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NEW CONSTRUCTION OF THE EAGON-NORTHCOTT COMPLEX

Oh-Jin Kang and Joohyung Kim*

Abstract. The authors [6] introduced the concept of a complete matrix of grade $g > 3$ to describe a structure theorem for complete intersections of grade $g > 3$. We show that a complete matrix can be used to construct the Eagon-Northcott complex [7]. Moreover, we prove that it is the minimal free resolution $\mathbb{F}$ of a class of determinantal ideals of $n \times (n+2)$ matrices $X = (x_{ij})$ such that entries of each row of $X = (x_{ij})$ form a regular sequence and the second differential map of $\mathbb{F}$ is a matrix f defined by the complete matrices of grade $n+2$.

1. Introduction

Let k be a field containing the field $\mathbb{Q}$ of rational numbers and let $R = k[x_{ij} | 1 \le i \le m, 1 \le j \le n]$ be the polynomial ring over a field k with indeterminates x_{ij}. Eagon and Northcott [7] defined a free complex from a matrix over a commutative ring with identity which is a generalization of the standard Koszul complex. As an application of it, they constructed the minimal free resolution of $R/I_t(X)$, where $t = \min(m, n)$. Also Buchsbaum and Rim [4] separately constructed the minimal free resolution of the class of the determinantal ideals. Buchsbaum [5] used the multilinear algebra to give other version of the Eagon and Northcott complex. Buchsbaum and Eisenbud [2] noted that the Eagon-Northcott and Buchsbaum-Rim complexes are constructed by the multilinear algebra, that is, the complexes are described in terms of tensor products of

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*Corresponding author.

exterior, symmetric and divided power algebras. On the other hand, using the representation theory of the general linear groups, Lascoux [12], Pragacz and Weyman [13], and Roberts [14] constructed the minimal free resolution of $R/I_t(X)$ for any m, n, t, where R contains the field $\mathbb{Q}$ of the rational numbers. Akin, Buchsbaum and Weyman [1] developed the characteristic free representation theory of the general linear groups and constructed the minimal free resolution of $R/I_t(X)$ over $R = \mathbb{Z}$ in the case of $t = \min(m, n) - 1$. Roberts [14] proved that there exists a minimal free resolution of $R/I_t(X)$ over $R = \mathbb{Z}$ if and only if the Betti numbers of $R/I_t(X)$ is independent of the characteristic of the base field. Hashimoto and Kurano [10] used this proof to show that there exists a minimal free resolution of $R/I_t(X)$ over $R = \mathbb{Z}$ in the case of $m = n = t + 2$. Hishimoto [8, 9] also extended this result to the case of $t = \min(m, n) - 2$ and proved that there is no minimal free resolution of $R/I_t(X)$ over $R = \mathbb{Z}$ in the case of $2 \leq t \leq \min(m, n) - 3$. Recently, Kang and Ko [11] introduced a complete matrix of grade 4 to describe a structure theorem for the complete intersections of grade 4 and Choi, Kang and Ko [6] extended this to a structure theorem for the complete intersections of grade $g > 3$. In this paper, we introduce a matrix f defined by complete matrices $f(i)$ of grade $n + 2$,

$$f = \begin{bmatrix} f(1)^t & -f(2)^t & \cdots & (-1)^{i+1} f(i) & \cdots & (-1)^{n+1} f(n)^t \end{bmatrix}$$

and define the ideal $\mathcal{D}_{n+1}(f)$ associated with f, which is generated by the maximal minors of the $n \times (n+2)$ matrix $D(f) = (x_{ij})$, where x_{ij} is the $(n+1)$st root of the jth $(n+1) \times (n+1)$ diagonal submatrix S_{ij} of the complete matrix $f(i)$ of grade $n + 2$.

The main purpose of this paper is to construct a minimal free resolution $\mathbb{F}$ of a class of the determinantal ideals generated by the maximal minors of an $n \times (n + 2)$ matrix $D(f)$, such that entries of each row of $D(f)$ form a regular sequence and the second differential map of $\mathbb{F}$ is a matrix f defined by the complete matrices of grade $n + 2$. This method gives us a case of constructing the minimal free resolution of a class of the determinantal ideals of an $n \times (n + 2)$ matrix. Among classes of determinantal ideals generated by the maximal minors of $p \times q$ matrices $Y = (y_{ij})$ with $p < q$ and indeterminates y_{ij}, except the class mentioned above it is not easy to find one of them which has the minimal free resolution such that the second differential map of it has a matrix defined by complete matrices of grade q and each row of Y forms a regular sequence of length q.

2. A minimal free resolution of a class of determinantal ideals

Let k be a field containing the field $\mathbb{Q}$ of rational numbers and let $R = k[x_{ij}|1 \leq i \leq n, 1 \leq j \leq n+2]$ be the polynomial ring over a field k with indeterminates x_{ij}. Choi, Kang and Ko [6] introduced a complete matrix of grade g to describe a structure theorem for complete intersections of grade $g > 3$. Choi, Kang and Ko [6] also showed that the second differential map of the Koszul complex defined by a regular sequence $\mathbf{x} = x_1, x_2, \ldots, x_g$ satisfies the conditions of Proposition 4.3 and Theorem 4.4 [6]. By using them and the induction on $g > 3$ we can define a complete matrix of grade $g > 3$ from the second differential map of the Koszul complex defined by a regular sequence $\mathbf{x}$. Theorem 4.4 [6] enables us to define a complete matrix of grade g. By the induction on g, we call $\bar{T}_k$ given in Theorem 4.4 [6] a complete matrix of grade $g-1$ for each k. For more background information we refer the reader to [6, 11].

Definition 2.1. [6] Let R be a commutative ring with identity. Let $g > 3$ and $t = \binom{g}{2}$ be integers. A $g \times t$ matrix f over R is said to be *complete of grade* g if
(1) f has g disjoint pairs (S, T) of a $g \times (g-1)$ submatrix S and a $g \times (t-g+1)$ submatrix T ;
(2) By removing a row and interchanging columns, each pair (S, T) can be reduced to a pair $(\bar{S}, \bar{T})$, where $\bar{S}$ is a $(g-1)\times(g-1)$ diagonal matrix with $det(\bar{S}) = x^{g-1}$ for some x in R, up to sign, and $\bar{T}$ is the complete matrix of grade $g-1$ with grade $\mathcal{K}_{g-2}(\bar{T}) = g-1$.

