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COMMON FIXED POINT OF GENERALIZED ASYMPTOTIC POINTWISE (QUASI-) NONEXPANSIVE MAPPINGS IN HYPERBOLIC SPACES

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Abstract. We prove a fixed point theorem for generalized asymptotic pointwise nonexpansive mapping in the setting of a hyperbolic space. A one-step iterative scheme approximating common fixed point of two generalized asymptotic pointwise (quasi-) nonexpansive mappings in this setting is provided. We obtain $\triangle-$convergence and strong convergence theorems of the iterative scheme for two generalized asymptotic pointwise nonexpansive mappings in the same setting. Our results generalize and extend some related results in the literature.

1. Introduction and Preliminaries

A hyperbolic space [11] is a metric space (X, d) with a mapping $W : X^2 \times I \to X$ such that

(1) $d\left(u, W\left(x, y, \lambda\right)\right) \leq \lambda d(u, x) + (1-\lambda) d(u, y)$;
(2) $d(W(x, y, \lambda_1), W(x, y, \lambda_2)) = \left|\lambda_1 - \lambda_2\right| d(x, y)$;
(3) $W(x, y, \lambda) = W(y, x, 1-\lambda)$;
(4) $d(W(u, v, \lambda), W(x, y, \lambda)) \leq \lambda d(u, x) + (1-\lambda)\, d(v, y)$

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for all $u, v, x, y \in X$ and $\lambda, \lambda_1, \lambda_2 \in I = [0,1]$. A nonempty subset D of X is convex if $W(x, y, \lambda) \in D$ for all $x, y \in D$ and $\lambda \in I$.

A hyperbolic space X is uniformly convex [13] if for all $u, x, y \in X$, $r > 0$ and $\varepsilon \in (0, 2]$, there exists a $\delta > 0$ such that

$$d\left(W\left(x, y, \frac{1}{2}\right), u\right) \leq (1-\delta)r < r,$$

whenever $d(x, u) \leq r, d(y, u) \leq r$ and $d(x, y) \geq r\varepsilon$. Note that uniformly convex hyperbolic spaces contain Banach spaces as well as $CAT(0)$ spaces (see for instance, [5, 9] and references therein).

Let $\{x_n\}$ be a bounded sequence in X. Define

$$r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n)$$

for all $x \in X$. The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is defined as

$$r(\{x_n\}) = \inf_{x\in X} r\left(x, \{x_n\}\right).$$

and the asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A(\{x_n\}) = \{y \in X : r(y, \{x_n\}) = r(\{x_n\})\}.$$

A point $x \in D$ is a fixed point of $T : D \to D$ if $Tx = x$ and $Fix\,(T)$ denotes the set of all fixed points of T.

We say that T is

(i) an *asymptotic pointwise nonexpansive* mapping if there exists a sequence of mappings $\alpha_n : D \to [0, \infty)$ with $\lim_{n\to\infty} \alpha_n(x) = 0$ for all $x \in D$, such that

$$d(T^n x, T^n y) \leq (1 + \alpha_n(x))d(x, y) \quad \text{for all} \quad x, y \in D, \ n \in \mathbb{N};$$

(ii) a *generalized asymptotic pointwise nonexpansive* mapping if there exist two sequences of mappings $\alpha_n : D \to [0, \infty)$ and $\beta_n : D \to [0, \infty)$ with $\lim_{n\to\infty} \alpha_n(x) = 0 = \lim_{n\to\infty} \beta_n(x)$ for all $x \in D$, such that for all $y \in D$ and $n \in \mathbb{N}$,

$$d(T^n x, T^n y) \leq d(x, y) + \alpha_n(x)d(x, y) + \beta_n(x);$$

(iii) a *generalized asymptotic pointwise quasi-nonexpansive* mapping if there exist two sequences of mappings $\alpha_n : D \to [0, \infty)$ and $\beta_n : D \to [0, \infty)$ with $\lim_{n\to\infty} \alpha_n(x) = 0 = \lim_{n\to\infty} \beta_n(x)$ for all $x \in D$, such that for all $p \in Fix(T)$ and $n \in \mathbb{N}$,

$$d(T^n x, p) \leq d(x, p) + \alpha_n(p)d(x, p) + \beta_n(p);$$

(iv) a *pointwise Lipschitzian* mapping if there exists a bounded function $\alpha : D \to [0, \infty)$ such that for all $x, y \in D$,

$$d(Tx, Ty) \leq \alpha(x)d(x, y).$$

We denote the class of all generalized asymptotic pointwise nonexpansive mappings from D to itself by $\mathcal{G}(D)$, the class of all generalized asymptotic pointwise quasi-nonexpansive mappings from D to itself by $\mathcal{Q}(D)$, and the class of all pointwise Lipschitzian mappings from D to itself by $\mathcal{Z}(D)$.

