Korean J. Math. **26** (2018), No. 3, pp. 467–481
https://doi.org/10.11568/kjm.2018.26.3.467

POSTPROCESSING FOR THE RAVIART–THOMAS MIXED FINITE ELEMENT APPROXIMATION OF THE EIGENVALUE PROBLEM

Kwang-Yeon Kim

Abstract. In this paper we present a postprocessing scheme for the Raviart–Thomas mixed finite element approximation of the second order elliptic eigenvalue problem. This scheme is carried out by solving a primal source problem on a higher order space, and thereby can improve the convergence rate of the eigenfunction and eigenvalue approximations. It is also used to compute a posteriori error estimates which are asymptotically exact for the L^2 errors of the eigenfunctions. Some numerical results are provided to confirm the theoretical results.

1. Introduction

Given a bounded polygonal domain $\Omega \subset \mathbb{R}^2$, we consider the following second order elliptic eigenvalue problem: find $(p, \lambda) \in H^1(\Omega) \times \mathbb{R}$ such that $\int_\Omega \rho p^2 \, d\boldsymbol{x} = 1$ and

$$(1) \qquad \begin{cases} -\operatorname{div}(A\nabla p) + cp = \lambda \rho p & \text{in } \Omega, \\ p = 0 & \text{on } \partial\Omega. \end{cases}$$

Received May 10, 2018. Revised September 2, 2018. Accepted September 7, 2018.
2010 Mathematics Subject Classification: 65N25, 65N30.
Key words and phrases: eigenvalue problem, mixed finite element method, superconvergence, postprocessing, asymptotic exactness.
This study is supported by 2015 Research Grant from Kangwon National University (No. D1000412-01-01).

Here $A \in W^{1,\infty}(\Omega)^{2\times 2}$ is a symmetric and uniformly positive definite matrix, and the functions $c, \rho \in W^{1,\infty}(\Omega)$ satisfy $c \geq 0$ and $\rho \geq \rho_0 > 0$ for some constant ρ_0. The Dirichlet boundary condition is assumed for simplicity only, and subsequent results are easily extended to other boundary conditions.

Finite element methods for the eigenvalue problem (1) have been extensively investigated in literature; we refer to the excellent survey papers [2, 3] for a general discussion. The mixed finite element method of (1) involving the vector function $\boldsymbol{u} = A\nabla p$ as well as p is useful when computing vibration modes of a fluid in a displacement formulation and has been studied, for example, in [1, 4, 5, 8, 14]. In particular, in [7, 9, 13], the supercloseness property of the mixed finite element method proved in [12] for the elliptic source problem has been extended to the elliptic eigenvalue problem (1).

One of the most effective ways to improve the accuracy of finite element approximations of the eigenfunctions and eigenvalues is to further solve an extra auxiliary source problem on a finer mesh [17] or on a higher order space [16]. Chen et al. [7] proposed such a postprocessing method to improve the convergence rate of the lowest order Raviart–Thomas mixed finite element approximation. In [7] the auxiliary source problem is solved by the mixed finite element method, and so requires a superconvergent eigenfunction approximation $\Pi_h p_h$.

In this paper we present a postprocessing method which can improve the convergence rate of the Raviart–Thomas mixed finite element method of any order for the eigenvalue problem (1). The idea is the same as that of [7], but we solve the auxiliary source problem using a higher order conforming finite element for the primal formulation. In doing so, our method does not need any superconvergent eigenfunction approximation like $\Pi_h p_h$. Moreover, we can obtain a posteriori error estimates for the L^2 errors of the eigenfunctions which are asymptotically exact when the eigenfunctions are smooth enough. We remark that residual-based error estimators were derived in [1, 8, 11] for the lowest order Raviart–Thomas element but they possess no such property, although easier to compute.

The rest of the paper is organized as follows. In Section 2 we introduce the Raviart–Thomas mixed finite element approximation of the eigenvalue problem and recall some previous results. In Section 3 we present our postprocessing method which is used to improve the eigenvalue approximation and to construct a posteriori error estimates. Finally, in Section 4 we report some numerical results which confirm the theoretical results.

