SOME GROWTH PROPERTIES BASED ON (p,q)-TH ORDER OF THE INTEGER TRANSLATION OF THE COMPOSITE ENTIRE AND MEROMORPHIC FUNCTIONS

Ritam Biswas

Abstract. In this paper the growth properties of the composition of integer translated entire and meromorphic functions in terms of their (p,q)-th order are discussed and based upon that some new results are developed.

1. Introduction, definitions and notations

Investigation of the behaviour of meromorphic and entire functions and their composition with the help of various growth indicators defined for them has been the most important phenomenon in the theory of complex numbers since years. Here we intend to develop some new results based upon the basic discussion about the growth properties of composition of integer translated entire and meromorphic functions. Consider $f(z)$ to be an *entire function* defined in the finite complex plane $\mathbb{C}$. The *maximum modulus function* corresponding to *entire* $f(z)$ is defined as $M(r; f) = \max\{|f(z)| : |z| = r\}$. If $f(z)$ be *meromorphic*, a different function $T(r, f)$ termed as *Nevanlinna's Characteristic function* of $f(z)$, may be defined for it, which plays the same role as *maximum modulus*

function in the following way:

$$T(r, f) = N(r, f) + m(r, f),$$

where the function $N(r, a; f)$ $(\overline{N}(r, a; f))$ known as *counting function* of a-points (distinct a-points) of *meromorphic* f is defined as

$$N(r, a; f) = \int_0^r \frac{n(t, a; f) - n(0, a; f)}{t} dt + n(0, a; f) \log r$$

$$(\overline{N}(r, a; f) = \int_0^r \frac{\overline{n}(t, a; f) - \overline{n}(0, a; f)}{t} dt + \overline{n}(0, a; f) \log r),$$

moreover we denote by $n(r, a; f)$ $(\overline{n}(r, a; f))$ the number of a-points (distinct a-points) of f in $|z| \leq r$ and an ∞-point is a *pole* of $f(z)$. In many occasions $N(r, \infty; f)$ and $\overline{N}(r, \infty; f)$ are denoted by $N(r, f)$ and $\overline{N}(r, f)$ respectively.

Further, the function $m(r, \infty; f)$ alternatively denoted by $m(r, f)$ known to be the *proximity function* of $f(z)$ is defined as follows:

$$m(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| f(re^{i\theta}) \right| d\theta,$$

where

$$\log^+ x = \max\{\log x, 0\}$$

for all $x \geq 0$, where $\log x$ means the logarithm of x.

Also $m(r, \frac{1}{f-a})$ may be denoted by $m(r, a; f)$.

If $f(z)$ be an entire function, then the *Nevanlinna's Characteristic function* $T(r, f)$ of $f(z)$ becomes

$$T(r, f) = m(r, f).$$

Further let $f(z)$ be a *meromorphic function* and $n \in \mathbb{N}$, where $\mathbb{N}$ is the set of all natural numbers, then the *translation* of $f(z)$ be denoted by $f(z + n)$. For each $n \in \mathbb{N}$, one may obtain a function with some properties. Let us consider this family by $f_n(z)$ where

$$f_n(z) = \{f(z + n) : n \in \mathbb{N}\}.$$

We recall that if α is a regular point of an analytic function $f(z)$ and if $f(\alpha) = 0$, then α is called a zero of $f(z)$. The point $z = \alpha$ is called a zero of $f(z)$ of multiplicity m (m being a positive integer) if in some

neighbourhood of α, $f(z)$ can be expanded in a Taylor's series of the form $f(z) = \sum\limits_{n=m}^{\infty} a_n (z - \alpha)^n$ where $a_m \neq 0$.

It is clear that the number of zeros of $f(z)$ may be changed in a finite region after translation but it remains unaltered in the open complex plane $\mathbb{C}$, i.e.,

$$(1) \qquad\qquad N(r, f(z + n)) = N(r, f) + e_n,$$

where e_n is a residue term such that $e_n \to 0$ as $r \to \infty$.

Also

$$m(r, f(z + n)) \;=\; \frac{1}{2\pi} \int\limits_0^{2\pi} \log^+ \left| f(re^{i\theta} + n) \right| d\theta$$

$$(2) \qquad i.e., m(r, f(z + n)) \;=\; m(r, f) + e_n',$$

where e_n' (may be distinct from e_n) be such that $e_n' \to 0$ as $r \to \infty$.

Therefore from (1) and (2), one may obtain that

$$N(r, f(z + n)) + m(r, f(z + n)) \;=\; N(r, f) + e_n + m(r, f) + e_n'$$
$$i.e., T(r, f(z + n)) \;=\; T(r, f) + e_n + e_n'.$$

Now if n varies, then the Nevanlinna's Characteristic function for the family $f_n(z)$, where $f_n(z) = \{f(z + n) : n \in \mathbb{N}\}$ for the meromorphic function f is

$$(3) \qquad\qquad T(r, f_n) = nT(r, f) + \sum_n (e_n + e_n').$$

Similarly, one can define a family for each $m \in \mathbb{N}$, $g_m(z) = \{g(z + m) : m \in \mathbb{N}\}$ where $g(z)$ is an entire function. Then the composition $f_n \circ g_m$ is defined.

Let $f_n \circ g_m = h_t$, where h is a meromorphic function and $t \in \mathbb{N}$. So h_t can be expressed as $h_t = \{h(z + t) : t \in \mathbb{N}\}$.

Then by (3)

$$T(r, h_t) = tT(r, h) + \sum_t (e_t + e_t')$$

where $e_t, e_t' \to 0$ as $r \to \infty$.

$$(4) \qquad\qquad i.e., T(r, f_n \circ g_m) = tT(r, f \circ g) + \sum_t (e_t + e_t').$$

Before introducing the growth definitions for the integer translation of composite entire and meromorphic functions and some basic relations related to it, the following notations and definitions are needed to be mentioned [we do not explain the standard definitions and notations in the theory of entire and meromorphic functions as those are available in [9] and [4]]:

$$\log^{[-k]} x = \exp^{[k]} x; \text{ for } k = 1, 2, 3, \cdots$$
$$\log^{[0]} x = x;$$
$$\log^{[k]} x = \log\left(\log^{[k-1]} x\right) \text{ for } k = 1, 2, 3, \cdots.$$

and

$$\exp^{[-k]} x = \log^{[k]} x; \text{ for } k = 1, 2, 3, \cdots$$
$$\exp^{[0]} x = x;$$
$$\exp^{[k]} x = \exp\left(\exp^{[k-1]} x\right) \text{ for } k = 1, 2, 3, \cdots.$$

We now recall the following definitions.

DEFINITION 1. The order ρ_f and lower order λ_f of a meromorphic function f are defined as

$$\rho_f = \limsup_{r \to \infty} \frac{\log T(r, f)}{\log r} \text{ and } \lambda_f = \liminf_{r \to \infty} \frac{\log T(r, f)}{\log r} \text{ respectively.}$$

If f is entire then one can easily verify that

$$\rho_f = \limsup_{r \to \infty} \frac{\log^{[2]} M(r, f)}{\log r} \text{ and } \lambda_f = \liminf_{r \to \infty} \frac{\log^{[2]} M(r, f)}{\log r}.$$

DEFINITION 2. [8] Let l be an integer ≥ 2. The generalized order $\rho_f^{[l]}$ and generalized lower order $\lambda_f^{[l]}$ of an entire function f are defined as

$$\rho_f^{[l]} = \limsup_{r \to \infty} \frac{\log^{[l]} M(r, f)}{\log r} \text{ and } \lambda_f^{[l]} = \liminf_{r \to \infty} \frac{\log^{[l]} M(r, f)}{\log r} \text{ respectively.}$$

If f is meromorphic, one can easily verify that

$$\rho_f^{[l]} = \limsup_{r \to \infty} \frac{\log^{[l-1]} T(r, f)}{\log r} \text{ and } \lambda_f^{[l]} = \liminf_{r \to \infty} \frac{\log^{[l-1]} T(r, f)}{\log r}.$$

Juneja, Kapoor and Bajpai [5] defined the (p, q) th order and (p, q) th lower order of an entire function f respectively as follows:

$$\rho_f(p, q) = \limsup_{r \to \infty} \frac{\log^{[p]} M(r, f)}{\log^{[q]} r} \text{ and } \lambda_f(p, q) = \liminf_{r \to \infty} \frac{\log^{[p]} M(r, f)}{\log^{[q]} r},$$

where p, q are positive integers with $p > q$.