A sequence $\{x_n\}$ in D is (i) an *approximate* common fixed point sequence for the mappings $T_1, T_2 : D \to D$ if

$$\lim_{n\to\infty} d(x_n, T_i x_n) = 0, \quad i = 1, 2;$$

(ii) $\triangle-$*convergent* to $x \in D$ if x is the unique asymptotic center of every subsequence $\{u_n\}$ of $\{x_n\}$ [8].

A subset D of a metric space X is a Chebyshev set if to each point $x \in X$, there exists a unique point $z \in D$ such that $d(x, z) = dist(x, D)$, where

$$dist(x, D) = \inf\{d(x, y) : y \in D\}).$$

If D is a Chebyshev set, one can define the nearest point projection $P : X \to D$ by assigning z to x.

Different iterative schemes have been used to approximate fixed points of certain nonlinear mappings; for instance, Mann and Ishikawa iterations [6, 12]. Khan and Takahashi [10] used the modified Ishikawa iterative scheme (two-step) to approximate common fixed points of two asymptotically nonexpansive mappings in a uniformly convex Banach space. These schemes have been under extensive study in hyperbolic spaces/convex metric spaces (see, for example, [5]). Khan [7], in uniformly convex Banach spaces, has approximated common fixed points of two asymptotically quasi-nonexpansive mappings through the following one-step iterative scheme:

$$x_1 \in D, \ x_{n+1} = (1 - \mu_n - \nu_n)\, x_n + \mu_n T_1^n x_n + \nu_n T_2^n x_n \tag{1.1}$$

where $0 < a \leq \mu_n, \ \nu_n \leq b < 1$ and satisfy $\mu_n + \nu_n < 1$.

In a hyperbolic space, the scheme is translated to the following:

$$x_1 \in D, \ x_{n+1} = W\left(T_1^n x_n, W\left(T_2^n x_n, x_n, \frac{\nu_n}{1 - \mu_n}\right), \mu_n\right) \tag{1.2}$$

where $0 < a \leq \mu_n,\ \nu_n \leq b < 1$ and satisfy $\mu_n + \nu_n < 1$.

If $T_2 = I$ (the identity mapping) in (1.2), it becomes Mann iterative scheme:

$$x_1 \in D,\ x_{n+1} = W\left(T^n x_n, x_n, \mu_n\right) \tag{1.3}$$

where $0 < a \leq \mu_n \leq b < 1$.

Recently, Fukhar-ud-din [3] employed iterative scheme (1.2) to approximate common fixed point of two strongly asymptotically (quasi-) nonexpansive mappings in the setting of uniformly convex hyperbolic space. In this paper we use (1.2) to approximate common fixed point of two generalized asymptotic pointwise nonexpansive mappings in the same setting.

The following known lemmas are needed for the development of our results.

Lemma 1.1. [14] *Let the sequences* $\{a_n\}, \{u_n\}$ *and* $\{v_n\}$ *of nonnegative numbers satisfy:*

$$a_{n+1} \leq (1 + u_n)a_n + v_n.$$

If both $\sum_{n=1}^{\infty} u_n$ *and* $\sum_{n=1}^{\infty} v_n$ *converge, then* $\lim_{n\to\infty} a_n$ *exists.*

Lemma 1.2. [4] *Every nonempty closed and convex subset* D *of a hyperbolic space* X *is a Chebyshev set.*

Lemma 1.3. [2] *Let* D *be a nonempty, closed and convex subset of a complete uniformly convex hyperbolic space* X. *Then every bounded sequence* $\{x_n\}$ *in* X *has a unique asymptotic center with respect to* D.

Lemma 1.4. [2] *Let* D *be a nonempty, closed and convex subset of a uniformly convex hyperbolic space* X. *Let* $\{x_n\}$ *be a bounded sequence in* D *such that* $A_D(\{x_n\}) = \{x\}$ and $r(\{x_n\}) = \rho$. *If* $\{y_m\}$ *is another sequence in* D *such that* $\lim_{m\to\infty} r(y_m, \{x_n\}) = \rho$ *(a real number), then* $\lim_{m\to\infty} y_m = x$.