2. Mixed Finite Element Method and Supercloseness Property

By introducing the vector function $\boldsymbol{u} = A\nabla p$, we can rewrite the eigenvalue problem (1) in the following mixed form

$$\begin{cases} A^{-1}\boldsymbol{u} - \nabla p = 0 & \text{in } \Omega, \\ -\operatorname{div} \boldsymbol{u} + cp = \lambda \rho p & \text{in } \Omega, \\ p = 0 & \text{on } \partial\Omega. \end{cases} \tag{2}$$

Define the function space

$$H(\operatorname{div}, \Omega) = \{\boldsymbol{v} \in L^2(\Omega)^2 : \operatorname{div} \boldsymbol{v} \in L^2(\Omega)\}$$

equipped with the norm

$$\|\boldsymbol{v}\|^2_{H(\operatorname{div},\Omega)} = \|\boldsymbol{v}\|^2_{0,\Omega} + \|\operatorname{div} \boldsymbol{v}\|^2_{0,\Omega}.$$

Then the weak formulation for the mixed form (2) reads as follows: find $(\boldsymbol{u}, p, \lambda) \in H(\operatorname{div}, \Omega) \times L^2(\Omega) \times \mathbb{R}$ such that $\int_\Omega \rho p^2 \, d\boldsymbol{x} = 1$ and

$$\begin{cases} \displaystyle\int_\Omega A^{-1}\boldsymbol{u} \cdot \boldsymbol{v} \, d\boldsymbol{x} + \int_\Omega \operatorname{div} \boldsymbol{v} \, p \, d\boldsymbol{x} = 0 & \forall \boldsymbol{v} \in H(\operatorname{div}, \Omega), \\ \displaystyle -\int_\Omega \operatorname{div} \boldsymbol{u} \, q \, d\boldsymbol{x} + \int_\Omega cpq \, d\boldsymbol{x} = \lambda \int_\Omega \rho p q \, d\boldsymbol{x} & \forall q \in L^2(\Omega). \end{cases} \tag{3}$$

From [2] we know that the eigenvalue problem (3) has an increasing sequence of positive eigenvalues $\{\lambda_m\}_{m=1}^\infty$ and the associated eigenfunctions $\{(\boldsymbol{u}_m, p_m)\}_{m=1}^\infty$ such that $\lim_{m\to\infty} \lambda_m = \infty$ and $\int_\Omega p_i p_j \, d\boldsymbol{x} = \delta_{ij}$.

The finite element discretization of (3) is constructed by first introducing a shape-regular family of triangulations $\{\mathcal{T}_h\}_{h>0}$ such that $\overline{\Omega} = \bigcup_{K\in\mathcal{T}_h} \overline{K}$ for each $h > 0$ and $h = \max_{K\in\mathcal{T}_h} h_K$, where h_K denotes the diameter of K. For an integer $k \geq 0$, the k-th order Raviart–Thomas mixed element over the triangulation $\mathcal{T}_h$ is defined by (see [6])

$$\boldsymbol{V}_h = \{\boldsymbol{v} \in H(\operatorname{div}, \Omega) : \boldsymbol{v}|_K \in RT_k(K) \ \ \forall K \in \mathcal{T}_h\},$$
$$W_h = \{q \in L^2(\Omega) : q|_K \in \mathcal{P}_k(K) \ \ \forall K \in \mathcal{T}_h\},$$

where $\mathcal{P}_k(K)$ is the space of all polynomials on K whose total degree is not greater than k, and

$$RT_k(K) = (\mathcal{P}_k(K))^2 + (x_1, x_2)\mathcal{P}_k(K).$$

Now we define the Raviart–Thomas mixed finite element method of the eigenvalue problem (3): find $(\boldsymbol{u}_h, p_h, \lambda_h) \in \boldsymbol{V}_h \times W_h \times \mathbb{R}$ such that $\int_\Omega \rho p_h^2 \, d\boldsymbol{x} = 1$ and

(4)

$$\begin{cases} \displaystyle\int_\Omega A^{-1}\boldsymbol{u}_h \cdot \boldsymbol{v}_h \, d\boldsymbol{x} + \int_\Omega \operatorname{div} \boldsymbol{v}_h \, p_h \, d\boldsymbol{x} = 0 & \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \\ \displaystyle -\int_\Omega \operatorname{div} \boldsymbol{u}_h \, q_h \, d\boldsymbol{x} + \int_\Omega c p_h q_h \, d\boldsymbol{x} = \lambda_h \int_\Omega \rho p_h q_h \, d\boldsymbol{x} & \forall q_h \in W_h. \end{cases}$$

The abstract theory of [2,4,5,14] shows that if the eigenfunctions $(\boldsymbol{u}, p)$ of (3) belong to $H^t(\Omega)^2 \times H^{t+1}(\Omega)$ for $t > 0$, then the following a priori error estimates hold with $s = \min\{k+1, t\}$

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{H(\operatorname{div},\Omega)} + \|p - p_h\|_{0,\Omega} \leq Ch^s, \qquad |\lambda - \lambda_h| \leq Ch^{2s} \tag{5}$$

for sufficiently small h. Hereafter C represents a generic positive constant which do not depend on the mesh size h.

