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SIMPLIFYING AND FINDING ORDINARY DIFFERENTIAL EQUATIONS IN TERMS OF THE STIRLING NUMBERS

Feng Qi*, Jing-Lin Wang, and Bai-Ni Guo

Abstract. In the paper, by virtue of techniques in combinatorial analysis, the authors simplify three families of nonlinear ordinary differential equations in terms of the Stirling numbers of the first kind and establish a new family of nonlinear ordinary differential equations in terms of the Stirling numbers of the second kind.

1. Motivation and main results

In [3, Theorem 1], it was acquired inductively and recursively that the nonlinear differential equations

$$G^{(n)}(t) = \frac{(-1)^n (n-1)!}{(1+t)^n} \sum_{j=2}^{n+1} (j-1)! H_{n-1,j-2} G^j(t) \tag{1}$$

for $n \in \mathbb{N}$ have a solution $G(t) = \frac{1}{\ln(1+t)}$, where $H_{n,0} = 1$ for $n \in \mathbb{N}$, $H_{n,1} = H_n = \sum_{k=1}^{n} \frac{1}{k}$ for $n \in \mathbb{N}$, and

$$H_{n,j} = \sum_{k=j}^{n} \frac{H_{k-1,j-1}}{k}, \quad 2 \le j \le n. \tag{2}$$

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∗Corresponding author.

In [4, Theorem 2.1], it was inductively and recursively obtained that the family of differential equations

$$F^{(n)}(t) = \frac{(-1)^{n-1}}{(1+\lambda t)^n} \sum_{k=1}^{n} (k-1)!\lambda^{n-k}(n-1)!H_{n-1,k-1}e^{-kF(t)}$$

for $n \in \mathbb{N}$ have a solution

$$F(t) = \ln\left[1 + \frac{\ln(1+\lambda t)}{\lambda}\right]. \tag{3}$$

In [2, Theorem 2.1], as did in [3, 4], it was obtained inductively and recursively that the differential equations

$$(n-1)!e^{-nF(t)} = (-1)^{n-1} \sum_{k=1}^{n} \lambda^{n-k}(1+\lambda t)^k a_k(n) F^{(k)}(t), \quad n \geq 1$$

have a solution $F(t)$ defined by (3), where $a_n(n) = 1$, $a_1(n) = 1$, and

$$a_k(n) = \sum_{i_{k-1}=0}^{n-k} \sum_{i_{k-2}=0}^{n-k-i_{k-1}} \cdots \sum_{i_1=0}^{n-k-i_{k-1}-\cdots-i_2} k^{i_{k-1}} \cdots 2^{i_1}.$$

We notice that, although the same method used in proofs of [2, Theorem 2.1], [3, Theorem 1], and [4, Theorem 2.1] is effectual, however, proofs of those main results in [2, Theorem 2.1], [3, Theorem 1], and [4, Theorem 2.1] are much long and tedious, and formulations of those main results in those papers [2–4] are less meaningful and significant.

The aim of this paper is, by virtue of techniques in combinatorial analysis, to simply and concisely verify and extend the above-mentioned [2, Theorem 2.1], [3, Theorem 1] and [4, Theorem 2.1].

Our main results can be stated as the following two theorems.

THEOREM 1.1. *For $n \in \mathbb{N}$, the nth derivative of the function $G(t)$ satisfies*

$$G^{(n)}(t) = \frac{1}{(1+t)^n} \sum_{k=1}^{n} (-1)^k k! s(n,k) G^{k+1}(t) \tag{4}$$

and

$$\sum_{k=1}^{n} S(n,k)(1+t)^k G^{(k)}(t) = \frac{(-1)^n n!}{[\ln(1+t)]^{n+1}}, \tag{5}$$

where the Stirling numbers of the first and second kinds $s(n,k)$ *and* $S(n,k)$ *can be generated by*

$$\frac{[\ln(1+x)]^k}{k!} = \sum_{n=k}^{\infty} s(n,k)\frac{x^n}{n!} \quad \text{and} \quad \frac{(e^x-1)^k}{k!} = \sum_{n=k}^{\infty} S(n,k)\frac{x^n}{n!}.$$

THEOREM 1.2. *For* $n \in \mathbb{N}$, *the nth derivative of the function* $F(t)$ *defined in* (3) *satisfies*

$$F^{(n)}(t) = \left(\frac{\lambda}{1+\lambda t}\right)^n \sum_{k=1}^{n} \frac{(-1)^{k-1}(k-1)!}{\lambda^k} s(n,k) e^{-kF(t)} \tag{6}$$

and

$$\sum_{k=1}^{n} S(n,k)\left(\frac{1}{\lambda}+t\right)^k F^{(k)}(t) = \frac{(-1)^{n-1}(n-1)!}{[\lambda+\ln(1+\lambda t)]^n}. \tag{7}$$

2. Simple proofs of main results

Now we are in a position to prove our main results.