When f is meromorphic, one can easily verify that

$$\rho_f(p, q) = \limsup_{r \to \infty} \frac{\log^{[p-1]} T(r, f)}{\log^{[q]} r} \text{ and } \lambda_f(p, q) = \liminf_{r \to \infty} \frac{\log^{[p-1]} T(r, f)}{\log^{[q]} r},$$

where p, q are positive integers and $p > q$.

The definitions of the growth indicators mentioned above are also valid for the composite meromorphic and entire functions $f \circ g$ where f is meromorphic and g is entire, as in this case $f \circ g$ is also a meromorphic function.

Now in the line of the definitions of (p, q)-th order and (p, q)-th lower order of a meromorphic function f we can define the (p, q)-th order and (p, q)-th lower order of the function f_n, as

$$\rho_{f_n}(p, q) = \limsup_{r \to \infty} \frac{\log^{[p-1]} T(r, f_n)}{\log^{[q]} r}$$

and

$$\lambda_{f_n}(p, q) = \liminf_{r \to \infty} \frac{\log^{[p-1]} T(r, f_n)}{\log^{[q]} r},$$

where p, q are positive integers and $p > q$.

Now from (3) we obtain

$$\log^{[p-1]} T(r, f_n) = \log^{[p-1]}[nT(r, f) + \sum_t (e_t + e'_t)]$$

$$i.e., \log^{[p-1]} T(r, f_n) = \log^{[p-1]}\left[nT(r, f)\left\{1 + \frac{\sum_t (e_t + e'_t)}{nT(r, f)}\right\}\right]$$

$$i.e., \limsup_{r \to \infty} \frac{\log^{[p-1]} T(r, f_n)}{\log^{[q]} r} = \limsup_{r \to \infty} \frac{\log^{[p-1]} T(r, f) + O(1)}{\log^{[q]} r}.$$

Therefore

(5) $$\rho_{f_n}(p, q) = \rho_f(p, q).$$

Similarly, we can show that

(6) $$\lambda_{f_n}(p, q) = \lambda_f(p, q).$$

2. Lemmas

In this section we present some lemmas which will be frequently used in the next section.

LEMMA 1. [1] *If f is a meromorphic function and g is an entire function then for all sufficiently large values of r,*

$$T(r, f \circ g) \leqslant \{1 + o(1)\} \frac{T(r, g)}{\log M(r, g)} T(M(r, g), f).$$

LEMMA 2. [2] *Let f be a meromorphic function and g be an entire function and suppose that $0 < \mu < \rho_g \leq \infty$. Then for a sequence of values of r tending to infinity,*

$$T(r, f \circ g) \geq T(\exp(r^\mu), f).$$

LEMMA 3. [3] *Let g be an entire function. Then for any $\delta(> 0)$ the function $r^{\lambda_g^{[l]} + \delta - \lambda_g^{[l]}(r)}$ is ultimately an increasing function of r.*

LEMMA 4. [5] *Let f be an entire function with non zero finite generalised order $\rho_f^{[l]}$ (non zero finite generalised lower order $\lambda_f^{[l]}$). If $p - q = l - 1$, then the (p, q)- th order $\rho_f(p, q)$ (lower (p, q)- th order $\lambda_f(p, q)$) of f will be equal to 1. If $p - q \neq l - 1$ then $\rho_f(p, q)$ ($\lambda_f(p, q)$) is either zero or infinity.*

REMARK 1. Lemma 4 is still valid for meromorphic f under the same conditions stated as above.

3. Theorems

In this section we present the main results of the paper.

THEOREM 1. *Let g be entire function and h, k be two transcendental entire functions such that $\lambda_h(a, b) > 0$, $\lambda_k(c, d) > 0$ and $\rho_g(m, n) < \lambda_k(c, d)$ where m, n, a, b, c, d are all positive integers with $m > n$, $a > b$ and $c > d$. Let h_s, k_t, f_u, g_v be the family of integer translations of*

the functions h, k, f, g respectively where $s, t, u, v \in N$. Then for every meromorphic function f with $0 < \rho_f(p, q) < \infty$ and for any two positive integers p, q with $p > q$

$$(i) \lim_{r \to \infty} \frac{\log^{[a-1]} T(\exp^{[d-1]} r, h_s \circ k_t)}{\log^{[p-1]} T(r, f_u \circ g_v)} = \infty, \text{ if } q \geqslant m \text{ and } b < c,$$

$$(ii) \lim_{r \to \infty} \frac{\log^{[a-1]} T(\exp^{[d-1]} r, h_s \circ k_t)}{\log^{[p+m-q-2]} T(r, f_u \circ g_v)} = \infty, \text{ if } q < m \text{ and } b < c,$$

$$(iii) \lim_{r \to \infty} \frac{\log^{[a+c-b-2]} T(\exp^{[d-1]} r, h_s \circ k_t)}{\log^{[p-1]} T(r, f_u \circ g_v)} = \infty, \text{ if } q \geqslant m \text{ and } b \geqslant c,$$

$$\text{and } (iv) \lim_{r \to \infty} \frac{\log^{[a+c-b-2]} T(\exp^{[d-1]} r, h_s \circ k_t)}{\log^{[p+m-q-2]} T(r, f_u \circ g_v)} = \infty, \text{ if } q < m \text{ and } b \geqslant c.$$

Proof. Let $f_u \circ g_v = h_t$, where h is a meromorphic function and $t \in \mathbb{N}$. So h_t can be expressed as

$$h_t = \{h(z + t) : t \in \mathbb{N}\}.$$

Then by (3) we obtain

$$T(r, h_t) = tT(r, h) + \sum_t (e_t + e_t'),$$

where $e_t, e_t' \to 0$ as $r \to \infty$,

$$(7) \qquad i.e., T(r, f_u \circ g_v) = tT(r, f \circ g) + \sum_t (e_t + e_t').$$

Now in view of Lemma 1 and the inequality $T(r, g) \leqslant \log^+ M(r, g)$ we get from (7) for all sufficiently large values of r that

$$T(r, f_u \circ g_v) \leq t\{1 + o(1)\} T(M(r, g), f) + \sum_t (e_t + e_t')$$

$$i.e., \ \log^{[p-1]} T(r, f_u \circ g_v) \leq \log^{[p-1]} T(M(r, g), f) + O(1)$$

$$(8) \qquad i.e., \ \log^{[p-1]} T(r, f_u \circ g_v) \leq (\rho_f(p, q) + \varepsilon) \log^{[q]} M(r, g) + O(1).$$

Now the following cases may arise :
Case I. Let $q \geqslant m$. Then we have from (8) for all sufficiently large values of r,

$$(9) \qquad \log^{[p-1]} T(r, f_u \circ g_v) \leqslant (\rho_f(p, q) + \varepsilon) \log^{[m-1]} M(r, g) + O(1).$$

