Korean J. Math. **26** (2018), No. 4, pp. 545–559
https://doi.org/10.11568/kjm.2018.26.4.545

POSITION VECTOR OF A DEVELOPABLE q-SLANT RULED SURFACE

Onur Kaya† and Mehmet Önder

Abstract. In this paper, we study the position vector of a developable q-slant ruled surface in the Euclidean 3-space E^3 in means of the Frenet frame of a q-slant ruled surface. First, we determinate the natural representations for the striction curve and ruling of a q-slant ruled surface. Then we obtain general parameterization of a developable q-slant ruled surface with respect to the conical curvature of the surface. Finally, we introduce some examples for the obtained result.

1. Introduction

One of the most important and fascinating subject of differential geometry is special curves or special surfaces. Generally, special curves are such curves whose curvatures satisfy some special conditions. The well-known of special curves is general helix in the Euclidean 3-space E^3. A general helix is a regular curve whose tangent line makes a constant angle with a fixed straight line called the axis of the general helix. Therefore, a general helix can be equivalently defined as one whose tangent indicatrix is a circle or an arc of a circle on the unit sphere. Moreover, a general helix is characterized by the property that the function κ/τ

Received April 18, 2018. Revised October 5, 2018. Accepted October 7, 2018.
2010 Mathematics Subject Classification: 53A25, 14J26.
Key words and phrases: Position vector; Slant ruled surface; Developable ruled surface.
† This research was supported by Manisa Celal Bayar University Scientific Research Projects Coordination Unit (Project No: 2014/07).

is constant where κ and τ are curvature and torsion of the curve, respectively [6]. Of course, there exist some other special curves in the space. Recently, Izumiya and Takeuchi have introduced another special curve by a similar way of the definition of a helix. They have defined this new curve as slant helix which is a curve whose principal normal lines make a constant angle with a fixed direction and they have given a characterization of slant helix in the Euclidean 3-space E^3 [8]. Since the definitions of helix and slant helix are similar, we conclude that the principal normal indicatrix of a slant helix is a circle or an arc of a circle on the unit sphere. Slant helices have been studied by some mathematicians and new types of these curves have been introduced. Kula and Yaylı have studied the tangent indicatrix and the binormal indicatrix of a slant helix and obtained that the spherical images of a slant helix are helices lying on unit sphere [13]. Later, Kula and et al have introduced some new results characterizing slant helices in E^3 [14]. Moreover, Önder and et al have defined a new type of slant helix called B_2-slant helix in Euclidean 4-space E^4 and given the characterizations of this curve [18].

In the case of surfaces, ruled surface is a type of special surfaces which is generated by a continuous movement of a line along a curve. Önder has generalized the theory of general helix and slant helix to ruled surfaces and called slant ruled surfaces in E^3 [16]. He has defined the slant ruled surfaces by the property that the vectors of the Frenet frame of a ruled surface make constant angles with fixed directions and he has given characterizations for a ruled surface to be a slant ruled surface. Also, he has obtained that the striction curves of developable slant ruled surfaces are helices or slant helices. Later, Önder and Kaya have defined Darboux slant ruled surfaces in E^3 such as the Darboux vector of the ruled surface makes a constant angle with a fixed direction and they have given characterizations for a ruled surface to be a Darboux slant ruled surface [15].

Another different problem of the differential geometry is to determine a curve or a surface. The determination of parametric representation of the position vector of an arbitrary space curve or an arbitrary surface is still an open problem in the Euclidean space E^3 and in the Minkowski space E_1^3. It is not easy to solve this problem in general case. However, this problem has been solved in some special cases only for curves such as the curve lies on a special plane, or as the curve is a helix i.e., both the curvature κ and the torsion τ of the curve are non-vanishing constants or the curve is a general helix, i.e., the function κ/τ is constant [1–5,7,9,10]. Of course, the determination of a parametric representation of a surface is more complicated and difficult since the surface has two parameters. Moreover, similar to the curves, this determination can be given only in some special cases. Kaya and Önder investigated the determination problem for h-slant ruled surfaces which are defined in [16] and gave parametric representations for developable h-slant ruled surfaces [12].

The aim of this paper is to determine the parametric representation of a q-slant ruled surface in the Euclidean 3-space E^3. For this purpose, first we give a brief summary of ruled surfaces and slant ruled surfaces in Section 2. The Section 3 contains the determination of position vector of a q-slant ruled surface in E^3. In the last section, Section 4, some examples for the obtained results are given.