Lemma 1.5. [4] *Let* X *be a uniformly convex hyperbolic space. Let* $x \in X$ *and* $\{a_n\}$ *be a sequence in* $[b, c]$ *for some* $b, c \in (0, 1)$. *If* $\{u_n\}$ and $\{v_n\}$ *are sequences in* X *such that* $\limsup_{n\longrightarrow\infty} d(u_n, x) \leq r$, $\limsup_{n\longrightarrow\infty} d(v_n, x) \leq r$ *and* $\lim_{n\longrightarrow\infty} d(W(u_n, v_n, a_n), x) = r$ *for some* $r \geq 0$, *then* $\lim_{n\to\infty} d(u_n, v_n) = 0$.

2. Main Results

In this section, we prove some convergence theorems of iterative scheme (1.2) under some suitable conditions. Before stating the convergence theorems, we prove the existence of fixed point of a generalized asymptotic pointwise nonexpansive mapping in a complete and uniformly convex hyperbolic space.

THEOREM 2.1. *Let D be a nonempty bounded, closed and convex subset of a complete and uniformly convex hyperbolic space X and the mapping $T \in \mathcal{G}(D)$ be continous. Then T has a fixed point.*

Proof. Let $z_0 \in D$ and $z_n = T^n z_0$ for $n \geq 1$. Let ξ_0 and ρ be the asymptotic center and the asymptotic radius of $\{z_n\}$ in D, respectively. Let $\xi_j = T^j \xi_0$ for $j \geq 1$. We show that $r(\xi_j, \{z_n\}) = \limsup_{n\to\infty} d(\xi_j, z_n) \to \rho$ as $j \to \infty$.

Let j and n be integers. For $j < n$, we have that

$$d(\xi_j, z_n) = d\left(T^j \xi_0, T^j z_{n-j}\right) \leq (1 + \alpha_j(\xi_0)) d(\xi_0, z_{n-j}) + \beta_j(\xi_0). \quad (2.1)$$

Let $\varepsilon > 0$. By the definition of ξ_0 and ρ, there exists an integer n_0 such that

$$d(\xi_0, z_n) < \rho + \frac{\varepsilon}{3} \text{ for all } n \geq n_0. \quad (2.2)$$

For a fixed $j \geq 1$, (2.2) can be written as

$$d(\xi_0, z_{n-j}) < \rho + \frac{\varepsilon}{3} \text{ for all } n \geq n_0 + j.$$

Taking $\limsup_{n\to\infty}$ on both sides in (2.1) and using (2.2), we obtain

$$r(\xi_j, \{z_n\}) \leq (1 + \alpha_j(\xi_0)) \left(\rho + \frac{\varepsilon}{3}\right) + \beta_j(\xi_0) \text{ for } j \geq 1.$$

Since $\lim_{j\to\infty} \alpha_j(\xi_0) = 0 = \lim_{j\to\infty} \beta_j(\xi_0)$, there is an integer n_1 such that

$$(1 + \alpha_j(\xi_0)) \left(\rho + \frac{\varepsilon}{3}\right) + \beta_j(\xi_0) < \rho + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \rho + \varepsilon \text{ for all } j \geq n_1.$$

Therefore

$$r(\xi_j, \{z_n\}) - \rho < \varepsilon \text{ for all } j \geq n_1.$$

That is, $r(\xi_j, \{x_n\}) \to \rho$ as $j \to \infty$. By Lemma 1.3, $\xi_j \to \xi_0$ as $j \to \infty$. As T is continuous, $T\xi_j \to T\xi_0$ as $j \to \infty$. Finally, by metric triangle inequality we get

$$\begin{aligned} d(\xi_0, T\xi_0) &\leq d(\xi_0, T\xi_j) + d(T\xi_j, T\xi_0) \\ &= d(\xi_0, \xi_{j+1}) + d(T\xi_j, T\xi_0). \end{aligned}$$

Hence $\xi_0 = T\xi_0$. □

We need the following lemmas in the proof of the main theorems.

LEMMA 2.2. *Let D be a a nonempty bounded, closed and convex subset of a convex hyperbolic space X and let $T_1, T_2 \in \mathcal{Q}(D)$ be such that $\mathcal{F} = Fix(T_1) \cap Fix(T_2) \neq \phi$. Then for the sequence $\{x_n\}$ in (1.2), we have that*
(i) $\lim_{n\to\infty} d(x_n, p)$ *exists for each* $p \in \mathcal{F}$ *and*
(ii) $\lim_{n\to\infty} dist(x_n, \mathcal{F})$ *exists.*