Let n be an integer with $n \geq 2$ and $x_{i1}, x_{i2}, \ldots, x_{in+2}$ a regular sequence on R for $i = 1, 2, \ldots, n$. First we construct a complete matrix of grade $n+2$. Let j and k be integers with $1 \leq j \leq n+1$ and $1 \leq k \leq n+2$, respectively. We define $f(i,j,k)$ to be a $1 \times (n+2-j)$ matrix whose the lth entry is given by

$$f(i,j,k)_l = \begin{cases} (-1)^{j+1}x_{ij} & \text{if } j < k \text{ and } l = k-j \\ 0 & \text{if } j < k \text{ and } l \neq k-j \\ (-1)^{l+j}x_{ij+l} & \text{if } j = k \\ 0 & \text{if } j > k. \end{cases} \tag{2.1}$$

Then we observe easily from (2.1) that if $j > k$, then $f(i,j,k)$ is a zero matrix and, if $j = k$, then $f(i,j,k)$ has the form of

$$f(i,j,k) = \begin{bmatrix} (-1)^{1+j}x_{i1+j} & (-1)^{2+j}x_{i2+j} & \cdots & (-1)^{n+2}x_{in+2} \end{bmatrix},$$

and, if $j < k$, then the $(k-j)$th entry of $f(i,j,k)$ is $(-1)^{j+1}x_{ij}$ and other entries are equal to zero. Let $s = \binom{n+2}{2}$. Define $f(i,k)$ to be an $s \times 1$ matrix given by

$$f(i,k) = \begin{bmatrix} f(i,1,k) & f(i,2,k) & \cdots & f(i,n+1,k) \end{bmatrix}^t . \tag{2.2}$$

We also define $f(i)$ to be an $(n+2) \times s$ matrix given by

$$f(i) = \begin{bmatrix} f(i,1) & f(i,2) & \cdots & f(i,n+2) \end{bmatrix}^t . \tag{2.3}$$

The following theorem shows that $f(i)$ is a complete matrix of grade $n+2$.

Theorem 2.2. *With notations as above, we have*
(1) *Every row of $f(i)$ has exactly $(n+1)$ nonzero entries.*
(2) *Every column of $f(i)$ has exactly two nonzero entries.*
(3) *Pairs of positive integers which represent the positions of the two nonzero entries in any two columns of $f(i)$ are all distinct.*

Proof. (1) It suffices to show that $f(i,k)$ has exactly $(n+1)$ nonzero entries for each k. It follows from (2.1) that if $k = 1$ and $j = k$, then every entry of $f(i,j,k)$ is nonzero and if $j > k$, then $f(i,j,k)$ is a zero matrix. Hence we can get from (2.2) that the number of nonzero entries of $f(i,1)^t$ is equal to $n+1$. It follows from (2.1) that if $k > 1$, then the number of nonzero entries of $f(i,j,k)$ is equal to 1 for $j < k$ and every entry of $f(i,k,k)$ is nonzero. Moreover $f(i,j,k)$ is a zero matrix for $j > k$. Hence the number of nonzero entries of $f(i,k)$ is equal to $n+1$.
(2) It follows from (2.2) and (2.3) that if $\mathbf{r}_k(f(i))$ is the kth row of $f(i)$, then we have

$$\mathbf{r}_k(f(i)) = \begin{bmatrix} f(i,1,k) & f(i,2,k) & \cdots & f(i,n+1,k) \end{bmatrix} .$$

Let $\mathbf{c}_l(f(i))$ be the lth column of $f(i)$. We show that the number of nonzero entries of $\mathbf{c}_l(f(i))$ is equal to 2. We observe from (2.1) that if l is an integer with $1 \le l \le n+1$, then the first and $(l+1)$th entries of $\mathbf{c}_l(f(i))$ are nonzero and other entries are zero: if l is an integer with $n+2 \le l \le 2n+1$, then the second and $(l-(n+1)+2)$th entries of $\mathbf{c}_l(f(i))$ are nonzero and other entries are zero. Continuing this way, we get the following : if

$$\phi(m) = \sum_{q=1}^{m-1}(n+2-q)$$

and if l is an integer with $1+\phi(h) \leq l \leq \phi(h+1)$ for $h = 1, 2, \ldots, n+1$, then the hth and $(l-\phi(h)+h)$th entries of $\mathbf{c}_l(f(i))$ are nonzero and other entries are zero. Thus we get the desired result.
(3) It follows from the observation in part (2). □

We describe a submatrix of an $m \times n$ matrix h for the following example. Let $h(i_1, i_2, \ldots, i_p | j_1, j_2, \ \ldots, j_q)$ be the $p \times q$ submatrix of h consisting of the pq entries at the intersection of rows $i_1, i_2, \ldots, i_p$ with columns $j_1, j_2, \ldots, j_q$, where $1 \leq i_1 < i_2 < \cdots < i_p \leq m$ and $1 \leq j_1 < j_2 < \cdots < j_q \leq n$.

Now we give an example to illustrate Theorem 2.2.