2. Ruled Surfaces in the Euclidean 3-space

In this section, we give a brief summary of the geometry of ruled surfaces and q-slant ruled surfaces in E^3.

A ruled surface S is a special surface generated by a continuous moving of a line along a curve $\vec{k}(u)$ and has the parametrization

$$\vec{r}(u,v) = \vec{k}(u) + v\vec{q}(u) \tag{1}$$

where the curve $\vec{k} = \vec{k}(u)$ is called base curve or generating curve of the surface and $\vec{q} = \vec{q}(u)$ is a unit direction vector of an oriented line in E^3 whose various positions are called rulings. The distribution parameter of S is the function $d = d(u)$ which is defined by

$$d = \frac{\left|\dot{\vec{k}}, \vec{q}, \dot{\vec{q}}\right|}{\langle \dot{\vec{q}}, \dot{\vec{q}} \rangle} \tag{2}$$

where $\dot{\vec{k}} = \frac{d\vec{k}}{du}$, $\dot{\vec{q}} = \frac{d\vec{q}}{du}$. If $\left|\dot{\vec{k}}, \vec{q}, \dot{\vec{q}}\right| = 0$, then the normal vectors do not change along a ruling, i.e., the tangent planes are identical at all points of the same ruling. Such a ruling is called a torsal ruling. If $\left|\dot{\vec{k}}, \vec{q}, \dot{\vec{q}}\right| \neq 0$, then the tangent planes of the surface S are distinct at all points of same ruling. Such rulings are called non-torsal [11].

DEFINITION 2.1. [11] A ruled surface whose all rulings are torsal is called a developable ruled surface. The remaining ruled surfaces are called skew ruled surfaces.

From (2), it is clear that a ruled surface is developable if and only if at all its points the distribution parameter is zero.

Let $\vec{m}$ be unit normal vector of a ruled surface S given by the parametrization (1). Then we have

$$\vec{m} = \frac{\vec{r}_u \times \vec{r}_v}{\|\vec{r}_u \times \vec{r}_v\|} = \frac{\left(\dot{\vec{k}} + v\dot{\vec{q}}\right) \times \vec{q}}{\sqrt{\left\langle \dot{\vec{k}} + v\dot{\vec{q}}, \dot{\vec{k}} + v\dot{\vec{q}} \right\rangle - \left\langle \dot{\vec{k}}, \vec{q} \right\rangle^2}} \tag{3}$$

If v infinitely decreases, then along a ruling $u = u_1$, the unit normal $\vec{m}$ approaches a limiting direction. This direction is called the asymptotic normal (or central tangent) direction and from (3) defined by

$$\vec{a} = \lim_{v \to \pm\infty} \vec{m}\,(u_1, v) = \frac{\vec{q} \times \dot{\vec{q}}}{\left\|\dot{\vec{q}}\right\|}.$$

The point at which the unit normal of S is perpendicular to $\vec{a}$ is called the striction point (or central point) C and the set of striction points of all rulings is called striction curve of the surface.

The vector $\vec{h}$ defined by $\vec{h} = \vec{a} \times \vec{q}$ is called central normal vector. Then the orthonormal system $\left\{C; \vec{q}, \vec{h}, \vec{a}\right\}$ is called Frenet frame of the ruled surface S where C is the central point and $\vec{q}, \vec{h}, \vec{a}$ are the unit vectors of ruling, the central normal vector and the central tangent vector, respectively.

The set of all bound vectors $\vec{q}(u)$ at the origin O constitutes a cone which is called directing cone of the ruled surface S. The end points of unit vectors $\vec{q}(u)$ trace a spherical curve $\vec{k}_1$ on the unit sphere S^2 and this curve is called *spherical image* of ruled surface S, whose arc length is denoted by s_1. A ruled surface and its directing cone have the same Frenet frame $\left\{\vec{q}, \vec{h}, \vec{a}\right\}$ and the derivative formulae of this frame with respect to the arc length s_1 are given as follows

$$\begin{bmatrix} d\vec{q}/ds_1 \\ d\vec{h}/ds_1 \\ d\vec{a}/ds_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & \kappa \\ 0 & -\kappa & 0 \end{bmatrix} \begin{bmatrix} \vec{q} \\ \vec{h} \\ \vec{a} \end{bmatrix} \tag{4}$$

where $\kappa(s_1) = \|d\vec{a}/ds_1\|$ is called the conical curvature of the directing cone (For details [11]).