Proof. Let $p \in \mathcal{F}$. Since $T_1, T_2 \in \mathcal{Q}(D)$ and $p \in \mathcal{F}$, there exist sequences $\{\alpha_n(p)\} \subset [0, \infty)$ ($\alpha_n(p) = \max\left\{\alpha_n^{T_1}(p), \alpha_n^{T_2}(p)\right\}$) and $\{\beta_n(p)\} \subset [0, \infty)$ ($\beta_n(p) = \max\left\{\beta_n^{T_1}(p), \beta_n^{T_2}(p)\right\}$) such that

$$\sum_{n=1}^{\infty}(\alpha_n(p) - 1) < \infty, \quad \sum_{n=1}^{\infty}(\beta_n(p) - 1) < \infty$$

and

$$d(T_1^n x, y) \leq (1+\alpha_n(y))d(x, y)+\beta_n(y), \quad d(T_2^n x, y) \leq (1+\alpha_n(y))d(x, y)+\beta_n(y)$$

for $n \geq 1$, $x \in D$ and $y \in \mathcal{F}$.

From (1.2), it follows that

$$\begin{aligned} & d(x_{n+1}, p) \\ &= d\left(W\left(T_1^n x_n, W\left(T_2^n x_n, x_n, \frac{\nu_n}{1-\mu_n}\right), \mu_n\right), p\right) \\ &\leq \mu_n d(T_1^n x_n, p) + (1-\mu_n) d\left(W\left(T_2^n x_n, x_n, \frac{\nu_n}{1-\mu_n}\right), p\right) \\ &\leq \mu_n d(T_1^n x_n, p) + \nu_n d(T_2^n x_n, p) + (1-\mu_n-\nu_n) d(x_n, p) \\ &\leq \mu_n[(1+\alpha_n(p))d(x_n, p) + \beta_n(p)] + \nu_n[(1+\alpha_n(p))d(x_n, p) + \beta_n(p)] \\ &\quad +(1-\mu_n-\nu_n)[(1+\alpha_n(p))d(x_n, p) + \beta_n(p)] \\ &= (1+\alpha_n(p))d(x_n, p) + \beta_n(p). \end{aligned}$$

By Lemma 1.4, $\lim_{n\to\infty} d(x_n, p)$ exists for each $p \in \mathcal{F}$. This implies that $\lim_{n\to\infty} dist(x_n, \mathcal{F})$ exists. □

LEMMA 2.3. *Let X be a uniformly convex hyperbolic space and $T_1, T_2 \in \mathcal{Q}(D) \cap \mathcal{Z}(D)$ be such that $\mathcal{F} = Fix\,(T_1) \cap Fix\,(T_2) \neq \phi$. If $\mu_n, \nu_n \in [1-\delta, \delta]$ for some $\delta \in (0,1)$, then $\{x_n\}$ in (1.2), is an approximate common fixed point sequence of T_1 and T_2.*

Proof. Reconsider the sequences $\{\alpha_n(p)\}$ and $\{\beta_n(p)\}$ introduced in the proof of Lemma 2.2.

To establish the result, we have to show that

$$\lim_{n\to\infty} d\,(x_n, T_i x_n) = 0, \qquad i = 1, 2.$$

By Lemma 2.2, $\lim_{n\to\infty} d(x_n, p)$ exists for each $p \in \mathcal{F}$. Assume that $\lim_{n\to\infty} d(x_n, p) = c$. The proof is trivial if $c = 0$. If $c > 0$, then $\lim_{n\to\infty} d(x_n, p) = c$ can be expressed as

$$\lim_{n\to\infty} d\left(W\left(T_1^n x_n, W\left(T_2^n x_n, x_n, \frac{\nu_n}{1-\mu_n}\right), \mu_n\right), p\right) = c. \tag{2.3}$$

Since T_1 is a generalized pointwise asymptotic quasi-nonexpansive, we have

$$d(T_1^n x_n, p) \leq (1+\alpha_n(p))d(x_n, p) + \beta_n(p)$$

and therefore $\limsup_{n\to\infty} d(T_1^n x_n, p) \leq c$. The inequality

$$\begin{aligned}
& d\left(W\left(T_2^n x_n, x_n, \frac{\nu_n}{1-\mu_n}\right), p\right) \\
\leq\ & \frac{\nu_n}{1-\mu_n} d(T_1^n x_n, p) + \left(1 - \frac{\nu_n}{1-\mu_n}\right) d(x_n, p) \\
\leq\ & \frac{\nu_n}{1-\mu_n}[(1+\alpha_n(p))d(x_n, p) + \beta_n(p)] \\
& + \left(1 - \frac{\nu_n}{1-\mu_n}\right) d(x_n, p) \\
\leq\ & \frac{\nu_n}{1-\mu_n}[(1+\alpha_n(p))d(x_n, p) + \beta_n(p)] \\
& + \left(1 - \frac{\nu_n}{1-\mu_n}\right) [(1+\alpha_n(p))d(x_n, p) + \beta_n(p)] \\
=\ & (1+\alpha_n(p))d(x_n, p) + \beta_n(p)
\end{aligned}$$

implies that

$$\limsup_{n\to\infty} d\left(W\left(T_2^n x_n, x_n, \frac{\nu_n}{1-\mu_n}\right), p\right) \leq c. \tag{2.4}$$