Example 2.3. Let $x_{i1}, x_{i2}, \ldots, x_{i4}$ be a regular sequence on a commutative ring with identity. For $i = 1, 2$, we define $f(i, j, k)$ to be a $1 \times (4-j)$ matrix as follows

$$\begin{aligned}
&f(i,1,1) = \begin{bmatrix} x_{i2} & -x_{i3} & x_{i4} \end{bmatrix}, \ f(i,2,1) = \begin{bmatrix} 0 & 0 \end{bmatrix}, \ f(i,3,1) = \begin{bmatrix} 0 \end{bmatrix}, \\
&f(i,1,2) = \begin{bmatrix} x_{i1} & 0 & 0 \end{bmatrix}, \ f(i,2,2) = \begin{bmatrix} -x_{i3} & x_{i4} \end{bmatrix}, \ f(i,3,2) = \begin{bmatrix} 0 \end{bmatrix}, \\
&f(i,1,3) = \begin{bmatrix} 0 & x_{i1} & 0 \end{bmatrix}, \ f(i,2,3) = \begin{bmatrix} -x_{i2} & 0 \end{bmatrix}, \ f(i,3,3) = \begin{bmatrix} x_{i4} \end{bmatrix}, \\
&f(i,1,4) = \begin{bmatrix} 0 & 0 & x_{i1} \end{bmatrix}, \ f(i,2,4) = \begin{bmatrix} 0 & -x_{i2} \end{bmatrix}, \ f(i,3,4) = \begin{bmatrix} x_{i3} \end{bmatrix}.
\end{aligned}$$

Then we have

$$\begin{aligned}
&f(i,1) = \begin{bmatrix} x_{i2} & -x_{i3} & x_{i4} & 0 & 0 & 0 \end{bmatrix}^t, \ f(i,2) = \begin{bmatrix} x_{i1} & 0 & 0 & -x_{i3} & x_{i4} & 0 \end{bmatrix}^t, \\
&f(i,3) = \begin{bmatrix} 0 & x_{i1} & 0 & -x_{i2} & 0 & x_{i4} \end{bmatrix}^t, \ f(i,4) = \begin{bmatrix} 0 & 0 & x_{i1} & 0 & -x_{i2} & x_{i3} \end{bmatrix}^t
\end{aligned}$$

and

$$f(i) = \begin{bmatrix} f(i,1)^t \\ f(i,2)^t \\ f(i,3)^t \\ f(i,4)^t \end{bmatrix} = \begin{bmatrix} x_{i2} & -x_{i3} & x_{i4} & 0 & 0 & 0 \\ x_{i1} & 0 & 0 & -x_{i3} & x_{i4} & 0 \\ 0 & x_{i1} & 0 & -x_{i2} & 0 & x_{i4} \\ 0 & 0 & x_{i1} & 0 & -x_{i2} & x_{i3} \end{bmatrix}.$$

It is easy to show that $f(i)$ is a complete matrix of grade 4 for each i. We observe that every row of $f(i)$ contains exactly three nonzero and three zero entries. For $k = 1, 2, 3, 4$, let S_k be the 4×3 submatrix of $f(i)$ formed by the three columns which entries of the kth row of $f(i)$ are nonzero and let T_k be the 4×3 submatrix of $f(i)$ formed by the three columns which entries of the kth row of $f(i)$ are zero. That is,

$$S_1 = f(i)(1,2,3,4|1,2,3), \quad T_1 = f(i)(1,2,3,4|4,5,6),$$
$$S_2 = f(i)(1,2,3,4|1,4,5), \quad T_2 = f(i)(1,2,3,4|2,3,6),$$
$$S_3 = f(i)(1,2,3,4|2,4,6), \quad T_3 = f(i)(1,2,3,4|1,3,5),$$
$$S_4 = f(i)(1,2,3,4|3,5,6), \quad T_4 = f(i)(1,2,3,4|1,2,4).$$

Let $\bar{S}_k$ be the 3×3 submatrix of S_k obtained by deleting the kth row of S_k for each k. Then $\bar{S}_k$ is a diagonal matrix with determinant x_k^3. Let $\bar{T}_k$ be the 3×3 matrix obtained by exchanging the first and third columns of T_k and by deleting the kth row of T_k. Then $\bar{T}_k$ becomes an alternating matrix when the second column is multiplied by -1, and $\mathrm{Pf}_2(\mathcal{A}(\bar{T}_k))$ has grade 3 for each k.(The definition of $\mathcal{A}(\bar{T}_k)$ has appeared in (3.1) [11].) Hence $f(i)$ is a complete matrix of grade 4. Actually it is the second differential map in the Koszul complex defined by a regular sequence $\mathbf{x}_i = x_{i1}, -x_{i2}, x_{i3}, -x_{i4}$.

The following proposition plays an important role in defining an $\binom{n+2}{2} \times n(n+2)$ matrix defined by complete matrices of grade $n+2$.

Proposition 2.4. *With notations as above, if $x_{i1}, x_{i2}, \ldots, x_{in+2}$ is a regular sequence on R for each i, then $f(i)$ is a complete matrix of grade $n+2$.*

Proof. It suffices to show that the conditions of Proposition 4.3 and Theorem 4.4 [6] are satisfied. Since $x_{i1}, x_{i2}, \ldots, x_{in+2}$ is a regular sequence for each i, so is $x_{i1}, -x_{i2}, x_{i3}, -x_{i4}, \ldots, (-1)^{k+1}x_{ik}, \ldots, (-1)^{n+3}x_{in+2}$. Thus

$$\mathbf{y}_{i1} = -x_{i2}, x_{i3}, -x_{i4}, \ldots, (-1)^{k+1}x_{ik}, \ldots, (-1)^{n+3}x_{in+2}$$

is a regular sequence. We observe that $f(i)$ has the form

$$\begin{aligned} f(i) &= \left[\begin{array}{c|c} l_1(i) & \mathbf{0} \\ \hline l_2(i) & f(i)_1 \end{array}\right], \\ l_1(i) &= \begin{bmatrix} -x_{i2} & x_{i3} & -x_{i4} & \cdots & (-1)^{k+1}x_{ik} & \cdots & (-1)^{n+3}x_{in+2} \end{bmatrix}, \\ l_2(i) &= \mathrm{diag}\{x_{i1}, x_{i1}, \cdots, x_{i1}\} \\ f(i)_1 &= \text{the second differential map in the Koszul complex } \mathbb{K}(\mathbf{y}_{i1}). \end{aligned} \tag{2.4}$$

Let $s = \binom{n+2}{2}$. It follows from part (1) of Theorem 2.2 that every row of $f(i)$ has exactly $(n+1)$ nonzero and $(s-n-1)$ zero entries. Let S_k be the $(n+2) \times (n+1)$ submatrix of $f(i)$ formed by $(n+1)$ columns which entries of the kth row of $f(i)$ are nonzero. Let T_k be the $(n+2) \times (n+1)$

submatrix of $f(i)$ formed by $(s-n-1)$ columns which entries of the kth row of $f(i)$ are zero. We have shown that the conditions of Proposition 4.3 [6] are satisfied. It follows from Theorem 2.2 that the conditions of Theorem 4.4 [6] are satisfied. □

Now we are ready to give a matrix defined by complete matrices $f(i)$ of grade $n+2$.