Now from the definition of (m, n)-th order of g we get for arbitrary positive ε and for all sufficiently large values of r,

$$\log^{[m]} M(r, g) \leqslant (\rho_g(m, n) + \varepsilon) \log^{[n]} r$$

(10) $i.e.,\ \log^{[m]} M(r, g) \leq (\rho_g(m, n) + \varepsilon) \log r.$

Also for all sufficiently large values of r it follows from (10) that

(11) $\log^{[m-1]} M(r, g) \leqslant r^{(\rho_g(m,n)+\varepsilon)}.$

So from (9) and (11) it follows for all sufficiently large values of r that

(12) $\log^{[p-1]} T(r, f_u \circ g_v) \leqslant (\rho_f(p, q) + \varepsilon) r^{(\rho_g(m,n)+\varepsilon)} + O(1).$

Case II. Let $q < m$. Then we get from(8) for all sufficiently large values of r that

(13)
$\log^{[p-1]} T(r, f_u \circ g_v) \leqslant (\rho_f(p, q) + \varepsilon) \exp^{[m-q]} \log^{[m]} M(r, g) + O(1).$

Again from (10) for all sufficiently large values of r,

$$\exp^{[m-q]} \log^{[m]} M(r, g) \leqslant \exp^{[m-q]} \log r^{(\rho_g(m,n)+\varepsilon)}$$

(14) $i.e.,\ \exp^{[m-q]} \log^{[m]} M(r, g) \leqslant \exp^{[m-q-1]} r^{(\rho_g(m,n)+\varepsilon)}.$

Now from (13) and (14) we obtain for all sufficiently large values of r that

$$\log^{[p-1]} T(r, f_u \circ g_v) \leqslant (\rho_f(p, q) + \varepsilon) \exp^{[m-q-1]} r^{(\rho_g(m,n)+\varepsilon)} + O(1)$$

$$i.e., \log^{[p]} T(r, f_u \circ g_v) \leqslant \exp^{[m-q-2]} r^{(\rho_g(m,n)+\varepsilon)} + O(1)$$

$$i.e., \log^{[p+m-q-2]} T(r, f_u \circ g_v) \leqslant \log^{[m-q-2]} \exp^{[m-q-2]} r^{(\rho_g(m,n)+\varepsilon)} + O(1)$$

(15) $i.e., \log^{[p+m-q-2]} T(r, f_u \circ g_v) \leqslant r^{(\rho_g(m,n)+\varepsilon)} + O(1).$

Since $\rho_g(m, n) < \lambda_k(c, d)$, we can choose $\varepsilon\ (> 0)$ in such a way that

(16) $\rho_g(m, n) + \varepsilon < \lambda_k(c, d) - \varepsilon.$

Similarly, if we consider $h \circ k = w$ for some entire function w then in the line of (7) we can obtain

(17) $T(r, h_s \circ k_t) = iT(r, h \circ k) + \sum_i (e_i + e_i'),$

where $i \in \mathbb{N}$ and $e_i,\ e_i' \to 0$ as $r \to \infty$. Now using the inequality $T(r, h \circ k) \geqslant \frac{1}{3} \log M\left\{\frac{1}{8} M(\frac{r}{4}, k) + o(1), h\right\} \{cf.\ [6]\}$ and by the definition

of (p, q)-th lower order of an entire function we obtain for all large values of r that

$$\log^{[a-1]} T(\exp^{[d-1]} r, h_s \circ k_t)$$

$$\geqslant \log^{[a]} M \left\{ \frac{1}{8} M(\frac{\exp^{[d-1]} r}{4}, k) + \circ(1), h \right\} + O(1)$$

$$i.e., \ \log^{[a-1]} T(\exp^{[d-1]} r, h_s \circ k_t)$$

$$\geqslant (\lambda_h(a, b) - \varepsilon) \log^{[b]} \left\{ \frac{1}{8} M(\frac{\exp^{[d-1]} r}{4}, k) \right\} + O(1)$$

$$(18) i.e., \ \log^{[a-1]} T(\exp^{[d-1]} r, h_s \circ k_t)$$

$$\geqslant (\lambda_h(a, b) - \varepsilon) \log^{[b]} M(\frac{\exp^{[d-1]} r}{4}, k) + O(1).$$

Case III. Let $b < c$. Then from (18) it follows for all sufficiently large values of r that

$$(19) \quad \log^{[a-1]} T(\exp^{[d-1]} r, h_s \circ k_t)$$

$$\geqslant (\lambda_h(a, b) - \varepsilon) \exp^{[c-b-1]} \log^{[c-1]} M(\frac{\exp^{[d-1]} r}{4}, k) + O(1).$$

Now from the definition of (c, d)-th lower order of k we obtain for arbitrary positive $\varepsilon \ (> 0)$ and for all sufficiently large values of r that

$$\log^{[c]} M(\frac{\exp^{[d-1]} r}{4}, k) \quad \geqslant \quad (\lambda_k(c, d) - \varepsilon) \log^{[d]} (\frac{\exp^{[d-1]} r}{4})$$

$$i.e., \quad \log^{[c]} M(\frac{\exp^{[d-1]} r}{4}, k) \quad \geqslant \quad (\lambda_k(c, d) - \varepsilon) \log r + O(1)$$

$$(20) \ i.e., \quad \log^{[c]} M(\frac{\exp^{[d-1]} r}{4}, k) \quad \geqslant \quad \log r^{(\lambda_k(c,d)-\varepsilon)} + O(1).$$

Also for all large values of r we get from (20) that

$$(21) \quad \log^{[c-1]} M(\frac{\exp^{[d-1]} r}{4}, k) \geqslant r^{(\lambda_k(c,d)-\varepsilon)} + O(1).$$

Now from (19) and (21) it follows for all sufficiently large values of r that
(22)
$$\log^{[a-1]} T(\exp^{[d-1]} r, h_s \circ k_t) \geqslant (\lambda_h(a, b) - \varepsilon) \exp^{[c-b-1]} r^{(\lambda_k(c,d)-\varepsilon)} + O(1).$$

Case IV. Let $b \geqslant c$. Then from (18) we obtain for all sufficiently large values of r,

(23) $\log^{[a-1]} T(\exp^{[d-1]} r, h_s \circ k_t)$

$$\geqslant (\lambda_h(a,b) - \varepsilon) \log^{[b-c]} \log^{[c]} \left\{ M(\frac{\exp^{[d-1]} r}{4}, k) \right\} + O(1).$$

Now from (20) and (23) we have for all sufficiently large values of r,

$\log^{[a-1]} T(\exp^{[d-1]} r, h_s \circ k_t)$

$$\geqslant (\lambda_h(a,b) - \varepsilon) \log^{[b-c]} \log r^{(\lambda_k(c,d)-\varepsilon)} + O(1)$$

i.e., $\log^{[a-1]} T(\exp^{[d-1]} r, h_s \circ k_t)$

$$\geqslant (\lambda_h(a,b) - \varepsilon) \log^{[b-c+1]} r^{(\lambda_k(c,d)-\varepsilon)} + O(1)$$

(24)
i.e., $\log^{[a]} T(\exp^{[d-1]} r, h_s \circ k_t) \geqslant \log^{[b-c+2]} r^{(\lambda_k(c,d)-\varepsilon)} + O(1).$

Since $a > b$ and a and b both are positive integers, $a \geq b+1$. So $a-b \geq 1$. Again since c is also a positive integer, $c \geq 1$. Therefore $a - b + c \geq 2$ i.e., $a + c - b - 2 \geq 0$. So from (24) we get for all sufficiently large values of r,