From (2.3) and (2.4), we observe that sequences $\{T_1^n x_n\}$ and $\left\{W\left(T_2^n x_n, x_n, \frac{\nu_n}{1-\mu_n}\right)\right\}$ satisfy all conditions in Lemma 1.5. Therefore

$$\lim_{n\to\infty} d\left(T_1^n x_n, W\left(T_2^n x_n, x_n, \frac{\nu_n}{1-\mu_n}\right)\right) = 0. \tag{2.5}$$

Observe that

$$\begin{aligned} d(x_{n+1}, T_1^n x_n) &= d\left(W\left(T_1^n x_n, W\left(T_2^n x_n, x_n, \frac{\nu_n}{1-\mu_n}\right), \mu_n\right), T_1^n x_n\right) \\ &\leq (1-\mu_n) d\left(W\left(T_2^n x_n, x_n, \frac{\nu_n}{1-\mu_n}\right), T_1^n x_n\right) \\ &\leq (1-a) d\left(W\left(T_2^n x_n, x_n, \frac{\nu_n}{1-\mu_n}\right), T_1^n x_n\right). \end{aligned}$$

Taking the limit on both sides in the above inequality and using (2.5), we have

$$\lim_{n\to\infty} d(x_{n+1}, T_1^n x_n) = 0. \tag{2.6}$$

By triangle inequality, we get that

$$\begin{aligned} d(x_{n+1}, p) \leq\ & d\left(x_{n+1}, T_1^n x_n\right) + d\left(T_1^n x_n, W\left(T_2^n x_n, x_n, \frac{\nu_n}{1-\mu_n}\right)\right) \\ & + d\left(W\left(T_2^n x_n, x_n, \frac{\nu_n}{1-\mu_n}\right), p\right). \end{aligned}$$

The above inequality, together with (2.5) and (2.6), provides that

$$c \leq \liminf_{n\to\infty} d\left(W\left(T_2^n x_n, x_n, \frac{\nu_n}{1-\mu_n}\right), p\right).$$

Combining (2.4) and (2.5), we obtain

$$\lim_{n\to\infty} d\left(W\left(T_2^n x_n, x_n, \frac{\nu_n}{1-\mu_n}\right), p\right) = c.$$

Again Lemma 1.5 with $x = p, r = c, a_n = \frac{\nu_n}{1-\mu_n}, u_n = T_2^n x_n, v_n = x_n$ and (2.6) give that

$$\lim_{n\to\infty} d(x_n, T_2^n x_n) = 0. \tag{2.7}$$

Next we calculate

$$\begin{aligned} d(x_{n+1},x_n) &\le d\left(W\left(T_1^n x_n, W\left(T_2^n x_n, x_n, \frac{\nu_n}{1-\mu_n}\right), \mu_n\right), x_n\right) \\ &\le \mu_n d(T_1^n x_n, x_n) + (1-\mu_n)\, d\left(W\left(T_2^n x_n, x_n, \frac{\nu_n}{1-\mu_n}\right), x_n\right) \\ &\le \mu_n d\left(T_1^n x_n, x_n\right) + \nu_n d\left(T_2^n x_n, x_n\right) \\ &\le \mu_n \left\{d\left(x_{n+1}, T_1^n x_n\right) + d\left(x_{n+1}, x_n\right)\right\} + \nu_n d\left(T_2^n x_n, x_n\right). \end{aligned}$$

Re-arranging the terms in the above inequality, we have

$$\begin{aligned} d(x_{n+1},x_n) &\le \frac{\mu_n}{1-\mu_n} d\left(x_{n+1}, T_1^n x_n\right) + \frac{\nu_n}{1-\mu_n} d\left(T_2^n x_n, x_n\right) \\ &\le \frac{b}{1-b}\left(d\left(x_{n+1}, T_1^n x_n\right) + d\left(T_2^n x_n, x_n\right)\right). \end{aligned}$$