Let now chose the base curve as the striction curve. Then the parametrization of ruled surface S is given by

$$\vec{r}(s,v) = \vec{c}(s) + v\vec{q}(s), \quad \|\vec{q}(s)\| = 1, \tag{5}$$

where s is the arc length parameter of striction curve. If S is a developable ruled surface then the tangent vectors of striction curve coincide with the rulings, i.e., $\frac{d\vec{c}}{ds} = \vec{q}$. Then, for the tangent vector of the striction curve we have

$$\frac{d\vec{c}}{ds_1} = f(s_1)\vec{q}(s_1) \tag{6}$$

where $\frac{d\vec{c}}{ds} = \vec{q}$ and $f(s_1) = \frac{ds}{ds_1}$ [11].

Definition 2.2. [16] Let S be a ruled surface in E^3 given by the parametrization

$$\vec{r}(s,v) = \vec{c}(s) + v\vec{q}(s), \quad \|\vec{q}(s)\| = 1,$$

where $\vec{c}(s)$ is striction curve of S and s is arc length parameter of $\vec{c}(s)$. Let the Frenet frame of S be $\left\{\vec{q}, \vec{h}, \vec{a}\right\}$. Then S is called a q-slant ruled surface if the ruling makes a constant angle θ with a fixed non-zero unit direction $\vec{u}$ in the space E^3, i.e.,

$$\langle \vec{q}, \vec{u} \rangle = \cos\theta = \text{constant}; \quad \theta \neq \frac{\pi}{2}. \tag{7}$$

Theorem 2.1. [17] *Let S be a regular ruled surface in E^3 with Frenet frame $\left\{\vec{q}, \vec{h}, \vec{a}\right\}$ and conical curvature $\kappa \neq 0$. Then S is a q-slant ruled surface if and only if the function κ is constant.*

3. Position Vectors of Developable q-Slant Ruled Surfaces

In this section, we give characterizations for the position vectors of slant ruled surfaces. First, we give the following theorem:

Theorem 3.1. *Let S be a developable q-slant ruled surface, $\left\{\vec{q}, \vec{h}, \vec{a}\right\}$ and $\kappa \neq 0$ be the Frenet frame and the conical curvature of the surface*

S, respectively. Then, the position vector of the striction curve of S with respect to the arc length parameter s_1 is given by:

$$\vec{c}(s_1) = \left(\kappa z(s_1) + \frac{z''(s_1)}{\kappa}\right)\vec{q}(s_1) - \frac{z'(s_1)}{\kappa}\vec{h}(s_1) + z(s_1)\vec{a}(s_1)$$

where

(8)

$$\begin{aligned} z(s_1) &= c_1\cos\left(\sqrt{1+\kappa^2}s_1\right) + c_2\sin\left(\sqrt{1+\kappa^2}s_1\right) \\ &+ \frac{\kappa}{\sqrt{1+\kappa^2}}\left[\left(\int\cos\left(\sqrt{1+\kappa^2}s_1\right)\left(\int f(s_1)ds_1\right)ds_1\right)\sin\left(\sqrt{1+\kappa^2}s_1\right)\right. \\ &\left. - \left(\int\sin\left(\sqrt{1+\kappa^2}s_1\right)\left(\int f(s_1)ds_1\right)ds_1\right)\cos\left(\sqrt{1+\kappa^2}s_1\right)\right] \end{aligned}$$

and c_1, c_2 are arbitrary constants, $f(s_1) = \frac{ds}{ds_1}$.

Proof. Since the Frenet frame $\left\{\vec{q}, \vec{h}, \vec{a}\right\}$ is linearly independent, the position vector of striction curve can be given by

$$\vec{c}(s_1) = x(s_1)\vec{q}(s_1) + y(s_1)\vec{h}(s_1) + z(s_1)\vec{a}(s_1) \tag{9}$$

where $x(s_1)$, $y(s_1)$ and $z(s_1)$ are differentiable functions of s_1. Since S is a developable ruled surface, from (6) we have

$$\vec{c}'(s_1) = f(s_1)\vec{q}(s_1) \tag{10}$$

where $\vec{c}'(s_1) = d\vec{c}/ds_1$. By differentiating (9) with respect to s_1 and using equality (10), we obtain the following system,

$$\begin{cases} x' - y - f = 0, \\ x + y' - z\kappa = 0, \\ y\kappa + z' = 0. \end{cases} \tag{11}$$