DEFINITION 2.5. *Let R be a commutative ring with identity. Let n be an integer with $n \geq 2$ and $x_{i1}, x_{i2}, \ldots, x_{in+2}$ a regular sequence on R for each $i(1 \leq i \leq n)$. Let $f(i)$ be an $(n+2) \times \binom{n+2}{2}$ complete matrix of grade $n+2$ defined in* (2.3). *We define f to be an $\binom{n+2}{2} \times n(n+2)$ matrix given by*

$$f = \begin{bmatrix} f(1)^t & -f(2)^t & \cdots & (-1)^{i+1}f(i)^t & \cdots & (-1)^{n+1}f(n)^t \end{bmatrix}.$$

We call f the matrix defined by complete matrices $f(i)$ of grade $n+2$. The following example illustrates Definition 2.5.

EXAMPLE 2.6. Let $x_{i1}, x_{i2}, \ldots, x_{i4}$ be a regular sequence on a commutative ring with identity for each i. Then, as shown in Example 2.3, $f(i)^t$ has the form

$$f(i)^t = \begin{bmatrix} x_{i2} & x_{i1} & 0 & 0 \\ -x_{i3} & 0 & x_{i1} & 0 \\ x_{i4} & 0 & 0 & x_{i1} \\ 0 & -x_{i3} & -x_{i2} & 0 \\ 0 & x_{i4} & 0 & -x_{i2} \\ 0 & 0 & x_{i4} & x_{i3} \end{bmatrix}.$$

We have proved that $f(1)$ and $f(2)$ are complete matrices of grade 4. The matrix f given by $f = \begin{bmatrix} f(1)^t & -f(2)^t \end{bmatrix}$, that is,

$$f = \begin{bmatrix} x_{12} & x_{11} & 0 & 0 & -x_{22} & -x_{21} & 0 & 0 \\ -x_{13} & 0 & x_{11} & 0 & x_{23} & 0 & -x_{21} & 0 \\ x_{14} & 0 & 0 & x_{11} & -x_{24} & 0 & 0 & -x_{21} \\ 0 & -x_{13} & -x_{12} & 0 & 0 & x_{23} & x_{22} & 0 \\ 0 & x_{14} & 0 & -x_{12} & 0 & -x_{24} & 0 & x_{22} \\ 0 & 0 & x_{14} & x_{13} & 0 & 0 & -x_{24} & -x_{23} \end{bmatrix}$$

is a 6×8 matrix defined by complete matrices $f(1)$ and $f(2)$ of grade 4.

The following proposition is a consequence of Theorem 4.4 [6].

PROPOSITION 2.7. *With notations as above, if f is a matrix defined by complete matrices $f(i)$ of grade $n+2$, then f has exactly $n(n+2)$*

$(n+1)\times(n+1)$ diagonal submatrices S_{ij} of which the determinant is the $(n+1)$st power of x_{ij}.

Proposition 2.7 enables us to define an ideal associated with the matrix f defined by complete matrices $f(i)$ of grade $n+2$, called the determinantal ideal of an $n\times(n+2)$ matrix $D(f)$.

DEFINITION 2.8. *Let R be a commutative ring with identity. Let f be a matrix defined by complete matrices $f(i)$ of grade $n+2$ in Definition 2.5 and x_{ij} the $(n+1)$st root of the determinant of the $(n+1)\times(n+1)$ diagonal submatrix S_{ij} of f mentioned in Proposition 2.7. Let $D(f)=(x_{ij})$ be an $n\times(n+2)$ matrix. Let X_{ij} be the element of R defined by*

$$X_{ij}=\begin{cases}\det A_{ij} & \text{if } i<j\\ 0 & \text{if } i=j\\ -\det A_{ji} & \text{if } i>j\end{cases}$$

,where A_{ij} is the submatrix of $D(f)$ obtained by deleting two columns i and j of $D(f)$. Define $\mathcal{D}_{n+1}(f)$ to be an ideal generated by elements X_{ij}, that is,

$$\mathcal{D}_{n+1}(f)=(X_{12},X_{13},\ldots,X_{n+1n+2}).$$

Now we construct the minimal free resolution $\mathbb{F}$ of $R/\mathcal{D}_{n+1}(f)$ such that the second differential map of $\mathbb{F}$ is an $\binom{n+2}{2}\times n(n+2)$ matrix f defined by complete matrices $f(i)$ of grade $n+2$.

Let $s=\binom{n+2}{2}$. Let f_1 be a map from R^s to R defined by

$$f_1=\begin{bmatrix}X_{12} & X_{13} & \cdots & X_{n+1n+2}\end{bmatrix}: R^s\to R,$$

and f_2 a map from $R^{n(n+2)}$ to R^s defined by

$$f_2=f:R^{n(n+2)}\to R^s.$$

For $n\geq 2$, let

$$s(n)=n(n+2)-s+1=\binom{n+1}{2}.$$

Finally we construct a map f_3 from $R^{s(n)}$ to $R^{n(n+2)}$ such that

$$\mathbb{F}:0\longrightarrow R^{s(n)}\xrightarrow{f_3}R^{n(n+2)}\xrightarrow{f_2}R^s\xrightarrow{f_1}R \tag{2.5}$$

is a minimal free resolution of $R/\mathcal{D}_{n+1}(f)$. Since $\mathbf{x}_i=x_{i1},x_{i2},\ldots,x_{in+2}$ is a regular sequence,