Taking lim sup on both sides in the above inequality and using (2.4) and (2.7) in it, we have

$$\lim_{n\to\infty} d(x_{n+1},x_n) = 0.$$

Since

$$d(x_n, T_1^n x_n) \le d(x_n, x_{n+1}) + d(x_{n+1}, T_2^n x_n),$$

therefore it follows that

$$\lim_{n\to\infty} d(x_n, T_1^n x_n) = 0.$$

Now

$$\begin{aligned} d\left(x_{n+1}, T_1 x_{n+1}\right) &\le d\left(x_{n+1}, T_1^{n+1} x_{n+1}\right) + d\left(T_1^{n+1} x_{n+1}, T_1 x_{n+1}\right) \\ &\le d\left(x_{n+1}, T_1^{n+1} x_{n+1}\right) + \alpha(x_{n+1}) d\left(x_{n+1}, T_1^n x_{n+1}\right) \\ &\le d\left(x_{n+1}, T_1^{n+1} x_{n+1}\right) + \alpha(x_{n+1}) d\left(x_n, T_1^n x_n\right) \\ &\quad + \alpha(x_{n+1})\left[d\left(x_{n+1}, x_n\right) + d\left(T_1^n x_n, T_1^n x_{n+1}\right)\right] \\ &\le d\left(x_{n+1}, T_1^{n+1} x_{n+1}\right) + \alpha(x_{n+1}) d\left(x_n, T_1^n x_n\right) \\ &\quad + \alpha(x_{n+1})\left(1 + \alpha(x_{n+1})\right) d\left(x_{n+1}, x_n\right). \end{aligned}$$

Since α is a bounded function, we have

$$\lim_{n\to\infty} d\left(x_n, T_1 x_n\right) = 0.$$

Similarly

$$\lim_{n\to\infty} d\left(x_n, T_2 x_n\right) = 0.$$

Hence the result. □

Since every generalized asymptotic pointwise quasi-nonexpansive mapping is generalized asymptotic pointwise nonexpansive, our $\triangle-$convergence result in the setting of a uniformly convex hyperbolic space is as follows.

THEOREM 2.4. *Let D be a nonempty subset of a uniformly convex hyperbolic space X and $T_1, T_2 \in \mathcal{G}(D) \cap \mathcal{Z}(D)$ be such that $\mathcal{F} = Fix\,(T_1) \cap Fix\,(T_2) \neq \phi$. If $\alpha_n, \beta_n \in [1-\delta, \delta]$ for some $\delta \in (0,1)$, then $\{x_n\}$ in (1.2) $\triangle-$converges to a point in $\mathcal{F}$.*

Proof. It follows from Lemma 2.2 that $\{x_n\}$ is bounded in D. By Lemma 1.1, $\{x_n\}$ has a unique asymptotic center, that is, $A(\{x_n\}) = \{x\}$. Let $\{u_n\}$ be a subsequence of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. Suppose $u \notin D$. As D is a Chebyshev set by Lemma 1.2, we can define a nearest point projection $P : X \to D$.Then $d\,(Pu, u_n) < d\,(u, u_n) \Longrightarrow r\,(Pu, \{u_n\}) < r\,(u, \{u_n\}) \Longrightarrow u$ is not the asymptotic center of $\{u_n\}$, a contradiction. Hence $u \in D$. Next, we show that $u \in \mathcal{F}$. By Lemma 2.3, we have

$$\lim_{n\to\infty} d\,(u_n, T_i u_n) = 0, \qquad i = 1, 2.$$

In fact, for $m, n \geq 1$,

$$\begin{aligned} d\,(T_1^m u, u_n) &\leq d\,(T_1^m u, T_1^m u_n) \\ &\quad + d\left(T_1^m u_n, T_1^{m-1} u_n\right) + \ldots + d\left(T_1^2 u_n, T u_n\right) + d\,(T_1 u_n, u_n) \\ &\leq (1+\alpha_m(u))d\,(u, u_n) + \beta_m(u) + d\,(T_1 u_n, u_n) \\ &\quad + (m-2)\alpha(u_n) d\,(T_1 u_n, u_n). \end{aligned}$$

Taking $\limsup_{n\to\infty}$ on both sides of the above inequality, we obtain

$$\limsup_{n\to\infty} d\,(T_1^m u, u_n) \leq (1+\alpha_m(u)) \limsup_{n\to\infty} d\,(u, u_n) + \beta_m(u).$$

Therefore

$$\limsup_{m\to\infty} r\,(T_1^m u, \{u_n\}) \leq r\,(u, \{u_n\}).$$

By the definition of $A(\{u_n\})$,

$$r\,(u, \{u_n\}) \leq \liminf_{m\to\infty} r\,(T_1^m u, \{u_n\}).$$

That is,

$$\lim_{m\to\infty} r\,(T_1^m u, \{u_n\}) = r\,(u, \{u_n\}).$$

By Lemma 1.3, $\lim_{m\to\infty} T_1^m u = u$. Since T_1 is continuous,

$$T_1 u = T_1\left(\lim_{m\to\infty} T_1^m u\right) = \lim_{m\to\infty} T_1^{m+1} u = u.$$

This shows that $u \in Fix\,(T_1)$. Similarly, $u \in Fix\,(T_2)$ and hence $u \in \mathcal{F}$.