From the third equation of the system (11), we get

$$y' = -\frac{z'}{\kappa}. \tag{12}$$

By taking the derivative of (12) and substituting the obtained result in the second equation of the system (11) it follows

$$x = z\kappa + \left(\frac{z'}{\kappa}\right)'. \tag{13}$$

If we write the equations (12) and (13) in the first equation of the system (11), we obtain the following differential equation

$$\left(z\kappa+\left(\frac{z'}{\kappa}\right)'\right)'+\frac{z'}{\kappa}=f. \tag{14}$$

Since S is a q-slant ruled surface, we have that $\kappa=$ constant. Therefore, equation (14) becomes

$$z'''+\left(1+\kappa^2\right)z'=\kappa f \tag{15}$$

and by integrating (3) we obtain

$$z''+\left(1+\kappa^2\right)z=\int f ds_1. \tag{16}$$

The general solution of (16) is

$$\begin{aligned} z(s_1)=&\, c_1\cos\left(\sqrt{1+\kappa^2}s_1\right)+c_2\sin\left(\sqrt{1+\kappa^2}s_1\right)\\ &+\frac{\kappa}{\sqrt{1+\kappa^2}}\left[\left(\int\cos\left(\sqrt{1+\kappa^2}s_1\right)\left(\int f(s_1)ds_1\right)ds_1\right)\sin\left(\sqrt{1+\kappa^2}s_1\right)\right.\\ &\left.-\left(\int\sin\left(\sqrt{1+\kappa^2}s_1\right)\left(\int f(s_1)ds_1\right)ds_1\right)\cos\left(\sqrt{1+\kappa^2}s_1\right)\right] \end{aligned}$$

On the other hand, if we substitute (12) and (13) in (9) we get

$$\vec{c}(s_1)=\left(\kappa z(s_1)+\frac{z''(s_1)}{\kappa}\right)\vec{q}(s_1)-\frac{z'(s_1)}{\kappa}\vec{h}(s_1)+z(s_1)\vec{a}(s_1) \tag{17}$$

which is desired. □

From Theorem 3.1, we have the following corollaries:

Corollary 3.2. *Let S be a developable q-slant ruled surface with Frenet frame $\left\{\vec{q},\vec{h},\vec{a}\right\}$ and conical curvature $\kappa\neq 0$. Then the parametrization of the surface S is given by*

$$\vec{r}(s_1,v)=\left(\kappa z(s_1)+\frac{z''(s_1)}{\kappa}+v\right)\vec{q}(s_1)-\frac{z'(s_1)}{\kappa}\vec{h}(s_1)+z(s_1)\vec{a}(s_1) \tag{18}$$

where $z(s_1)$ is defined by (8).

Corollary 3.3. *If $z(s_1)=$ constant, then the position vector of the surface lies on the plane spanned by the vectors $\vec{q}$ and $\vec{a}$.*

Proof. From Theorem 3.1 we have

$$\vec{c}(s_1) = \left(\kappa z(s_1) + \frac{z''(s_1)}{\kappa}\right)\vec{q}(s_1) - \frac{z'(s_1)}{\kappa}\vec{h}(s_1) + z(s_1)\vec{a}(s_1) \tag{19}$$

Let $z(s_1)$ be constant. Then, by taking $z = m =$ constant, (19) becomes

$$\vec{c}(s_1) = m\left(\kappa\vec{q}(s_1) + \vec{a}(s_1)\right).$$

Therefore, the parametrization of the surface becomes

$$\vec{r}(s_1, v) = (m\kappa + v)\,\vec{q}(s_1) + m\vec{a}(s_1)$$

which means that the position vector of the surface S lies on the plane spanned by the vectors $\vec{q}$ and $\vec{a}$. □

Lemma 3.4. *Let S be a regular ruled surface with Frenet frame $\left\{\vec{q}, \vec{h}, \vec{a}\right\}$ and conical curvature function $\kappa \neq 0$. Then, the ruling of S satisfies the following third order differential equation:*

$$\kappa\vec{q}''' - \kappa'\vec{q}'' + \kappa\left(1 + \kappa^2\right)\vec{q}' - \kappa'\vec{q} = 0. \tag{20}$$