$$\mathcal{K}_{n+1}(f(i))=(x_{i1},-x_{i2},\ldots,(-1)^{k+1}x_{ik},\ldots,(-1)^{n+3}x_{in+2})$$

is a complete intersection of grade $n+2$ for each i. Hence by Theorem 4.10 [6], we have the Koszul complex $\mathbb{K}(\tilde{\mathbf{x}}_i)$ defined by the regular sequence $\tilde{\mathbf{x}}_i = x_{i1}, -x_{i2}, \ldots, (-1)^{k+1}x_{ik}, \ldots,\ (-1)^{n+3}x_{in+2}$ such that the second differential map of $\mathbb{K}(\tilde{\mathbf{x}}_i)$ is $f(i)$. For each i we define $\tilde{f}(i)$ to be an $(n+2)\times 1$ matrix given by

$$\tilde{f}(i) = \begin{bmatrix} x_{i1} & -x_{i2} & \cdots & (-1)^{k+1}x_{ik} & \cdots & (-1)^{n+3}x_{in+2}\end{bmatrix}^t.$$

We note that

$$\binom{n+1}{2} = \binom{n}{1} + \binom{n}{2}.$$

Let $h_1(i)$ be an $n(n+2)\times 1$ matrix defined as follows: we first divide $h_1(i)$ by n $(n+2)\times 1$ submatrices of it. The ith $(n+2)\times 1$ submatrix of it is $\tilde{f}(i)$ and other $(n+2)\times 1$ submatrices are zero matrices. Define h_1 to be an $n(n+2)\times n$ matrix given by

$$h_1 = \begin{bmatrix} h_1(1) & -h_1(2) & \cdots & (-1)^{i+1}h_1(i) & \cdots & (-1)^{n+1}h_1(n)\end{bmatrix}. \tag{2.6}$$

Similarly, we define h_2 to be an $n(n+2)\times\binom{n}{2}$ matrix given as follows: Let $h_2(k)$ be the kth column of h_2. We divide $h_2(k)$ by n $(n+2)\times 1$ submatrices of it. Let $P(n+2) = \{(i,j) | 1 \le i < j \le n+2\}$ be the set of pairs of integers. We set the lexicographic order on P, that is,

$$(1,2) < (1,3) < \cdots < (1,n+2) < (2,3) < (2,4) < \cdots < (n+1,n+2).$$

Let (k_1,k_2) be the kth element in $P(n+2)$. Define the k_1th $(n+2)\times 1$ submatrix of $h_2(k)$ to be a $(-1)^{k_2+1}\tilde{f}(k_2)$, the k_2th submatrix of it to be a $(-1)^{k_1}\tilde{f}(k_1)$, and other $(n+2)\times 1$ submatrices to be zero. Hence h_2 is of the form

$$h_2 = \begin{bmatrix} h_2(1) & h_2(2) & \cdots & h_2(i) & \cdots & h_2(n)\end{bmatrix}. \tag{2.7}$$

Finally, using (2.6) and (2.7), we define a map f_3 from $R^{s(n)}$ to $R^{n(n+2)}$ given by

$$f_3 = \begin{bmatrix} h_1 & h_2\end{bmatrix} : R^{s(n)} \to R^{n(n+2)}.$$

We show that the sequence $\mathbb{F}$ of free R-modules and R-maps defined in (2.5) is a complex.

LEMMA 2.9. *With reference to Definition* 2.8, *for* $k = 1, 2, \ldots, n+2$, *we have*

$$\sum_{j=1}^{n+2} (-1)^{j+1} x_{ij} X_{jk} = 0 \ \textit{ for each } i (1 \le i \le n).$$

Proof. Let f be an $s \times n(n+2)$ matrix defined by complete matrices $f(i)$ and $X = D(f) = (x_{ij})$ an $n \times (n+2)$ matrix defined in Definition 2.8, where $s = \binom{n+2}{2}$. Let $X^{(l)}$ be an $n \times (n+1)$ submatrix of $D(f)$ obtained by deleting the lth column of $D(f)$. Define $X(i)^{(k)}$ to be an $(n+1) \times (n+1)$ matrix given by

$$X(i)^{(k)} = \begin{bmatrix} \mathbf{r}_i(X^{(k)}) \\ X^{(k)} \end{bmatrix}$$

,where $\mathbf{r}_i(X^{(k)})$ is the ith row of $X^{(k)}$.
Since $\det X(i)^{(k)} = \sum_{j=1}^{n+2}(-1)^{j+1}x_{ij}X_{jk}$ and $\det X(i)^{(k)} = 0$ for each i, the result holds. □

The following lemma says that $\mathbb{F}$ defined in (2.5) becomes a free complex.

Lemma 2.10. *With notations as above,*
(1) $f(i)\tilde{f}(i) = 0$ *for each* i.
(2) $f_i f_{i+1} = 0$ *for* $i = 1, 2$.

Proof. (1) Clear.
(2) For $i = 1$, it is immediate from Lemma 2.9 and the definitions of f_1 and f_2. For $i = 2$, it is immediate from part (1) and the constructions of f_2 and f_3. □

To complete our main result we need two lemmas.

Lemma 2.11. *With notations as above,* $I_{s-1}(f)$ *contains some powers of* X_{ij}.