(25) $\log^{[a+c-b-2]} T(\exp^{[d-1]} r, h_s \circ k_t) \geqslant r^{(\lambda_k(c,d)-\varepsilon)} + O(1).$

Now combining (12) of Case I and (22) of Case III, it follows for all sufficiently large values of r that

$$\frac{\log^{[a-1]} T(\exp^{[d-1]} r, h_s \circ k_t)}{\log^{[p-1]} T(r, f_u \circ g_v)} \geq \frac{(\lambda_h(a,b) - \varepsilon) \exp^{[c-b-1]} r^{(\lambda_k(c,d)-\varepsilon)} + O(1)}{(\rho_f(p,q) + \varepsilon) r^{(\rho_g(m,n)+\varepsilon)} + O(1)}$$

i.e., $\liminf\limits_{r \to \infty} \dfrac{\log^{[a-1]} T(\exp^{[d-1]} r, h_s \circ k_t)}{\log^{[p-1]} T(r, f_u \circ g_v)} = \infty,$

from which the first part of the theorem follows.

Again combining (15) of Case II and (22) of Case III we obtain for all sufficiently large values of r that

$$\frac{\log^{[a-1]} T(\exp^{[d-1]} r, h_s \circ k_t)}{\log^{[p+m-q-2]} T(r, f_u \circ g_v)} \geq \frac{(\lambda_h(a,b) - \varepsilon) \exp^{[c-b-1]} r^{(\lambda_k(c,d)-\varepsilon)} + O(1)}{r^{(\rho_g(m,n)+\varepsilon)} + O(1)}$$

$$\textit{i.e.,} \quad \liminf\limits_{r \to \infty} \frac{\log^{[a-1]} T(\exp^{[d-1]} r, h_s \circ k_t)}{\log^{[p+m-q-2]} T(r, f_u \circ g_v)} = \infty$$

$$i.e., \quad \lim_{r \to \infty} \frac{\log^{[a-1]} T(\exp^{[d-1]} r, h_s \circ k_t)}{\log^{[p+m-q-2]} T(r, f_u \circ g_v)} = \infty.$$

This establishes the second part of the theorem.

Now in view of (12) of Case I and (25) of Case IV we get for all sufficiently large values of r that

(26)
$$\frac{\log^{[a+c-b-2]} T(\exp^{[d-1]} r, h_s \circ k_t)}{\log^{[p-1]} T(r, f_u \circ g_v)} \geq \frac{r^{(\lambda_k(c,d)-\varepsilon)} + O(1)}{(\rho_f(p,q) + \varepsilon) \, r^{(\rho_g(m,n)+\varepsilon)} + O(1)}.$$

So from (16) and (26) we obtain that

$$\liminf_{r \to \infty} \frac{\log^{[a+c-b-2]} T(\exp^{[d-1]} r, h_s \circ k_t)}{\log^{[p-1]} T(r, f_u \circ g_v)} = \infty,$$

from which the third part of the theorem follows.

Again combining (15) of Case II and (25) of Case IV, it follows for all sufficiently large values of r that

(27)
$$\frac{\log^{[a+c-b-2]} T(\exp^{[d-1]} r, h_s \circ k_t)}{\log^{[p+m-q-2]} T(r, f_u \circ g_v)} \geq \frac{r^{(\lambda_k(c,d)-\varepsilon)} + O(1)}{r^{(\rho_g(m,n)+\varepsilon)} + O(1)}.$$

Now in view of (16) we obtain from (27) that

$$\liminf_{r \to \infty} \frac{\log^{[a+c-b-2]} T(\exp^{[d-1]} r, h_s \circ k_t)}{\log^{[p+m-q-2]} T(r, f_u \circ g_v)} = \infty$$

$$i.e., \quad \lim_{r \to \infty} \frac{\log^{[a+c-b-2]} T(\exp^{[d-1]} r, h_s \circ k_t)}{\log^{[p+m-q-2]} T(r, f_u \circ g_v)} = \infty.$$

This proves the fourth part of the theorem.

Thus the theorem follows. $\qquad\qquad\square$

THEOREM 2. *Let h be meromorphic and g, k be entire such that $\lambda_h(a, b) > 0$, $0 < \rho_k < \infty$ and $\rho_g(m, n) < \rho_k$ where m, n, a, b are all positive integers with $m > n$ and $a > b$. Let h_s, k_t, f_u, g_v be the family of integer translations of the functions h, k, f, g respectively where $s, t, u, v \in \mathbb{N}$. Then for every meromorphic function f with $0 < \rho_f(p, q) < \infty$ and*

for any two positive integers p, q with $p > q$

$(i) \qquad \displaystyle\limsup_{r \to \infty} \frac{\log^{[a-1]} T(r, h_s \circ k_t)}{\log^{[p-1]} T(r, f_u \circ g_v) + \log^{[m]} M(r, g_v)} \quad = \quad \infty,$

$$\text{if } b \;=\; 1 \text{ and } q \geqslant m;$$

$(ii) \; \displaystyle\limsup_{r \to \infty} \frac{\log^{[a-1]} T(r, h_s \circ k_t)}{\log^{[p+m-q-2]} T(r, f_u \circ g_v) + \log^{[m]} M(r, g_v)} \quad = \quad \infty,$

$$\text{if } b \;=\; 1 \text{ and } q < m;$$

$(iii) \; \displaystyle\limsup_{r \to \infty} \frac{\log^{[a-1]} T(r, h_s \circ k_t)}{\log^{[p-1]} T(r, f_u \circ g_v) + \log^{[m]} M(r, g_v)} \quad = \quad \infty,$

$$\text{if } q \;\geqslant\; m \text{ and } 1 < b < n+1;$$

$(iv) \; \displaystyle\limsup_{r \to \infty} \frac{\log^{[a-1]} T(r, h_s \circ k_t)}{\log^{[p-1]} T(r, f_u \circ g_v) + \log^{[m]} M(r, g_v)} \geq \frac{\mu \lambda_h(a,b)}{(\rho_f(p,q) + 1)\, \rho_g(m,n)},$

$$\text{if } q \geq m, \; b = 2, n = 1 \text{ and } 0 < \mu < \rho_k;$$

$(v) \; \displaystyle\limsup_{r \to \infty} \frac{\log^{[a-1]} T(r, h_s \circ k_t)}{\log^{[p-1]} T(r, f_u \circ g_v) + \log^{[m]} M(r, g_v)} \geq \frac{\lambda_h(a,b)}{(\rho_f(p,q) + 1)\, \rho_g(m,n)},$

$$\text{if } q \geq m \text{ and } b > 2, n \geq 2;$$

$(vi) \; \displaystyle\limsup_{r \to \infty} \frac{\log^{[a-1]} T(r, h_s \circ k_t)}{\log^{[p+m+n-q-2]} T(r, f_u \circ g_v) + \log^{[m]} M(r, g_v)} = \infty,$

$$\text{if } q < m \text{ and } 1 < b < n+1;$$

$(vii) \; \displaystyle\limsup_{r \to \infty} \frac{\log^{[a-1]} T(r, h_s \circ k_t)}{\log^{[p+m+n-q-2]} T(r, f_u \circ g_v) + \log^{[m]} M(r, g_v)} \geq \frac{\mu \lambda_h(a,b)}{1 + \rho_g(m,n)},$

$$\text{if } q < m, \; b = n = 2 \text{ and } 0 < \mu < \rho_k$$

and

$(viii) \; \displaystyle\limsup_{r \to \infty} \frac{\log^{[a-1]} T(r, h_s \circ k_t)}{\log^{[p+m+n-q-2]} T(r, f_u \circ g_v) + \log^{[m]} M(r, g_v)} \geq \frac{\lambda_h(a,b)}{1 + \rho_g(m,n)},$

$$\text{if } q < m \text{ and } b = n > 2.$$

Proof. Since $\rho_g(m, n) < \rho_k$, we can choose $\varepsilon(> 0)$ in such a way that

$$(28) \qquad\qquad \rho_g(m, n) + \varepsilon < \mu < \rho_k - \varepsilon.$$

Let $h_s \circ k_t = w_l$, where $w = h \circ k$ is a meromorphic function and $l \in N$. So w_l can be written as

$$w_l = \{ w(z + l) : l \in N \}.$$

Then by (3) we obtain

$$T(r, w_l) = lT(r, w) + \sum_l (e_l + e_l'),$$

where $e_l, e_l' \to 0$ as $r \to \infty$,

(29) $$\qquad i.e., \; T(r, h_s \circ k_t) = lT(r, h \circ k) + \sum_l (e_l + e_l').$$