Proof of Theorem 1.1. In combinatorial analysis, the Faà di Bruno formula plays an important role and can be described in terms of the Bell polynomials of the second kind

$$\mathrm{B}_{n,k}(x_1,x_2,\dots,x_{n-k+1}) = \sum_{\substack{1\le i\le n-k+1\\ \ell_i\in\{0\}\cup\mathbb{N}\\ \sum_{i=1}^{n-k+1} i\ell_i=n\\ \sum_{i=1}^{n-k+1}\ell_i=k}} \frac{n!}{\prod_{i=1}^{n-k+1}\ell_i!} \prod_{i=1}^{n-k+1}\left(\frac{x_i}{i!}\right)^{\ell_i}$$

for $n \ge k \ge 0$, see [1, p. 134, Theorem A], by

$$\frac{\mathrm{d}^n}{\mathrm{d}\,t^n}[f\circ h(t)] = \sum_{k=0}^{n} f^{(k)}(h(t))\mathrm{B}_{n,k}\big(h'(t),h''(t),\dots,h^{(n-k+1)}(t)\big) \tag{8}$$

for $n \ge 0$, see [1, p. 139, Theorem C]. Setting $f(u)=\frac{1}{u}$ and $u=h(t)=\ln(1+t)$ in (8) gives

$$G^{(n)}(t) = \sum_{k=0}^{n}\left(\frac{1}{u}\right)^{(k)} \mathrm{B}_{n,k}\left(\frac{1}{1+t}, -\frac{1}{(1+t)^2}, \dots, (-1)^{n-k}\frac{(n-k)!}{(1+t)^{n-k+1}}\right)$$

$$
\begin{aligned}
&= \sum_{k=0}^{n} \frac{(-1)^k k!}{u^{k+1}} \left(\frac{1}{1+t}\right)^n (-1)^{n+k} \mathrm{B}_{n,k}(0!, 1!, 2!, \dots, (n-k)!) \\
&= \frac{1}{(1+t)^n} \sum_{k=0}^{n} \frac{(-1)^k k!}{\ln^{k+1}(1+t)} s(n,k) \\
&= \frac{1}{(1+t)^n} \sum_{k=0}^{n} (-1)^k k! s(n,k) G^{k+1}(t) \\
&= \frac{1}{(1+t)^n} \sum_{k=1}^{n} (-1)^k k! s(n,k) G^{k+1}(t)
\end{aligned}
$$

for $n \in \mathbb{N}$, where we used in the above lines the formula $s(n,0)=0$ for $n \in \mathbb{N}$ and the identities

$$
\mathrm{B}_{n,k}\big(abx_1, ab^2x_2, \dots, ab^{n-k+1}x_{n-k+1}\big) = a^k b^n \mathrm{B}_{n,k}(x_1, x_2, \dots, x_{n-k+1}) \tag{9}
$$

and

$$
\mathrm{B}_{n,k}(0!, 1!, 2!, \dots, (n-k)!) = (-1)^{n-k} s(n,k) \tag{10}
$$

listed in [1, p. 135, Theorem B]. This means that the nonlinear differential equations (4) have a solution $G(t)$.

The inversion theorem [10, Theorem 12.1, p. 171] reads that

$$
a_n = \sum_{\alpha=0}^{n} S(n,\alpha) b_\alpha \quad \text{if and only if} \quad b_n = \sum_{k=0}^{n} s(n,k) a_k. \tag{11}
$$

Applying this inversion theorem to the equality (4) results in

$$
(-1)^n n! G^{n+1}(t) = \sum_{k=1}^{n} S(n,k)(1+t)^k G^{(k)}(t), \quad n \in \mathbb{N}.
$$

The equation (5) is thus proved. The proof of Theorem 1.1 is complete. □

Second proof of the equation (4) *in Theorem 1.1.* In [6, Corollary 2.3], it was inductively and recursively procured that the Stirling numbers of the first kind $s(n,k)$ for $1 \le k \le n$ can be expressed as

$$
s(n,k) = (-1)^{n+k}(n-1)! \sum_{\ell_1=1}^{n-1} \frac{1}{\ell_1} \sum_{\ell_2=1}^{\ell_1-1} \frac{1}{\ell_2} \cdots \sum_{\ell_{k-2}=1}^{\ell_{k-3}-1} \frac{1}{\ell_{k-2}} \sum_{\ell_{k-1}=1}^{\ell_{k-2}-1} \frac{1}{\ell_{k-1}}. \tag{12}
$$