Proof. From the Frenet formulae (4) we have $\vec{q}' = \vec{h}$. By differentiating the last equation and using the Frenet formulae we obtain

$$\vec{a} = \frac{1}{\kappa}\left(\vec{q}'' + \vec{q}\right) \tag{21}$$

By taking the derivative of (21) and using the Frenet formulae again we get

$$\kappa\vec{q}''' - \kappa'\vec{q}'' + \kappa\left(1 + \kappa^2\right)\vec{q}' - \kappa'\vec{q} = 0.$$

which is desired. □

Corollary 3.5. *Let S be a developable ruled surface with Frenet frame $\left\{\vec{q}, \vec{h}, \vec{a}\right\}$ and conical curvature $\kappa \neq 0$. Then, the striction curve of S satisfies the following fourth order differential equation:*

$$\kappa\left(\frac{\vec{c}'(s_1)}{f(s_1)}\right)''' - \kappa'\left(\frac{\vec{c}'(s_1)}{f(s_1)}\right)'' + \kappa\left(1 + \kappa^2\right)\left(\frac{\vec{c}'(s_1)}{f(s_1)}\right)' - \kappa'\left(\frac{\vec{c}'(s_1)}{f(s_1)}\right) = 0.$$

Proof. From Lemma 3.4 we have that for a regular ruled surface

$$\kappa\vec{q}''' - \kappa'\vec{q}'' + \kappa\left(1 + \kappa^2\right)\vec{q}' - \kappa'\vec{q} = 0. \tag{22}$$

is satisfied. On the other hand, since S is a developable ruled surface from (10) we have

$$\vec{c}'(s_1) = f(s_1)\vec{q}(s_1). \tag{23}$$

By substituting (23) in (22) we obtain

$$\kappa\left(\frac{\vec{c}'(s_1)}{f(s_1)}\right)''' - \kappa'\left(\frac{\vec{c}'(s_1)}{f(s_1)}\right)'' + \kappa\left(1+\kappa^2\right)\left(\frac{\vec{c}'(s_1)}{f(s_1)}\right)' - \kappa'\left(\frac{\vec{c}'(s_1)}{f(s_1)}\right) = 0.$$

□

THEOREM 3.6. *Let S be a developable q-slant ruled surface with Frenet frame $\left\{\vec{q}, \vec{h}, \vec{a}\right\}$ and conical curvature $\kappa \neq 0$. Then, the position vector of the striction curve of S is given by:*

$$\begin{aligned}\vec{c}(s_1) = \frac{1}{\sqrt{1+\kappa^2}} &\left(\int \cos\left(\sqrt{1+\kappa^2}s_1\right) f(s_1)ds_1,\right.\\ &\left.\int \sin\left(\sqrt{1+\kappa^2}s_1\right) f(s_1)ds_1, \kappa\int f(s_1)ds_1\right)\end{aligned} \tag{24}$$

Proof. Since S is a developable q-slant ruled surface, from (6) and Theorem 2.1 we have$\vec{c}'(s_1) = f(s_1)\vec{q}(s_1)$ and $\kappa =$ constant, respectively. Then, the equation (20) becomes

$$\vec{q}''' + \left(1+\kappa^2\right)\vec{q}' = 0. \tag{25}$$

The ruling can be given by:

$$\vec{q} = q_1\vec{e}_1 + q_2\vec{e}_2 + q_3\vec{e}_3 \tag{26}$$

where $\vec{e}_1, \vec{e}_2, \vec{e}_3$ are the base vectors of $\mathbb{R}^3$. Since S is a q-slant ruled surface, the ruling makes a constant angle with a fixed direction. Without loss of generality, we can take the fixed straight line as $\vec{e}_3$. Therefore, we get

$$q_3 = \langle\vec{q}, \vec{e}_3\rangle = \cos\theta = n = \text{constant} \tag{27}$$

where θ is the constant angle between the unit vectors $\vec{q}, \vec{e}_3$. By using (27) and that $\vec{q}$ is a unit vector, we obtain

$$q_1^2 + q_2^2 = 1 - n^2 = \sin^2\theta \tag{28}$$

The general solution of (28) is given by

$$q_1 = \sin\theta\cos\left(t(s_1)\right), \quad q_2 = \sin\theta\sin\left(t(s_1)\right) \tag{29}$$

where $t = t(s_1)$ is an arbitrary function of s_1. Then from (29) and (26) we have

$$\vec{q} = \sin\theta\cos\left(t(s_1)\right)\vec{e}_1 + \sin\theta\sin\left(t(s_1)\right)\vec{e}_2 + \cos\theta\vec{e}_3.$$