Suppose $x \neq u$. By the uniqueness of asymptotic center,

$$\begin{aligned} r\,(u,\{u_n\}) &< r\,(x,\{u_n\}) \\ &\leq r\,(x,\{x_n\}) \\ &< r\,(u,\{x_n\}) \\ &= r\,(u,\{u_n\})\,, \end{aligned}$$

which is a contradiction. Hence $x = u$. This proves that x is the unique asymptotic center of $\{u_n\}$. This completes the proof. □

The following definitions are needed for strong convergence of the iterative scheme (1.2).

(a) Two mappings $T_1, T_2 : D \to D$ satisfy Condition (AV) if there exists a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(r) > 0$ for $r > 0$ such that

$$\frac{1}{2}\sum_{i=1}^{2} d(x, T_i x) \geq f(dist(x,\mathcal{F})) \text{ for all } x \in D.$$

(b) A mapping $T : D \to D$ is semi-compact if for a sequence $\{x_n\}$ in D with $\lim_{n\to\infty} d\,(x_n, Tx_n) = 0$, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that x_{n_i} converges to a point in D.

Theorem 2.5. *Let D be a nonempty closed and convex subset of a uniformly convex hyperbolic space X and $T_1, T_2 \in \mathcal{G}(D) \cap \mathcal{Z}(D)$. If $\alpha_n, \beta_n \in [1-\delta, \delta]$ for some $\delta \in (0,1)$ and T_1, T_2 satisfy Condition (AV), then $\{x_n\}$ defined in (1.2) strongly converges to a point in $\mathcal{F}$.*

Proof. As a consequence of Lemma 2.2, $\lim_{n\to\infty} dist(x_n,\mathcal{F})$ exists. By condition (AV), we have

$$\lim_{n\to\infty} f(dist(x_n,\mathcal{F})) \leq \frac{1}{2}\lim_{n\to\infty}\sum_{i=1}^{2} d(x_n, T_i x_n) = 0.$$

Since f is a nondecreasing function and $f(0) = 0$, we obtain

$$\lim_{n\to\infty} dist(x_n,\mathcal{F}) = 0.$$

Let $\varepsilon > 0$. There exists $n_0 \geq 1$ such that

$$d(x_n, F) < \frac{\varepsilon}{3}, \text{ for all } n \geq n_0.$$

In particular, $dist(x_{n_0}, \mathcal{F}) < \frac{\varepsilon}{3}$. Thus, there exists $p \in \mathcal{F}$ such that

$$d(x_{n_0}, p) < \frac{\varepsilon}{2}.$$

For $m, n \geq n_0$,

$$d(x_{m+n}, x_n) \leq d(x_{m+n}, p) + d(x_n, p) < 2d(x_{n_0}, p) < \varepsilon.$$

Thus, $\{x_n\}$ is a Cauchy sequence in the closed subset D of X. Hence, $x_n \to x \in D$. Since T_i is a pointwise Lipschitzian mapping, we have

$$\begin{aligned} d(x, T_i x) &\leq d(x, x_n) + d(x_n, T_i x_n) + d(T_i x_n, T_i x) \\ &\leq d(x_n, x) + d(x_n, T_i x_n) + \alpha(x) d(x_n, x) \\ &= (1 + \alpha(x)) d(x_n, x) + d(x_n, T_i x_n), \quad i = 1, 2. \end{aligned}$$

Since α is bounded, we obtain

$$d(x, T_i x) \leq (1 + \alpha(x)) d(x_n, x) + d(x_n, T_i x_n) \to 0.$$

Thus, $T_i x = x$ for each $i \in \{1, 2\}$. Hence x is a common fixed point of T_1 and T_2. □

THEOREM 2.6. *Let D be a nonempty closed and convex subset of a uniformly convex hyperbolic space X and $T_1, T_2 \in \mathcal{G}(D) \cap \mathcal{Z}(D)$. If $\alpha_n, \beta_n \in [1 - \delta, \delta]$ for some $\delta \in (0, 1)$ and either T_1 or T_2 is semi-compact, then $\{x_n\}$ defined in (1.2) strongly converges to a common fixed point of T_1 and T_2.*