Then by Lemma 2 and from (29) we obtain for a sequence of values of r tending to infinity,

$$T(r, h_s \circ k_t) \; \geq \; lT(\exp(r^\mu), h) + \sum_l (e_l + e_l'), \; \text{where } 0 < \mu < \rho_k \leq \infty$$

$$i.e., T(r, h_s \circ k_t) \; \geq \; lT(\exp(r^\mu), h)[1 + \frac{\sum_l (e_l + e_l')}{lT(\exp(r^\mu), h)}]$$

$$i.e., \; \log^{[a-1]} T(r, h_s \circ k_t) \; \geq \; \log^{[a-1]} T(\exp(r^\mu), h) + O(1)$$
$$i.e., \; \log^{[a-1]} T(r, h_s \circ k_t) \; \geq \; (\lambda_h (a, b) - \varepsilon) \log^{[b]} \exp(r^\mu) + O(1)$$

(30) $$\qquad i.e., \; \log^{[a-1]} T(r, h_s \circ k_t) \geq (\lambda_h (a, b) - \varepsilon) \log^{[b-1]}(r^\mu) + O(1).$$

Now the following two cases may arise :
Case I. Let $b = 1$. Then from (30) we get for a sequence of values of r tending to infinity that

(31) $$\qquad \log^{[a-1]} T(r, h_s \circ k_t) \geq (\lambda_h (a, b) - \varepsilon)(r^\mu) + O(1).$$

Case II. Let $b - 1 = l > 0$. Then from (30), it follows for a sequence of values of r tending to infinity that

(32) $$\qquad \log^{[a-1]} T(r, h_s \circ k_t) \geq (\lambda_h (a, b) - \varepsilon) \log^{[l]}(r^\mu) + O(1).$$

Now from the definition of (m, n)-th order of g and from the relation (5) we have for arbitrary positive ε and for all sufficiently large values of r,

$$\log^{[m]} M (r, g_v) \; \leqslant \; (\rho_{g_v}(m, n) + \varepsilon) \log^{[n]} r$$
(33) $$\qquad i.e., \; \log^{[m]} M (r, g_v) \; \leqslant \; (\rho_g(m, n) + \varepsilon) \log^{[n]} r.$$

Let $q \geqslant m$. Then we have from (8) and (33) for all sufficiently large values of r,

$$(34) \quad \log^{[p-1]} T(r, f_u \circ g_v) \leq (\rho_f(p, q) + \varepsilon)(\rho_g(m, n) + \varepsilon) \log^{[n]} r + O(1).$$

Now if $b = 1$ and $q \geqslant m$, we get from (12), (31), (33) and in view of (28) for a sequence of values of r tending to infinity,

$$\frac{\log^{[a-1]} T(r, h_s \circ k_t)}{\log^{[p-1]} T(r, f_u \circ g_v) + \log^{[m]} M(r, g_v)}$$

$$\geq \frac{(\lambda_h(a, b) - \varepsilon)(r^\mu) + O(1)}{(\rho_f(p, q) + \varepsilon) r^{(\rho_g(m,n)+\varepsilon)} + (\rho_g(m, n) + \varepsilon) \log^{[n]} r + O(1)}$$

$$i.e., \quad \limsup_{r \to \infty} \frac{\log^{[a-1]} T(r, h_s \circ k_t)}{\log^{[p-1]} T(r, f_u \circ g_v) + \log^{[m]} M(r, g_v)} = \infty,$$

which proves the first part of the theorem.

Again we obtain from (15), (28), (31) and (33) for a sequence of values of r tending to infinity when $b = 1$ and $q < m$

$$\frac{\log^{[a-1]} T(r, h_s \circ k_t)}{\log^{[p+m-q-2]} T(r, f_u \circ g_v) + \log^{[m]} M(r, g_v)}$$

$$\geq \frac{(\lambda_h(a, b) - \varepsilon)(r^\mu) + O(1)}{r^{(\rho_g(m,n)+\varepsilon)} + (\rho_g(m, n) + \varepsilon) \log^{[n]} r + O(1)}$$

$$i.e., \quad \limsup_{r \to \infty} \frac{\log^{[a-1]} T(r, h_s \circ k_t)}{\log^{[p+m-q-2]} T(r, f_u \circ g_v) + \log^{[m]} M(r, g_v)} = \infty.$$

This proves the second part of the theorem.

When $b > 1$ and $q \geq m$, from (32), (33) and (34) we get for a sequence of values of r tending to infinity,

$$\frac{\log^{[a-1]} T(r, h_s \circ k_t)}{\log^{[p-1]} T(r, f_u \circ g_v) + \log^{[m]} M(r, g_v)}$$

$$\geq \frac{(\lambda_h(a, b) - \varepsilon) \log^{[l]}(r^\mu) + O(1)}{(\rho_f(p, q) + \varepsilon)(\rho_g(m, n) + \varepsilon) \log^{[n]} r + (\rho_g(m, n) + \varepsilon) \log^{[n]} r + O(1)}$$

$$i.e., \quad \limsup_{r \to \infty} \frac{\log^{[a-1]} T(r, h_s \circ k_t)}{\log^{[p-1]} T(r, f_u \circ g_v) + \log^{[m]} M(r, g_v)} = \infty, \text{ if } 1 < b < n+1,$$

again

$$\limsup_{r \to \infty} \frac{\log^{[a-1]} T(r, h_s \circ k_t)}{\log^{[p-1]} T(r, f_u \circ g_v) + \log^{[m]} M(r, g_v)} \geq \frac{\mu \lambda_h(a, b)}{(\rho_f(p, q) + 1) \rho_g(m, n)},$$

$$\text{if } b = 2, n = 1 \text{ and } 0 < \mu < \rho_k$$

and also

$$\limsup_{r \to \infty} \frac{\log^{[a-1]} T(r, h_s \circ k_t)}{\log^{[p-1]} T(r, f_u \circ g_v) + \log^{[m]} M(r, g_v)} \geq \frac{\lambda_h(a, b)}{(\rho_f(p, q) + 1) \rho_g(m, n)},$$

$$\text{if } b > 2, n \geq 2 \text{ and } 0 < \mu < \rho_k.$$

This respectively proves the third, fourth and fifth part of the theorem. Again when $b > 1$ and $q < m$, combining $(15), (32)$ and (33) we obtain for a sequence of values of r tending to infinity,

$$\frac{\log^{[a-1]} T(r, h_s \circ k_t)}{\log^{[p+m+n-q-2]} T(r, f_u \circ g_v) + \log^{[m]} M(r, g_v)}$$