Proof. We note that $f_2 = f$. Let $v = f$. For each k with $1 \le k \le s$, we let v_k be the submatrix of v obtained by deleting the kth row of it. Let $P(n+2)$ be a set of pairs of integers defined as above and (k_1, k_2) the kth element in $P(n+2)$. We show that $I_{s-1}(v_k)$ contains ${X_{k_1k_2}}^{n+1}$. It is sufficient to show this for the case $k = 1$. The proof for other cases is similar. Let $n = 2$. Then $s - 1 = \binom{4}{2} - 1 = 5$, and v has the following form

$$v = \begin{bmatrix} x_{12} & x_{11} & 0 & 0 & -x_{22} & -x_{21} & 0 & 0 \\ -x_{13} & 0 & x_{11} & 0 & x_{23} & 0 & -x_{21} & 0 \\ x_{14} & 0 & 0 & x_{11} & -x_{24} & 0 & 0 & -x_{21} \\ 0 & -x_{13} & -x_{12} & 0 & 0 & x_{23} & x_{22} & 0 \\ 0 & x_{14} & 0 & -x_{12} & 0 & -x_{24} & 0 & x_{22} \\ 0 & 0 & x_{14} & x_{13} & 0 & 0 & -x_{24} & -x_{23} \end{bmatrix}.$$

We note that $X_{12} = x_{13}x_{24} - x_{14}x_{23}$. Let $v_1(i_1, i_2, \ldots, i_5)$ be the 5×5 submatrix of v_1 formed by five columns $i_1, i_2, \ldots, i_5$ of v_1. Then we have

$$\det v_1(1,2,3,5,6) = x_{14}{X_{12}}^2, \quad \text{and} \quad \det v_1(1,2,4,5,6) = x_{13}{X_{12}}^2.$$

Therefore

$$-x_{23}\det v_1(1,2,3,5,6) + x_{24}\det v_1(1,2,4,5,6) = {X_{12}}^3 \in I_5(v).$$

Now we consider the case $n > 2$. We have three cases to consider: $n = 3, n = 4$ and $n > 4$.

(a) $n = 3$. Then $s - 1 = 9$. We observe from Theorem 2.2 that for each i, every column and row of $f(i)^t$ contain exactly $n+1$ nonzero and two nonzero entries, respectively. The definition of v says that every column and row of v has exactly $n+1$ nonzero and $2n$ nonzero entries, respectively. Hence it follows from part (3) of Theorem 2.2 that v_1 has exactly $2n$ columns having n nonzero entries and exactly n^2 columns having $n+1$ nonzero entries. Let B_1 be the submatrix of v_1 formed by $2n$ columns of v_1 having exactly n nonzero entries. Then B_1 is an $(s-1) \times 2n$ matrix. For each i, let $l_1(i)$ be the sequence of the x_i's defined in (2.4) and $l_{11}(i)$ the submatrix of $l_1(i)$ obtained by deleting the first column of it. Then B_1 has the following form:

$$\begin{aligned} B_1 &= \begin{bmatrix} \overline{B_1} \\ \mathbf{0} \end{bmatrix}, \text{ where } \overline{B_1} = \begin{bmatrix} \overline{B_1(1)} & \overline{B_1(2)} & \cdots & \overline{B_1(n)} \end{bmatrix}, \\ \overline{B_1(i)} &= (-1)^{i+1} \begin{bmatrix} l_{11}(i)^t & \mathbf{0} \\ \mathbf{0} & l_{11}(i)^t \end{bmatrix}. \end{aligned} \tag{2.8}$$

It is easy to show that $\det(\overline{B_1}) = {X_{12}}^2$. Let $f(i)^t_1$ be the submatrix of $f(i)^t$ obtained by deleting the first row of it. It follows from Theorem 2.2 that $f(i)^t_1$ has three columns containing $n+1$ nonzero entries for each i such that by interchanging these three columns and then multiplying the second column by -1, the three rows of them form a 3×3 alternating matrix. We denote them by $(-1)^{i+1}f(1)_{i1}{}^t, (-1)^{i+1}f(2)_{i1}{}^t, (-1)^{i+1}f(3)_{i1}{}^t$, respectively. Let p, q, and r be integers with $1 \leq p, q, r \leq 3$ such that only two of them are equal. Let i, j, and k be integers with $1 \leq i, j, k \leq 3$ such that either all of them are distinct or only two of them are equal. Let $B_2(i_p, j_q, k_r)$ be the submatrix of v_1 formed by three columns $(-1)^{i+1}f(p)_{i1}{}^t, (-1)^{j+1}f(q)_{j1}{}^t$ and $(-1)^{k+1}f(r)_{k1}{}^t$ of v_1. Now we define $B(i_p, j_q, k_r)$ to be an $(s-1) \times (s-1)$ submatrix of v_1 given by

$$B(i_p, j_q, k_r) = \begin{bmatrix} B_1 & B_2(i_p, j_q, k_r) \end{bmatrix}.$$

We show that if $I_{s-1}(B)$ is an ideal generated by the determinants of the $(s-1)\times(s-1)$ submatrices $B(i_p, j_q, k_r)$ of v_1, then $I_{s-1}(B)$ contains ${X_{12}}^4$. We define $X(s_1, s_2, \ldots, s_a|t_1, t_2, \ldots, t_b)$ to be the submatrix of X formed by rows $s_1, s_2, \ldots, s_a$ and columns $t_1, t_2, \ldots, t_b$ of X. Now we set

$$D_{13} = \det X(2,3|4,5), D_{23} = \det X(1,3|4,5), D_{33} = \det X(1,2|4,5)$$
$$D_{14} = \det X(2,3|3,5), D_{24} = \det X(1,3|3,5), D_{34} = \det X(1,2|3,5).$$

Then we have

$$X_{12} = x_{13}D_{13} - x_{23}D_{23} + x_{33}D_{33} = -x_{14}D_{14} + x_{24}D_{24} - x_{34}D_{34}.$$

Let $\tilde{B}_2(i_p, j_q, k_r)$ be the submatrix of $B_2(i_p, j_q, k_r)$ formed by the last three rows of $B_2(i_p, j_q, k_r)$. Then it follows from (2.8) and the determinant of the block matrix that

$$\det B(i_p, j_q, k_r) = {X_{12}}^2 \det \tilde{B}_2(i_p, j_q, k_r).$$

The following simple computation shows that ${X_{12}}^2$ is a linear combination of elements $\det \tilde{B}_2(i_p, j_q, k_r)$:

$$\begin{aligned} &-x_{13}x_{14}D_{13}D_{14} - x_{23}x_{24}D_{23}D_{24} - x_{33}x_{34}D_{33}D_{34} \\ &= -\det B(1_3, 2_1, 3_1)\det B(1_3, 2_2, 3_2) - \det B(1_1, 2_3, 3_1)\det B(1_2, 2_3, 3_2) \\ &\quad - \det B(1_1, 2_1, 1_3)\det B(1_2, 2_2, 3_3), \end{aligned} \tag{2.9}$$