See also [5, p. 27, Remark 3.3]. Comparing (2) with (12), we observe that

$$(-1)^{n+k}s(n,k)=(n-1)!H_{n-1,k-1}, \quad n\ge k\ge 1. \tag{13}$$

See also [7, p. 9, Section 2.5]. Then substituting $(j-2)$ by $(k-1)$ and $(n-1)!H_{n-1,k-1}$ by $(-1)^{n+k}s(n,k)$ in (1) leads to (4). The second proof of the equation (4) is complete. □

Proof of Theorem 1.2. Setting $f(u)=\ln(1+u)$ and $u=h(t)=\frac{\ln(1+\lambda t)}{\lambda}$ in (8) gives

$$\begin{aligned}
F^{(n)}(t)&=\sum_{k=0}^{n}[\ln(1+u)]^{(k)}\\
&\quad\times \mathrm{B}_{n,k}\left(\frac{1}{1+\lambda t},-\frac{\lambda}{(1+\lambda t)^2},\dots,(-1)^{n-k}\frac{(n-k)!\lambda^{n-k}}{(1+\lambda t)^{n-k+1}}\right)\\
&=\sum_{k=0}^{n}[\ln(1+u)]^{(k)}\left(\frac{1}{1+\lambda t}\right)^{n}(-1)^{n+k}\lambda^{n-k}\mathrm{B}_{n,k}(0!,1!,2!,\dots,(n-k)!)\\
&=\left(\frac{1}{1+\lambda t}\right)^{n}\sum_{k=0}^{n}[\ln(1+u)]^{(k)}(-1)^{n+k}\lambda^{n-k}(-1)^{n-k}s(n,k)\\
&=\frac{1}{(1+\lambda t)^n}\left[\ln(1+u)\lambda^n s(n,0)+\sum_{k=1}^{n}\frac{(-1)^{k-1}(k-1)!}{(1+u)^k}\lambda^{n-k}s(n,k)\right]\\
&=\frac{1}{(1+\lambda t)^n}\sum_{k=1}^{n}\frac{(-1)^{k-1}(k-1)!}{\left[1+\frac{\ln(1+\lambda t)}{\lambda}\right]^k}\lambda^{n-k}s(n,k)\\
&=\frac{1}{(1+\lambda t)^n}\sum_{k=1}^{n}(-1)^{k-1}(k-1)!\lambda^{n-k}s(n,k)e^{-kF(t)}
\end{aligned}$$

for $n\in\mathbb{N}$, where we used in the above lines the formula $s(n,0)=0$ for $n\in\mathbb{N}$ and the identities (9) and (10). This means that the nonlinear differential equations (6) have a solution $F(t)$.

Applying the inversion theorem stated in (11) to the equation (6) results in

$$\frac{(-1)^{n-1}(n-1)!}{\lambda^n}e^{-nF(t)}=\sum_{k=1}^{n}S(n,k)\left(\frac{1+\lambda t}{\lambda}\right)^{k}F^{(k)}(t).$$

The identity (7) is proved. The proof of Theorem 1.2 is complete. □

3. Remarks

Finally we list several remarks on our main and the closely related results.

REMARK 3.1. The relation (13) shows that we should not call the quantities $H_{n,k}$ defined in (2) the generalized harmonic numbers, as did in [4, p. 745].

REMARK 3.2. Till now we can see that formulations and proofs of Theorems 1.1 and 1.2 in this paper are simpler, more concise, more meaningful, and more significant than those in the papers [2–4].

REMARK 3.3. This paper is an extended and simplified version of the preprints [8, 9].

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Feng Qi
[1]Institute of Mathematics, Henan Polytechnic University
Jiaozuo 454010, Henan, China

[2]College of Mathematics, Inner Mongolia University for Nationalities
Tongliao 028043, Inner Mongolia, China

[3]School of Mathematical Sciences, Tianjin Polytechnic University
Tianjin 300387, China
E-mail: `qifeng618@gmail.com`

Jing-Lin Wang
[1]Teda No. 2 Primary School
Economic and Technological Development Zone
Tianjin 300457, China

[2]Department of Mathematics, College of Science
Tianjin Polytechnic University
Tianjin 300387, China
E-mail: `jing-lin.wang@hotmail.com`

Bai-Ni Guo
School of Mathematics and Informatics
Henan Polytechnic University
Jiaozuo 454010, Henan, China
E-mail: `bai.ni.guo@gmail.com`