$$\geq \frac{(\lambda_h(a, b) - \varepsilon) \log^{[l]}(r^\mu) + O(1)}{\log^{[n]} r + (\rho_g(m, n) + \varepsilon) \log^{[n]} r + O(1)}$$

$$i.e., \ \limsup_{r \to \infty} \frac{\log^{[a-1]} T(r, h_s \circ k_t)}{\log^{[p+m+n-q-2]} T(r, f_u \circ g_v) + \log^{[m]} M(r, g_v)} = \infty, \text{ if } 1 < b < n+1,$$

also

$$\limsup_{r \to \infty} \frac{\log^{[a-1]} T(r, h_s \circ k_t)}{\log^{[p+m+n-q-2]} T(r, f_u \circ g_v) + \log^{[m]} M(r, g_v)} \geq \frac{\mu \lambda_h(a, b)}{1 + \rho_g(m, n)},$$

$$\text{if } b = 2, n = 1 \text{ and } 0 < \mu < \rho_k$$

and again

$$\limsup_{r \to \infty} \frac{\log^{[a-1]} T(r, h_s \circ k_t)}{\log^{[p+m+n-q-2]} T(r, f_u \circ g_v) + \log^{[m]} M(r, g_v)} \geq \frac{\lambda_h(a, b)}{1 + \rho_g(m, n)},$$

$$\text{if } b > 2, n \geq 2 \text{ and } 0 < \mu < \rho_k,$$

from which the sixth, seventh and eighth part of the theorem respectively follow. $\qquad \square$

THEOREM 3. *Let f and g be two non-constant entire functions such that*

(A) $\lambda_f(p,q) > 0$,

(B) $\rho_g^{[l]} < \infty$ and

(C) $\lambda_{f \circ g}(a,b) > 0$ where p, q and l are all positive integers with $p > q$, $a > b$. Let f_u, g_v be the family of integer translations of the functions f, g respectively where $u, v \in N$. Also let $a, b, l \geq 2$. Then

$$(i) \limsup_{r \to \infty} \frac{\log^{[p-1]} T(r, f_u \circ g_v) \cdot \log^{[a-1]} T(\exp^{[b-1]} r, f_u \circ g_v)}{\log^{[l-2]} T(r, g_v) \cdot \log^{[l-1]} T(r, g_v)}$$

$$\geq \quad \frac{\lambda_f(p,q)\lambda_{f \circ g}(a,b)}{4^{\lambda_g^{[l]}} \rho_g^{[l]}}, \ if \ q \leq l - 1$$

and

$$(ii) \limsup_{r \to \infty} \frac{\log^{[p-l+q]} T(r, f_u \circ g_v) \cdot \log^{[a-1]} T(\exp^{[b-1]} r, f_u \circ g_v)}{\log^{[l-2]} T(r, g_v) \cdot \log^{[l-1]} T(r, g_v)}$$

$$\geq \quad \frac{\lambda_{f \circ g}(a,b)}{4^{\lambda_g^{[l]}} \rho_g^{[l]}}, \ if \ q > l - 1.$$

Proof. Using the inequality $T(r, f \circ g) \geq \frac{1}{3} \log M\{\frac{1}{8} M(\frac{r}{4}, g) + \circ(1), f\}$ {cf. [6]} we have from (4) for all sufficiently large positive numbers of r in view of the relation $T(r, g) \leqslant \log^+ M(r, g)$,

$$\log^{[p-1]} T(r, f_u \circ g_v) \geq \log^{[p]} M\{\frac{1}{8} M(\frac{r}{4}, g) + \circ(1), f\} + O(1)$$

$$i.e., \ \log^{[p-1]} T(r, f_u \circ g_v) \geq (\lambda_f(p,q) - \varepsilon) \log^{[q]}\{\frac{1}{8} M(\frac{r}{4}, g)\} + O(1)$$

$$i.e., \ \log^{[p-1]} T(r, f_u \circ g_v) \geq (\lambda_f(p,q) - \varepsilon) \log^{[q]} M(\frac{r}{4}, g) + O(1)$$

$$(35) i.e., \ \log^{[p-1]} T(r, f_u \circ g_v) \geq (\lambda_f(p,q) - \varepsilon) \log^{[q-1]} T(\frac{r}{4}, g) + O(1).$$

Case I. Let $q \leq l - 1$.
Then from (35) we obtain for all sufficiently large positive numbers of r,

$$(36) \qquad \log^{[p-1]} T(r, f_u \circ g_v) \geq (\lambda_f(p,q) - \varepsilon) \log^{[l-2]} T(\frac{r}{4}, g) + O(1).$$

Case II. Let $q > l - 1$.

Then from (35) we have for all sufficiently large positive numbers of r,

$$\log^{[p-1]} T(r, f_u \circ g_v) \geq (\lambda_f(p,q) - \varepsilon) \exp^{[l-q-1]} \log^{[l-2]} T(\frac{r}{4}, g) + O(1)$$

i.e., $\quad \log^{[p]} T(r, f_u \circ g_v) \geq \exp^{[l-q]} \log^{[l-2]} T(\frac{r}{4}, g) + O(1)$

(37) *i.e.,* $\log^{[p-l+q]} T(r, f_u \circ g_v) \geq \log^{[l-2]} T(\frac{r}{4}, g) + O(1).$

Again from the definition of generalized lower proximate order, for given $\varepsilon(0 < \varepsilon < 1)$, we get for all large positive numbers of r,

(38) $$\log^{[l-2]} T(\frac{r}{4}, g) \geq (1 - \varepsilon) \left(\frac{r}{4}\right)^{\lambda_g^{[l]}\left(\frac{r}{4}\right)}.$$

From (3) we may obtain

(39) $$T(r, g_v) = vT(r, g) + \sum_v (e_v + e_v'),$$

where $e_v, e_v' \to 0$ as $r \to \infty$,

$$i.e., T(r, g_v) = vT(r, g)[1 + \frac{\displaystyle\sum_v (e_v + e_v')}{vT(r, g)}]$$

(40) *i.e.,* $\log^{[l-2]} T(r, g_v) = \log^{[l-2]} T(r, g) + O(1).$

Now for given $\varepsilon(0 < \varepsilon < 1)$, we get for a sequence of positive numbers of r tending to infinity that

$$\log^{[l-2]} T(r, g) \leq (1 + \varepsilon) r^{\lambda_g^{[l]}(r)},$$

so from (40) we get

$$\log^{[l-2]} T(r, g_v) \leq (1 + \varepsilon) r^{\lambda_g^{[l]}(r)} + O(1)$$

(41) *i.e.,* $\dfrac{\log^{[l-2]} T(r, g_v)}{4^{\lambda_g^{[l]}+\delta}} \leq \dfrac{(1 + \varepsilon) r^{\lambda_g^{[l]}(r)} + O(1)}{4^{\lambda_g^{[l]}+\delta}}.$

Therefore from (36) of Case I and (38) we have for all sufficiently large positive numbers of r,

$$\log^{[p-1]} T(r, f_u \circ g_v) \geq (\lambda_f(p,q) - \varepsilon)(1 - \varepsilon) \frac{\left(\frac{r}{4}\right)^{\lambda_g^{[l]}+\delta}}{\left(\frac{r}{4}\right)^{\lambda_g^{[l]}+\delta-\lambda_g^{[l]}\left(\frac{r}{4}\right)}} + O(1).$$