and

$$\begin{aligned} x_{13}x_{24}D_{13}D_{24} &= -x_{13}x_{24}x_{33}\det B(1_2, 2_1, 3_1) + x_{13}^2x_{35}\det B(2_1, 2_3, 3_1), \\ -x_{13}x_{34}D_{13}D_{34} &= -x_{13}x_{34}^2\det B(1_2, 2_1, 2_2) + x_{13}x_{24}x_{34}\det B(1_2, 2_2, 3_1), \\ x_{14}x_{23}D_{14}D_{23} &= x_{14}x_{23}x_{33}\det B(1_1, 2_2, 3_1) - x_{14}x_{23}^2\det B(1_1, 3_1, 3_2), \\ x_{23}x_{34}D_{14}D_{23} &= x_{15}x_{34}^2\det B(1_2, 2_2, 2_3) - x_{15}x_{2,3}x_{35}\det B(1_2, 2_2, 3_1), \\ -x_{14}x_{33}D_{14}D_{33} &= -x_{14}x_{33}^2\det B(1_1, 2_1, 2_2) - x_{13}x_{24}x_{35}\det B(1_2, 2_2, 3_1), \\ x_{24}x_{33}D_{24}D_{33} &= -x_{24}x_{33}^2\det B(1_1, 1_2, 2_1) + x_{13}x_{24}x_{33}\det B(1_1, 2_1, 3_2). \end{aligned} \tag{2.10}$$

Hence it follows from (2.9) and (2.10) that ${X_{12}}^4$ is contained in the ideal $I_{s-1}(B)$.

(b) $n = 4$. Then $s - 1 = 14$. As shown in the case of $n = 3$, Theorem 2.2 states that every column and row of v has exactly $n + 1$ nonzero and $2n$ nonzero entries, respectively. Let $B_1, \overline{B}_1$ and $l_{11}(i)$ be the matrices defined as in the case of $n = 3$. Direct computation shows that $\det(\overline{B}_1) = {X_{12}}^2$. Since $x_{i3}, -x_{i4}, x_{i5}, -x_{i6}$ is a regular subsequence of the regular sequence $x_{i1}, -x_{i2}, x_{i3}, -x_{i4}, x_{i5}, -x_{i6}$ for each i, by Theorem 3.5 [11], there exists a 4×6 complete submatrix $\check{f(i)}$ of a complete matrix

$f(i)$ of grade 6 such that $\mathcal{K}_3(\check{f(i)}) = x_{i3}, -x_{i4}, x_{i5}, -x_{i6}$ is a complete intersection of grade 4. $\check{f(i)}$ has the following form

$$\check{f(i)}^t = \begin{bmatrix} -x_{i4} & -x_{i3} & 0 & 0 \\ x_{i5} & 0 & -x_{i3} & 0 \\ -x_{i6} & 0 & 0 & -x_{i3} \\ 0 & x_{i5} & x_{i4} & 0 \\ 0 & -x_{i6} & 0 & x_{i4} \\ 0 & 0 & -x_{i6} & -x_{i5} \end{bmatrix}.$$

Hence v_1 contains exactly four columns such that the first three entries of the last six entries of them are nonzero, and the second three entries of them are zero (For example, see the first column of $\check{f(i)}^t$). Let $C_2(l_1, l_2, l_3)$ be the $(s-1) \times 3$ submatrix of v_1 formed by three columns l_1, l_2, l_3 of the above four columns in this order. Let $B_2(i_p, j_q, k_r)$ be the $(s-1) \times 3$ submatrix of v_1 defined as in the case of $n = 3$. Now we define $B(l_1, l_2, l_3, i_p, j_q, k_r)$ to be an $(s-1) \times (s-1)$ submatrix of v_1 given by

$$B(l_1, l_2, l_3, i_p, j_q, k_r) = \begin{bmatrix} B_1 & C_2(l_1, l_2, l_3) & B_2(i_p, j_q, k_r) \end{bmatrix}.$$

Let $\tilde{B}_2(i_p, j_q, k_r)$ be the submatrix of v_1 defined as in the case of $n = 3$. Let $\tilde{C}_2(l_1, l_2, l_3)$ be the submatrix of v_1 consisting of the second three nonzero entries of the last six entries of $\mathbf{c}(v_1)$ described as above. Then

$$\det B(l_1, l_2, l_3, i_p, j_q, k_r) = {X_{12}}^2 \det \tilde{C}_2(l_1, l_2, l_3) \det \tilde{B}_2(i_p, j_q, k_r).$$

Similarly to the case of $n = 3$, if $I_{s-1}(B)$ is the ideal generated by the determinants of the submatrices $B(l_1,\ l_2, l_3, i_p, j_q, k_r)$, then it contains ${X_{12}}^5$.

(c) $n > 4$. Similarly to the case of $n = 4$, we can see that if $I_{s-1}(B)$ is the ideal generated by the determinants of the submatrices of v_1 defined as in the case of $n = 4$, then $I_{s-1}(B)$ contains ${X_{12}}^{n+1}$. □

LEMMA 2.12. *With notations as above, $I_{s(n)}(f_3)$ contains some powers of X_{ij} for every $i < j$.*

Proof. Let $P(n+2)$ be the set of pairs of integers defined as above and (k_1, k_2) the kth element in $P(n+2)$. Let $w = f_3$ and w_k an $n^2 \times s(n)$ submatrix of w obtained by deleting rows $k_1, k_2, (n+2)+k_1, (n+2)+k_2, \ldots, (n-1)(n+2)+k_1, (n-1)(n+2)+k_2$ of w. Similarly to Lemma 2.11, we can show that $I_{s(n)}(w_k)$ contains ${X_{k_1k_2}}^n$ for each k. In the proof of Lemma 2.11 we used the column expansion of the determinant $X_{k_1k_2}$ but in this lemma we perform the row expansion of its determinant. □

The following theorem is our main result.