Proof. As a consequence of Lemma 2.2, $\lim_{n\to\infty} dist(x_n, \mathcal{F})$ exists. By Lemma 2.3,

$$\lim_{n\to\infty} d(x_n, T_i x_n) = 0, \qquad i = 1, 2.$$

Let T_1 be a semi-compact mapping. There exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that x_{n_i} converges to $x \in D$. Since T_1 and T_2 are pointwise Lipschitzian mappings, we have

$$\begin{aligned} d(x, T_i x) &\leq d(x, x_{n_i}) + d(x_{n_i}, T_i x_{n_i}) + d(T_i x_{n_i}, T_i x) \\ &\leq d(x_{n_i}, x) + d(x_{n_i}, T_i x_{n_i}) + \alpha(x) d(x_{n_i}, x) \\ &= (1 + \alpha(x)) d(x_{n_i}, x) + d(x_{n_i}, T_i x_{n_i}) \to 0, \qquad i = 1, 2. \end{aligned}$$

Hence $T_1 x = x = T_2 x$. This completes the proof. □

We have similar results for Mann iterations.

3. The case of nonself mappings

Recall that D is a retraction of X if there exists a continuous mapping $P : X \to D$ such that $Px = x$, for all $x \in D$. A mapping $P : X \to D$ is said to be a retraction if $P^2 = P$. Consequently, if P is a retraction, then $Py = y$ for all y in the range of P. Chidume et al. [1] defined nonself asymptotically nonexpansive mapping. Similarly, we define the following nonself mappings.

Let $P : X \to D$ be a nonexpansive retraction. A nonself mapping $T : D \to X$ is (i) a generalized asymptotic pointwise nonexpansive if there exist two sequences of mappings $\alpha_n : D \to [0, \infty)$ and $\beta_n : D \to [0, \infty)$ with $\lim_{n\to\infty} \alpha_n(x) = 0 = \lim_{n\to\infty} \beta_n(x)$ for all $x \in D$, such that for all $y \in D$ and $n \in \mathbb{N}$,

$$d(T\,(PT)^{n-1}\,x, T\,(PT)^{n-1}\,y) \le d(x, y) + \alpha_n(x)d(x, y) + \beta_n(x);$$

(ii) a pointwise Lipschitzian if there exists a bounded function $\alpha : D \to [0, \infty)$ such that for all $x, y \in D$,

$$d(T\,(PT)^{n-1}\,x, T\,(PT)^{n-1}\,y) \le \alpha(x)d(x, y).$$

Consider two nonself generalized asymptotic pointwise nonexpansive mappings T_1 and T_2. The one-step iterative scheme, with an intial guess $x_1 \in D$, is of the form

$$x_{n+1} = P\left(W\left(T_1\,(PT)^{n-1}\,x_n, PW\left(T_2\,(PT_2)^{n-1}\,x_n, x_n, \frac{\nu_n}{1-\mu_n}\right), \mu_n\right)\right) \tag{3.1}$$

where $0 < a \le \mu_n,\ \nu_n \le b < 1$ and satisfy $\mu_n + \nu_n < 1$.

Due to the elementary properties of P, we can obtain the results for nonself mappings. The following theorems are the nonself versions of above theorems.

Theorem 3.1. *Let X be a uniformly convex hyperbolic space and $T_1, T_2 : D \to X$ be nonself generalized asymptotic pointwise nonexpansive mappings on a nonempty closed and convex subset D of X. If $\mu_n, \nu_n \in [1-\delta, \delta]$ for some $\delta \in (0, 1)$, then $\{x_n\}$ in (3.1) is $\triangle-$convergent to a common fixed point of T_1 and T_2.*

Theorem 3.2. *Let X be a uniformly convex hyperbolic space and $T_1, T_2 : D \to X$ be nonself pointwise Lipschitzian and nonself generalized asymptotic pointwise nonexpansive mappings on a nonempty closed and* *convex subset C of X. If $\mu_n, \nu_n \in [1-\delta, \delta]$ for some $\delta \in (0,1)$ and T_1, T_2 satisfy Condition(AV), then $\{x_n\}$ in (3.1), strongly converges to a common fixed point of T_1 and T_2.*

Remark 3.3. Since uniformly convex Banach spaces and $CAT(0)$ spaces are uniformly convex hyperbolic spaces, all the above results are valid in uniformly convex Banach spaces and $CAT(0)$ spaces simultaneously and are new in the current literature.

Future Work: It will be interesting to generalize this work for asymptotic pointwise nonexpansive mappings in a modular space.

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