Since s_1 is the arc length parameter of the spherical curve $\vec{k}_1(s_1)$ drawn by $\vec{q}$, from last equality we have that $t' \sin\theta = 1$, which gives us

$$t(s_1) = \frac{1}{\sin\theta} s_1 + c$$

where c is a real constant. Thanks to parameter change $t \to t + c$, we obtain

$$t(s_1) = \frac{1}{\sin\theta} s_1. \tag{30}$$

On the other hand the components of the vector $\vec{q}$ satisfy (25). Therefore, by considering (29) and (26) in (25), we get the following system,

$$\begin{cases} \left[t''' - (t')^3 + (1+\kappa^2)\, t'\right] \sin t + (3t''t') \cos t = 0, \\ -\left[t''' - (t')^3 + (1+\kappa^2)\, t'\right] \cos t + (3t''t') \sin t = 0, \end{cases}$$

which can be reduced to

$$\begin{cases} t''t' = 0, \\ t''' - (t')^3 + (1+\kappa^2)\, t' = 0. \end{cases} \tag{31}$$

Since t is not constant, then the general solution of the first equation of system (31) can be given by:

$$t(s_1) = d_1 s_1 + d_2 \tag{32}$$

where d_1 and d_2 are arbitrary constants. Thanks to parameter change $t \to t + d_2$, (32) becomes

$$t(s_1) = d_1 s_1. \tag{33}$$

If we substitute (33) in the second equation of the system (31), we obtain

$$d_1 = \sqrt{1+\kappa^2}. \tag{34}$$

By using (33), (34) and (30) we obtain that $\sin\theta = \frac{1}{\sqrt{1+\kappa^2}}$. Then the parametrization of ruling $\vec{q}$ is obtained as follows

$$\vec{q}(s_1) = \frac{1}{\sqrt{1+\kappa^2}} \left(\cos\left(\sqrt{1+\kappa^2} s_1\right), \sin\left(\sqrt{1+\kappa^2} s_1\right), \kappa\right).$$

Since S is developable, from (6) we have that

$$\vec{c}(s_1) = \frac{1}{\sqrt{1+\kappa^2}} \left(\int \cos\left(\sqrt{1+\kappa^2} s_1\right) f(s_1) ds_1 \, , \int \sin\left(\sqrt{1+\kappa^2} s_1\right) f(s_1) ds_1, \kappa \int f(s_1) ds_1 \right)$$

□

Theorem 3.6 gives us the following corollary:

Corollary 3.7. *Let S be a developable q-slant ruled surface with Frenet frame $\left\{\vec{q}, \vec{h}, \vec{a}\right\}$ and conical curvature $\kappa \neq 0$. Then, the parametrization of the surface S is given by*

(35)

$$\vec{r}(s_1, v) = \frac{1}{\sqrt{1+\kappa^2}} \left(\int fg ds_1 + vg, \int fh ds_1 + vh, \left(\int f ds_1 + v \right) \kappa \right)$$

where $g(s_1) = \cos\left(\sqrt{1+\kappa^2} s_1\right)$ and $h(s_1) = \sin\left(\sqrt{1+\kappa^2} s_1\right)$.

Proof. Let S be a developable q-slant ruled surface. From Theorem 3.6 we have

$$(36) \qquad \begin{cases} \vec{q}(s_1) = \frac{1}{\sqrt{1+\kappa^2}} \left(\cos\left(\sqrt{1+\kappa^2} s_1\right), \sin\left(\sqrt{1+\kappa^2} s_1\right), \kappa\right), \\ \vec{c}(s_1) = \frac{1}{\sqrt{1+\kappa^2}} \left(\int \cos\left(\sqrt{1+\kappa^2} s_1\right) f(s_1) ds_1, \right. \\ \qquad \left. \int \sin\left(\sqrt{1+\kappa^2} s_1\right) f(s_1) ds_1, \kappa \int f(s_1) ds_1\right). \end{cases}$$

Since the parameterization of the surface S is given by

$$(37) \qquad \vec{r}(s_1, v) = \vec{c}(s_1) + v\vec{q}(s_1)$$

by substituting (36) in (3) we obtain

(38)

$$\begin{aligned} \vec{r}(s_1, v) = \frac{1}{\sqrt{1+\kappa^2}} \Bigg[& \int f(s_1) \cos\left(\sqrt{1+\kappa^2} s_1\right) ds_1 + v \cos\left(\sqrt{1+\kappa^2} s_1\right), \\ & \int f(s_1) \sin\left(\sqrt{1+\kappa^2} s_1\right) ds_1 + v \sin\left(\sqrt{1+\kappa^2} s_1\right), \\ & \kappa \int f(s_1) ds_1 + v\kappa \Bigg]. \end{aligned}$$

