa^2, \\ g_{12} &= \vec{\psi}_s \wedge \vec{\psi}_t = \alpha\kappa, \\ g_{22} &= \vec{\psi}_t \wedge \vec{\psi}_t = \alpha^2 + \beta^2. \end{aligned} \tag{27}$$

If we compute components of the seconde fundamental form l_{ij}, we have

$$\begin{aligned} l_{11} &= \vec{\psi}_{ss} \wedge \vec{\mathbf{N}}_\psi = -\kappa^2, \\ l_{12} &= \vec{\psi}_{st} \wedge \vec{\mathbf{N}}_\psi = -\alpha\kappa, \\ l_{22} &= \vec{\psi}_{tt} \wedge \vec{\mathbf{N}}_\psi = -\alpha^2 + \beta^2. \end{aligned} \tag{28}$$

The Gaussian curvature K_1 and the mean curvature H_1 are calculated and given by

$$\begin{aligned} K_1 &= \frac{l_{11}l_{22} - l_{12}^2}{g_{11}g_{22} - g_{12}^2} = -1, \\ H_1 &= \frac{l_{11}g_{22} - 2l_{12}g_{12} + l_{22}g_{11}}{2(g_{11}g_{22} - g_{12}^2)} = 0. \end{aligned} \tag{29}$$

The principal curvatures are calculated and given by

$$\begin{aligned} k_{11} &= H + \sqrt{H^2 - K} = 1, \\ k_{21} &= H - \sqrt{H^2 - K} = -1. \end{aligned} \tag{30}$$

4.2. Surfaces generated via spherical image of the normal vector field. The equation of surfaces generated by the normal spherical image is given by

$$\vec{\phi} = \vec{\mathbf{N}}(s,t) \tag{31}$$

The tangent space to the surface $\vec{\phi}$ is given by

$$\begin{aligned} \vec{\phi}_s &= -\kappa\vec{\mathbf{T}} + \tau\vec{\mathbf{N}}, \\ \vec{\phi}_t &= -\alpha\vec{\mathbf{T}} + \gamma\vec{\mathbf{B}}. \end{aligned} \tag{32}$$

The unit normal to $\vec{\phi}$ is given by

$$\vec{\mathbf{N}}_\phi = \frac{\vec{\psi}_s \wedge \vec{\phi}_t}{|\vec{\psi}_t \wedge \vec{\phi}_t|} = \vec{\mathbf{N}}. \tag{33}$$

Using equations (17), (18) and (32), the second derivative are calculated and given by

$$\begin{aligned} \vec{\phi}_{ss} &= -(\kappa^2 + \tau^2)\vec{\mathbf{N}} - \kappa_s\vec{\mathbf{T}} - \tau_s\vec{\mathbf{B}}, \\ \vec{\phi}_{st} &= -(\alpha^2 + \gamma^2)\vec{\mathbf{N}} + (\beta\gamma - \alpha_t)\vec{\mathbf{T}} + (-\alpha\beta + \gamma_t)\vec{\mathbf{B}}, \\ \vec{\phi}_{tt} &= -(\alpha^2 - \gamma^2)\vec{\mathbf{N}} + (\beta\gamma - \alpha_t)\vec{\mathbf{T}} + (-\alpha\beta + \gamma_t)\vec{\mathbf{B}}. \end{aligned} \tag{34}$$

If we compute components of the first fundamental form g_{ij}, we have

$$\begin{aligned} g_{11} &= \vec{\phi}_s \wedge \vec{\phi}_s = \tau^2 + \kappa^2, \\ g_{12} &= \vec{\phi}_s \wedge \vec{\phi}_t = \alpha\kappa + \gamma\tau, \\ g_{22} &= \vec{\phi}_t \wedge \vec{\phi}_t = \alpha^2 + \gamma^2. \end{aligned} \tag{35}$$

If we compute components of the seconde fundamental form l_{ij}, we have

$$\begin{aligned} l_{11} &= \vec{\phi}_{ss} \wedge \vec{\mathbf{N}}_{\phi} = -\tau^2 - \kappa^2, \\ l_{12} &= \vec{\phi}_{st} \wedge \vec{\mathbf{N}}_{\phi} = -\alpha\kappa - \gamma\tau, \\ l_{22} &= \vec{\phi}_{tt} \wedge \vec{\mathbf{N}}_{\phi} = \alpha^2 - \gamma^2. \end{aligned} \tag{36}$$