THEOREM 2.13. *Let R be a noetherian local ring with maximal ideal $\mathfrak{m}$. With the notation as above $\mathbb{F}$ is the minimal free resolution of $R/I_n(D(f))$ such that the second differential map of $\mathbb{F}$ is a matrix f defined by the complete matrices $f(i)$ of grade $n+2$.*

Proof. In Lemma 2.10 we proved that $\mathbb{F}$ is a complex. In Lemmas 2.11 and 2.12 we also showed that the rank and depth conditions in the Buchsbaum and Eisenbud's acyclicity criterion [3] are satisfied. Since $x_{i1}, x_{i2}, \ldots, x_{in+2}$ is a regular sequence for each i, every x_{ik} is contained in $\mathfrak{m}$ for every k. Hence $\mathbb{F}$ is minimal. It is obvious that the second differential map of $\mathbb{F}$ is f. $\square$

We finish this section with the following example illustrating Theorem 2.13.

EXAMPLE 2.14. Let $R = \mathbb{Q}[[x, y, z, u, w]]$ be the formal power series ring over the field $\mathbb{Q}$ of rational numbers with indeterminates x, y, z, u, w. We note that x, y, z, u, w is a regular sequence. Let $f(1), f(2)$ and $f(3)$ be 5×10 matrices given by

$$f(1) = \begin{bmatrix} -y & -x & 0 & 0 & 0 \\ z & 0 & -x & 0 & 0 \\ -u & 0 & 0 & -x & 0 \\ w & 0 & 0 & 0 & -x \\ 0 & z & y & 0 & 0 \\ 0 & -u & 0 & y & 0 \\ 0 & w & 0 & 0 & y \\ 0 & 0 & -u & -z & 0 \\ 0 & 0 & w & 0 & -z \\ 0 & 0 & 0 & w & u \end{bmatrix}^t, f(2) = \begin{bmatrix} u & z & 0 & 0 & 0 \\ -x & 0 & z & 0 & 0 \\ w & 0 & 0 & z & 0 \\ -y & 0 & 0 & 0 & z \\ 0 & -x & -u & 0 & 0 \\ 0 & w & 0 & -u & 0 \\ 0 & -y & 0 & 0 & -u \\ 0 & 0 & w & x & 0 \\ 0 & 0 & -y & 0 & x \\ 0 & 0 & 0 & -y & -w \end{bmatrix}^t,$$

$$f(3) = \begin{bmatrix} -x & -w & 0 & 0 & 0 \\ y & 0 & -w & 0 & 0 \\ -z & 0 & 0 & -w & 0 \\ u & 0 & 0 & 0 & -w \\ 0 & y & x & 0 & 0 \\ 0 & -z & 0 & x & 0 \\ 0 & u & 0 & 0 & x \\ 0 & 0 & -z & -y & 0 \\ 0 & 0 & u & 0 & -y \\ 0 & 0 & 0 & u & z \end{bmatrix}^t.$$

Then they are complete matrices of grade 5. Moreover $\tilde{f}(1), \tilde{f}(2)$, and $\tilde{f}(3)$ are given by

$$\begin{aligned}&\tilde{f}(1) = \begin{bmatrix} x & -y & z & -u & w \end{bmatrix}^t, \ \tilde{f}(2) = \begin{bmatrix} z & -u & x & -w & y \end{bmatrix}^t, \\ &\tilde{f}(3) = \begin{bmatrix} w & -x & y & -z & u \end{bmatrix}^t.\end{aligned}$$

Let f be a 10×15 matrix defined by complete matrices $f(1), f(2)$, and $f(3)$, that is

$$f = \begin{bmatrix} f(1)^t & -f(2)^t & f(3)^t \end{bmatrix}.$$

Then x, y, z, u, and w are the fourth roots of the determinants of the 4×4 diagonal submatrices of $f(i)$ for each i. Let $X = D(f)$ be a 3×5 matrix defined in Definition 2.8, that is,

$$D(f) = \begin{bmatrix} x & -y & z & -u & w \\ z & -u & x & -w & y \\ w & -x & y & -z & u \end{bmatrix}.$$

Let X_{ij} be the determinant of the submatrix of $D(f)$ obtained by deleting two columns i, j of it. Thus $\mathcal{D}_4(f)$ is generated by the elements X_{ij}. The minimal free resolution $\mathbb{F}$ of $R/\mathcal{D}_4(f)$ is

$$\mathbb{F} : 0 \longrightarrow R^6 \xrightarrow{f_3} R^{15} \xrightarrow{f_2} R^{10} \xrightarrow{f_1} R,$$

where

$$\begin{aligned}&f_1 = \begin{bmatrix} X_{12} & X_{13} & X_{14} & X_{15} & X_{23} & X_{24} & X_{25} & X_{34} & X_{35} & X_{45} \end{bmatrix}, \\ &f_2 = \begin{bmatrix} f(1)^t & -f(2)^t & f(3)^t \end{bmatrix}, \\ &f_3 = \begin{bmatrix} -\tilde{f}(1) & \mathbf{0} & \mathbf{0} & \mathbf{0} & \tilde{f}(3) & -\tilde{f}(2) \\ \mathbf{0} & \tilde{f}(2) & \mathbf{0} & \tilde{f}(3) & \mathbf{0} & \tilde{f}(1) \\ \mathbf{0} & \mathbf{0} & -\tilde{f}(3) & -\tilde{f}(2) & -\tilde{f}(1) & \mathbf{0} \end{bmatrix}.\end{aligned}$$

Clearly the second differential map of $\mathbb{F}$ is f and $\mathcal{D}_4(f) = I_3(\tilde{X})$, where

$$\tilde{X} = \begin{bmatrix} x & y & z & u & w \\ z & u & x & w & y \\ w & x & y & z & u \end{bmatrix}.$$

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Department of Mathematics
School of Natural Sciences
University of Incheon
402-749 Incheon, Korea
E-mail: ohkang@incheon.ac.kr

Department of Mathematics Education
Wonkwang University
570-749 Iksan, Korea
E-mail: joohyung@wku.ac.kr