Since by Lemma 3, $\left(\frac{r}{4}\right)^{\left\{\lambda_g^{[l]}+\delta-\lambda_g^{[l]}\left(\frac{r}{4}\right)\right\}}$ is ultimately an increasing function of r, it follows from above for all sufficiently large positive numbers of r that

$$(42) \qquad \log^{[p-1]} T(r, f_u \circ g_v) \geq (\lambda_f(p,q) - \varepsilon)(1-\varepsilon)\frac{(r)^{\lambda_g^{[l]}(r)}}{4^{\lambda_g^{[l]}+\delta}} + O(1).$$

Now from (41) and (42) we have for a sequence of positive numbers of r tending to infinity,

$$\log^{[p-1]} T(r, f_u \circ g_v) \;\geq\; (\lambda_f(p,q)-\varepsilon)\frac{(1-\varepsilon)}{(1+\varepsilon)}\frac{\log^{[l-2]} T(r, g_v)}{4^{\lambda_g^{[l]}+\delta}} + O(1)$$

$$i.e., \quad \frac{\log^{[p-1]} T(r, f_u \circ g_v)}{\log^{[l-2]} T(r, g_v)} \;\geq\; \frac{(1-\varepsilon)}{(1+\varepsilon)} \cdot \frac{(\lambda_f(p,q)-\varepsilon)}{4^{\lambda_g^{[l]}+\delta}} + \frac{O(1)}{\log^{[l-2]} T(r, g_v)}.$$

As $\varepsilon\,(>0)$ and $\delta\,(>0)$ are arbitrary, we obtain from above that

$$(43) \qquad\qquad \limsup_{r\to\infty}\frac{\log^{[p-1]} T(r, f_u \circ g_v)}{\log^{[l-2]} T(r, g_v)} \geq \frac{\lambda_f(p,q)}{4^{\lambda_g^{[l]}}}.$$

Similarly from (37) of Case II and (38) we get for all sufficiently large values of r,

$$\log^{[p-l+q]} T(r, f_u \circ g_v) \geq (1-\varepsilon)\frac{\left(\frac{r}{4}\right)^{\lambda_g^{[l]}+\delta}}{\left(\frac{r}{4}\right)^{\lambda_g^{[l]}+\delta-\lambda_g^{[l]}\left(\frac{r}{4}\right)}} + O(1).$$

In view of Lemma 3, $\left(\frac{r}{4}\right)^{\left\{\lambda_g^{[l]}+\delta-\lambda_g^{[l]}\left(\frac{r}{4}\right)\right\}}$ is ultimately an increasing function of r and therefore it follows from above for all sufficiently large values of r that

$$\log^{[p-l+q]} T(r, f_u \circ g_v) \geq (1-\varepsilon)\frac{(r)^{\lambda_g^{[l]}(r)}}{4^{\lambda_g^{[l]}+\delta}} + O(1).$$

Now from (41) and above we have for a sequence of positive numbers of r tending to infinity,

$$\log^{[p-l+q]} T(r, f_u \circ g_v) \;\geq\; \frac{(1-\varepsilon)}{(1+\varepsilon)} \cdot \frac{\log^{[l-2]} T(r, g_v)}{4^{\lambda_g^{[l]}+\delta}} + O(1)$$

$$i.e., \quad \frac{\log^{[p-l+q]} T(r, f_u \circ g_v)}{\log^{[l-2]} T(r, g_v)} \;\geq\; \frac{(1-\varepsilon)}{(1+\varepsilon)} \cdot \frac{1}{4^{\lambda_g^{[l]}+\delta}} + \frac{O(1)}{\log^{[l-2]} T(r, g_v)}.$$

Since $\varepsilon\,(> 0)$ and $\delta\,(> 0)$ are arbitrary, we obtain from the above that

$$(44) \qquad \limsup_{r \to \infty} \frac{\log^{[p-l+q]} T(r, f_u \circ g_v)}{\log^{[l-2]} T(r, g_v)} \geq \frac{1}{4\lambda_g^{[l]}}.$$

Now let $f_u \circ g_v = h_t$, where h is a meromorphic function and $h_t = \{h(z+t) : t \in N\}$. Then from (4) we obtain

$$T(\exp^{[b-1]} r, f_u \circ g_v) \;=\; tT(\exp^{[b-1]} r, f \circ g) + \sum_t (e_t + e_t')$$

$$i.e., \; T(\exp^{[b-1]} r, f_u \circ g_v) \;=\; tT(\exp^{[b-1]} r, f \circ g)[1 + \frac{\sum_t (e_t + e_t')}{tT(\exp^{[b-1]} r, f \circ g)}]$$

$$(45)$$
$$i.e., \; \log^{[a-1]} T(\exp^{[b-1]} r, f_u \circ g_v) = \log^{[a-1]} T(\exp^{[b-1]} r, f \circ g) + O(1).$$

Now for all large positive numbers of r,

$$\log^{[a-1]} T(\exp^{[b-1]} r, f \circ g) \geq (\lambda_{fog}(a, b) - \varepsilon) \log^{[b]}(\exp^{[b-1]} r)$$

$$i.e., \; \log^{[a-1]} T(\exp^{[b-1]} r, f \circ g) \geq (\lambda_{fog}(a, b) - \varepsilon) \log r.$$

Therefore from (45) we obtain

$$(46) \qquad \log^{[a-1]} T(\exp^{[b-1]} r, f_u \circ g_v) \geq (\lambda_{fog}(a, b) - \varepsilon) \log r$$

and using (39) we may establish that

$$\log^{[l-1]} T(r, g_v) \leq \left(\rho_g^{[l]} + \varepsilon\right) \log r.$$

Therefore from the above two inequalities we get for all large positive numbers of r that

$$\frac{\log^{[a-1]} T(\exp^{[b-1]} r, f_u \circ g_v)}{\log^{[l-1]} T(r, g_v)} \geq \frac{\lambda_{fog}(a, b) - \varepsilon}{\rho_g^{[l]} + \varepsilon}.$$

As $\varepsilon\,(> 0)$ is arbitrary, it follows from above that

$$(47) \qquad \liminf_{r \to \infty} \frac{\log^{[a-1]} T(\exp^{[b-1]} r, f_u \circ g_v)}{\log^{[l-1]} T(r, g_v)} \geq \frac{\lambda_{fog}(a, b)}{\rho_g^{[l]}}.$$

Now combining (43) and (47) we obtain that

$$\limsup_{r \to \infty} \frac{\log^{[p-1]} T(r, f_u \circ g_v) \cdot \log^{[a-1]} T(\exp^{[b-1]} r, f_u \circ g_v)}{\log^{[l-2]} T(r, g_v) \cdot \log^{[l-1]} T(r, g_v)}$$

$$\geq \quad \limsup_{r \to \infty} \frac{\log^{[p-1]} T(r, f_u \circ \dot{g}_v)}{\log^{[l-2]} T(r, g_v)} \cdot \liminf_{r \to \infty} \frac{\log^{[a-1]} T(\exp^{[b-1]} r, f_u \circ g_v)}{\log^{[l-1]} T(r, g_v)}$$

$$\geq \quad \frac{\lambda_f(p,q) \lambda_{f \circ g}(a,b)}{4^{\lambda_g^{[l]}} \rho_g^{[l]}}.$$