The Gaussian curvature K_2 and the mean curvature H_2 are calculated and given by

$$\begin{aligned} K_2 &= \frac{l_{11}l_{22} - l_{12}^2}{g_{11}g_{22} - g_{12}^2} = 1, \\ H_2 &= \frac{l_{11}g_{22} - 2l_{12}g_{12} + l_{22}g_{11}}{2(g_{11}g_{22} - g_{12}^2)} = -1. \end{aligned} \tag{37}$$

The principal curvatures are calculated and given by

$$\begin{aligned} k_{12} &= H + \sqrt{H^2 - K} = -1, \\ k_{22} &= H - \sqrt{H^2 - K} = -1. \end{aligned} \tag{38}$$

4.3. Surfaces generated via spherical image of the binormal vector field. The equation of surfaces generated by the binormal spherical image is given by

$$\vec{\varphi} = \vec{\mathbf{B}}(s,t) \tag{39}$$

The tangent space to the surface $\vec{\varphi}$ is given by

$$\begin{aligned} \vec{\varphi}_s &= -\tau\vec{\mathbf{N}}, \\ \vec{\varphi}_t &= \beta\vec{\mathbf{T}} - \gamma\vec{\mathbf{N}}. \end{aligned} \tag{40}$$

The unit normal to $\vec{\varphi}$ is given by

$$\vec{\mathbf{N}}_{\varphi} = \frac{\vec{\varphi}_s \wedge \vec{\varphi}_t}{|\vec{\varphi}_t \wedge \vec{\varphi}_t|} = \vec{\mathbf{B}}. \tag{41}$$

Using equations (17), (18) and (40), the second derivative are calculated and given by

$$\begin{aligned} \vec{\varphi}_{ss} &= \kappa\tau\vec{\mathbf{T}} - \tau^2\vec{\mathbf{B}} - \tau_s\vec{\mathbf{N}}, \\ \vec{\varphi}_{st} &= (\beta^2 - \gamma^2)\vec{\mathbf{B}} + (\alpha\gamma + \beta_t)\vec{\mathbf{T}} + (\alpha\beta - \gamma_t)\vec{\mathbf{N}}, \\ \vec{\varphi}_{tt} &= (\beta^2 - \gamma^2)\vec{\mathbf{B}} + (\alpha\gamma + \beta_t)\vec{\mathbf{T}} + (\alpha\beta - \gamma_t)\vec{\mathbf{N}}. \end{aligned} \tag{42}$$

If we compute components of the first fundamental form g_{ij}, we have

$$\begin{aligned} g_{11} &= \vec{\varphi}_s \wedge \vec{\varphi}_s = \tau^2, \\ g_{12} &= \vec{\varphi}_s \wedge \vec{\varphi}_t = \gamma\tau, \\ g_{22} &= \vec{\varphi}_t \wedge \vec{\varphi}_t = \beta^2 + \gamma^2. \end{aligned} \tag{43}$$

If we compute components of the seconde fundamental form l_{ij}, we have

$$\begin{aligned} l_{11} &= \vec{\varphi}_{ss} \wedge \vec{\mathbf{N}}_\varphi = -\tau^2, \\ l_{12} &= \vec{\varphi}_{st} \wedge \vec{\mathbf{N}}_\varphi = -\gamma\tau, \\ l_{22} &= \vec{\varphi}_{tt} \wedge \vec{\mathbf{N}}_\varphi = \beta^2 - \gamma^2. \end{aligned} \tag{44}$$

The Gaussian curvature K_3 and the mean curvature H_3 are calculated and given by

$$\begin{aligned} K_3 &= \frac{l_{11}l_{22} - l_{12}^2}{g_{11}g_{22} - g_{12}^2} = -1, \\ H_3 &= \frac{l_{11}g_{22} - 2l_{12}g_{12} + l_{22}g_{11}}{2(g_{11}g_{22} - g_{12}^2)} = 0. \end{aligned} \tag{45}$$

The principal curvatures are calculated and given by

$$\begin{aligned} k_{13} &= H + \sqrt{H^2 - K} = 1, \\ k_{23} &= H - \sqrt{H^2 - K} = -1. \end{aligned} \tag{46}$$

5. conclusion

In this paper, we linked the motion of spherical images with the motion of their curves. Surfaces generated by the evolution of spherical image of a space curve are constructed. Also geometric proprieties of these surface are obtained.

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