If we take $g(s_1) = \cos\left(\sqrt{1+\kappa^2} s_1\right)$ and $h(s_1) = \sin\left(\sqrt{1+\kappa^2} s_1\right)$ in (38), we get

$$\vec{r}(s_1, v) = \frac{1}{\sqrt{1+\kappa^2}} \left(\int fg ds_1 + vg, \int fh ds_1 + vh, \left(\int f ds_1 + v \right) \kappa \right)$$

□

COROLLARY 3.8. *Let S be a developable q-slant ruled surface with Frenet frame $\left\{\vec{q}, \vec{h}, \vec{a}\right\}$ and conical curvature $\kappa \neq 0$. Then, the parametrization of the surface S according to the arc length s of striction curve is given by:*

$$\vec{r}(s,v) = \frac{1}{\sqrt{1+\kappa^2}}\left(\int g ds + vg, \int h ds + vh, (s+m+v)\,\kappa\right)$$

where m is a constant.

Proof. Since $f = \frac{ds}{ds_1}$, from (35) we have the desired equation immediately. □

4. Examples

In this section we consider some special chosen of conical curvature κ and function $f(s_1)$ and obtain some examples of developable q-slant ruled surfaces by considering equality (35).

EXAMPLE 4.1. Let consider the ruled surface S with conical curvature $\kappa = \sqrt{3}$ and function $f = 2$. The parametrization of developable q-slant ruled surface S with axis $\vec{e}_3$ is obtained as follows

$$\begin{aligned}\vec{r}(s_1,v) = \Bigg(&\frac{1}{2}\sin(2s_1) + c_1 + \frac{1}{2}v\cos(2s_1), -\frac{1}{2}\cos(2s_1) + c_2 + \frac{1}{2}v\sin(2s_1),\\ &\sqrt{3}s_1 + c_3 + \frac{1}{2}v\sqrt{3}\Bigg)\end{aligned}$$

where c_1, c_2, c_3 are integration constants. The shape of surface S is given in Fig. 1.

EXAMPLE 4.2. If we take $\kappa = 2\sqrt{2}$ and $f = 3$, the parametrization of developable q-slant ruled surface S with axis $\vec{e}_3$ is obtained as follows

$$\begin{aligned}\vec{r}(s_1,v) = \Bigg(&\frac{1}{3}\sin(3s_1) + d_1 + \frac{1}{3}v\cos(3s_1), -\frac{1}{3}\cos(3s_1) + d_2 + \frac{1}{3}v\sin(3s_1),\\ &2\sqrt{2}s_1 + d_3 + \frac{2\sqrt{2}}{3}v\Bigg)\end{aligned}$$

where d_1, d_2, d_3 are integration constants. Then the shape of obtained surface is given in Fig. 2.

Example 4.3. If we take the conical curvature $\kappa = 1$ and function $f(s_1) = s_1^2$. Then the parametrization of developable q-slant ruled surface S with axis $\vec{e}_3$ is obtained as follows

$$\begin{aligned}\vec{r}(s_1, v) = \Bigg(& \frac{1}{2}s_1^2 \sin\left(\sqrt{2}s_1\right) - \frac{1}{2}\sin\left(\sqrt{2}s_1\right) + \frac{1}{2}\sqrt{2}s_1 \cos\left(\sqrt{2}s_1\right) + m_1 \\ & + \frac{1}{2}v\sqrt{2}\cos\left(\sqrt{2}s_1\right), \\ & -\frac{1}{2}s_1^2 \cos\left(\sqrt{2}s_1\right) + \frac{1}{2}\cos\left(\sqrt{2}s_1\right) + \frac{1}{2}\sqrt{2}s_1 \sin\left(\sqrt{2}s_1\right) \\ & + m_2 + \frac{1}{2}v\sqrt{2}\sin\left(\sqrt{2}s_1\right), \\ & \frac{1}{6}\sqrt{2}s_1^3 + m_3 + \frac{1}{2}v\sqrt{2}\Bigg)\end{aligned}$$

where m_1, m_2, m_3 are integration constants and the shape of surface is given in Fig. 3.