This proves the first part of the theorem.
Again combining from (44) and (47) we get that

$$\limsup_{r \to \infty} \frac{\log^{[p-l+q]} T(r, f_u \circ g_v) \cdot \log^{[a-1]} T(\exp^{[b-1]} r, f_u \circ g_v)}{\log^{[l-2]} T(r, g_v) \cdot \log^{[l-1]} T(r, g_v)}$$

$$\geq \quad \limsup_{r \to \infty} \frac{\log^{[p-l+q]} T(r, f_u \circ g_v)}{\log^{[l-2]} T(r, g_v)} \cdot \liminf_{r \to \infty} \frac{\log^{[a-1]} T(\exp^{[b-1]} r, f_u \circ g_v)}{\log^{[l-1]} T(r, g_v)}$$

$$\geq \quad \frac{1}{4^{\lambda_g^{[l]}}} \cdot \frac{\lambda_{f \circ g}(a,b)}{\rho_g^{[l]}}$$

$$i.e., \quad \limsup_{r \to \infty} \frac{\log^{[p-l+q]} T(r, f_u \circ g_v) \cdot \log^{[a-1]} T(\exp^{[b-1]} r, f_u \circ g_v)}{\log^{[l-2]} T(r, g_v) \cdot \log^{[l-1]} T(r, g_v)}$$

$$\geq \quad \frac{\lambda_{f \circ g}(a,b)}{4^{\lambda_g^{[l]}} \rho_g^{[l]}}.$$

Thus the second part of the theorem is established.   □

COROLLARY 1. *Under the same conditions of Theorem 3 if* $m - n = l - 1$ *and* $\rho_g(m,n) = 1$ *where* m *and* n *are any two positive integers then*

$$(i) \limsup_{r \to \infty} \frac{\log^{[p-1]} T(r, f_u \circ g_v) \cdot \log^{[a-1]} T(\exp^{[b-1]} r, f_u \circ g_v)}{\log^{[l-2]} T(r, g_v) \cdot \log^{[m-1]} T(\exp^{[n-1]} r, g_v)}$$

$$\geq \quad \frac{\lambda_f(p,q) \cdot \lambda_{f \circ g}(a,b)}{4^{\lambda_g^{[l]}}} \ for \ q \leq l - 1$$

and

$$(ii) \limsup_{r \to \infty} \frac{\log^{[p-l+q]} T(r, f_u \circ g_v) \cdot \log^{[a-1]} T(\exp^{[b-1]} r, f_u \circ g_v)}{\log^{[l-2]} T(r, g_v) \cdot \log^{[m-1]} T(\exp^{[n-1]} r, g_v)}$$

$$\geq \quad \frac{\lambda_{f \circ g}(a, b)}{4\lambda_g^{[l]}} \text{ for } q > l - 1.$$

Proof. Since $\rho_g(m, n) = 1$, by Lemma 4 it follows for all sufficiently large positive numbers of r that

$$\log^{[m-1]} T(\exp^{[n-1]} r, g_v) \quad \leq \quad (1 + \varepsilon) \log^{[n]} \exp^{[n-1]} r$$

(48)   *i.e.,* $\log^{[m-1]} T(\exp^{[n-1]} r, g_v) \quad \leq \quad (1 + \varepsilon) \log r.$

Therefore from (46) and (48) we get for all large positive numbers of r that

$$\frac{\log^{[a-1]} T(\exp^{[b-1]} r, f_u \circ g_v)}{\log^{[m-1]} T(\exp^{[n-1]} r, g_v)} \geq \frac{\lambda_{f \circ g}(a, b) - \varepsilon}{1 + \varepsilon}.$$

As $\varepsilon \, (> 0)$ is arbitrary, it follows from above that

(49) $$\liminf_{r \to \infty} \frac{\log^{[a-1]} T(\exp^{[b-1]} r, f_u \circ g_v)}{\log^{[m-1]} T(\exp^{[n-1]} r, g_v)} \geq \lambda_{f \circ g}(a, b).$$

Now combining (43) and (49) we obtain that

$$\limsup_{r \to \infty} \frac{\log^{[p-1]} T(r, f_u \circ g_v) \cdot \log^{[a-1]} T(\exp^{[b-1]} r, f_u \circ g_v)}{\log^{[l-2]} T(r, g_v) \cdot \log^{[m-1]} T(\exp^{[n-1]} r, g_v)}$$

$$\geq \quad \limsup_{r \to \infty} \frac{\log^{[p-1]} T(r, f_u \circ g_v)}{\log^{[l-2]} T(r, g_v)} \cdot \liminf_{r \to \infty} \frac{\log^{[a-1]} T(\exp^{[b-1]} r, f_u \circ g_v)}{\log^{[m-1]} T(\exp^{[n-1]} r, g_v)}$$

$$\geq \quad \frac{\lambda_f(p, q)\lambda_{f \circ g}(a, b)}{4\lambda_g^{[l]}}.$$

This proves the first part of the corollary.
Again combining from (44) and (49) we get that

$$\limsup_{r \to \infty} \frac{\log^{[p-l+q]} T(r, f_u \circ g_v) \cdot \log^{[a-1]} T(\exp^{[b-1]} r, f_u \circ g_v)}{\log^{[l-2]} T(r, g_v) \cdot \log^{[m-1]} T(\exp^{[n-1]} r, g_v)}$$

$$\geq \quad \limsup_{r \to \infty} \frac{\log^{[p-l+q]} T(r, f_u \circ g_v)}{\log^{[l-2]} T(r, g_v)} \cdot \liminf_{r \to \infty} \frac{\log^{[a-1]} T(\exp^{[b-1]} r, f_u \circ g_v)}{\log^{[m-1]} T(\exp^{[n-1]} r, g_v)}$$

$$\geq \quad \frac{1}{4\lambda_g^{[l]}} \cdot \lambda_{f \circ g}(a, b)$$

$$i.e., \ \limsup_{r\to\infty} \frac{\log^{[p-l+q]} T(r, f_u \circ g_v) \cdot \log^{[a-1]} T(\exp^{[b-1]} r, f_u \circ g_v)}{\log^{[l-2]} T(r, g_v) \cdot \log^{[m-1]} T(\exp^{[n-1]} r, g_v)}$$

$$\geq \ \frac{\lambda_{f\circ g}(a,b)}{4^{\lambda_g^{[l]}}} \ .$$

Thus the second part of the corollary is established. $\square$

Acknowledgement

References

[1] W. Bergweiler, *On the Nevanlinna Characteristic of a composite function,* Complex Variables **10** (1988), 225–236.

[2] W. Bergweiler, *On the growth rate of composite meromorphic functions*, Complex Variables **14** (1990), 187–196.

[3] S. K. Datta and T. Biswas, *Growth rates of composite entire and meromorphic functions,* Bull. Allahabad Math. Soc. **27** (1) (2012), 55–94.

[4] W. K. Hayman, *Meromorphic Functions*, The Clarendon Press, Oxford,1964.

[5] O. P. Juneja, G. P. Kapoor and S. K. Bajpai, *On the (p,q)-order and lower (p,q)-order of an entire function*, J. Reine Angew. Math. **282** (1976), 53–67.

[6] K. Niino and N. Suita, *Growth of a composite function of entire functions,* Kodai Math. J. **3** (1980), 374–379.

[7] K. Niino and C. C. Yang, *Some growth relationships on factors of two composite entire functions*, Factorization Theory of Meromorphic Functions and Related Topics, Marcel Dekker, Inc. (New York and Basel), 1982, 95–99.

[8] D. Sato, *On the rate of growth of entire functions of fast growth*, Bull. Amer. Math. Soc. **69** (1963), 411–414.

[9] G. Valiron, *Lectures on the General Theory of Integral Functions,* Chelsea Publishing Company, 1949.

Ritam Biswas
Department of Mathematics
Murshidabad College of Engineering and Technology
Banjetia, Berhampore, P.O.- Cossimbazar Raj,
PIN-742102, West Bengal, India.
E-mail: ritamiitr@yahoo.co.in