Example 4.4. Let now consider the ruled surface S with conical curvature $\kappa = 2$ and function $f(s_1) = e^{s_1}$. The parametrization of developable q-slant ruled surface S with axis $\vec{e}_3$ is obtained as follows

$$\begin{aligned}\vec{r}(s_1, v) = \Bigg(& \frac{1}{30}\sqrt{5}e^{s_1} \cos\left(\sqrt{5}s_1\right) + \frac{1}{6}e^{s_1} \sin\left(\sqrt{5}s_1\right) + n_1 + \frac{\sqrt{5}}{5}v\cos\left(\sqrt{5}s_1\right), \\ & -\frac{1}{6}e^{s_1}\cos\left(\sqrt{5}s_1\right) + \frac{1}{30}\sqrt{5}e^{s_1}\sin\left(\sqrt{2}s_1\right) + n_2 + \frac{\sqrt{5}}{5}v\sin\left(\sqrt{5}s_1\right), \\ & \frac{2}{5}\sqrt{5}e^{s_1} + n_3 + \frac{2}{5}\sqrt{5}v\Bigg)\end{aligned}$$

where n_1, n_2, n_3 are integration constants and the shape of surface is given in Fig. 4.

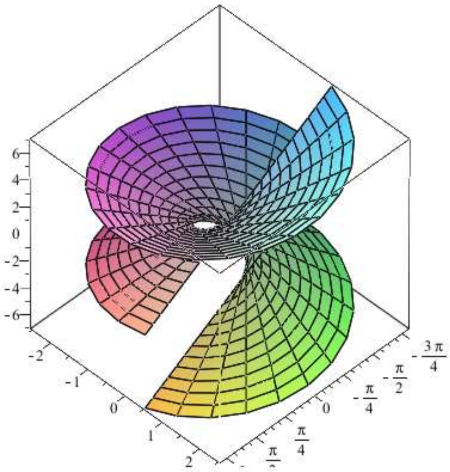

Figure 1

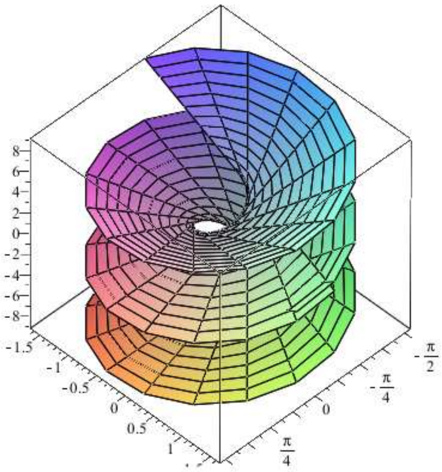

Figure 2

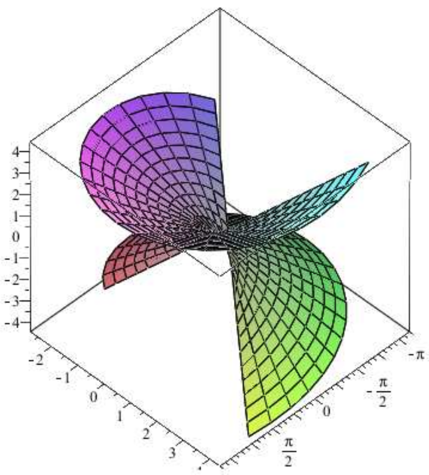

Figure 3

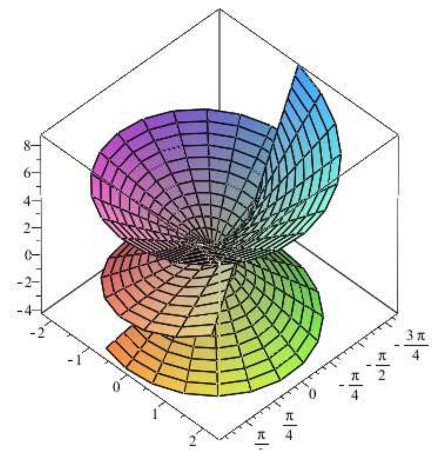

Figure 4

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Onur Kaya
Department of Mathematics
Manisa Celal Bayar University
Manisa 45140, Turkey
E-mail: `onur.kaya@cbu.edu.tr`

Mehmet Önder
Independent Researcher
Delibekirli Village, Tepe Street
Kırıkhan, Hatay 31440, Turkey
E-mail: `mehmetonder197999@gmail.com`