Finally, we state the supercloseness result from [7,9,13] between the eigenfunction approximation p_h in (4) and the L^2-projection $P_h p$ of the exact eigenfunction p in W_h. To this end, following [13], we assume that the elliptic source problem

$$\begin{cases} -\operatorname{div}(A\nabla w) + cw = g & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega \end{cases}$$

satisfies the following regularity estimate for some $\gamma \in (0, 1]$

$$\|w\|_{1+\gamma,\Omega} \leq C\|g\|_{0,\Omega}. \tag{6}$$

According to the results of [10], we have $\gamma = 1$ if Ω is convex and $\gamma = \frac{\pi}{\omega} - \epsilon$ for any $\epsilon > 0$ if Ω is a nonconvex polygon whose maximum interior angle is $\omega < 2\pi$.

Theorem 1. *(See [13, Corollary 3.3]) Assume that the eigenfunctions $(\boldsymbol{u}, p)$ of (3) belong to $H^t(\Omega)^2 \times H^{t+1}(\Omega)$ and the regularity estimate (6) holds (with $\gamma \leq t$). Then the following supercloseness result holds with $s = \min\{k+1, t\}$*

$$\|P_h p - p_h\|_{0,\Omega} \leq Ch^{s+\gamma}$$

for sufficiently small h.

Remark 1. By virtue of Theorem 1, the a priori error estimate $\|p - p_h\|_{0,\Omega} = O(h^s)$ given in (5) may be improved to

$$\begin{aligned} \|p - p_h\|_{0,\Omega} &\le \|p - P_h p\|_{0,\Omega} + \|P_h p - p_h\|_{0,\Omega} \\ &\le C h^{\min\{k+1,t+1\}} + C h^{\min\{k+1,t\}+\gamma} \\ &\le C h^{\min\{k+1,t+\gamma\}}. \end{aligned} \tag{7}$$

This was observed in Remark 4.2 of [13].

Based on the supercloseness result of Theorem 1, the authors of [7] were able to construct a superconvergent eigenfunction approximation $\Pi_h p_h$ which is continuous piecewise linear and satisfies

$$\|p - \Pi_h p_h\|_0 \le C h^{2\gamma}$$

for the lowest order case $k = 0$ and $s = t = \gamma$. Then, in order to improve the convergence rate of the eigenpair approximation $(\boldsymbol{u}_h, p_h, \lambda_h)$, they proposed a postprocessing method which solves the auxiliary source problem obtained from (4) by replacing p_h by $\Pi_h p_h$ in the right-hand side of the second equation, i.e., $\lambda_h \int_\Omega \rho p_h q_h \, d\boldsymbol{x} \to \lambda_h \int_\Omega \rho (\Pi_h p_h) q_h \, d\boldsymbol{x}$. The auxiliary source problem thus obtained is solved by the mixed finite element method on a finer mesh [17] or on a higher order space [16]. In doing so, construction of a superconvergent eigenfunction approximation $\Pi_h p_h$ is essential.

3. Postprocessing for Improving the Eigenpair Approximation

In this section we propose a postprocessing method to improve the eigenpair approximation (p_h, λ_h) computed by the mixed finite element method (4) for any oder $k \ge 0$.

Postprocessing method: Let $\Psi_h \subset H_0^1(\Omega)$ be the standard $\mathcal{P}_{k+2}$ conforming finite element space on $\mathcal{T}_h$ and define $\psi_h \in \Psi_h$ to be the solution of the auxiliary source problem

$$a(\psi_h, \phi_h) = \lambda_h b(p_h, \phi_h) \qquad \forall \phi_h \in \Psi_h, \tag{8}$$

where

$$a(\psi, \phi) = \int_\Omega (A \nabla \psi \cdot \nabla \phi + c \psi \phi) \, d\boldsymbol{x}, \qquad b(\psi, \phi) = \int_\Omega \rho \psi \phi \, d\boldsymbol{x}.$$

It is important to note that the auxiliary source problem is solved using a higher order conforming finite element for the primal formulation, whereas a higher-order mixed finite element method is used in [7]. Clearly, our method is much more economical. Moreover, our method does not demand any additional construction of a superconvergent eigenfunction approximation like $\Pi_h p_h$. Finally, as will be shown below, our method not only improves the convergence rate of the eigenpair approximation but also yields asymptotically exact a posteriori error estimates for both $\|\boldsymbol{u} - \boldsymbol{u}_h\|_{0,\Omega}$ and $\|p - p_h\|_{0,\Omega}$ if the eigenfunctions are smooth enough. All of these assertions are based on the following error estimates for ψ_h.

THEOREM 2. *Let $\psi_h \in \Psi_h$ be defined by (8) and assume that the regularity estimate (6) holds. Then we have for any $\chi_h \in \Psi_h$*

$$\|p-\psi_h\|_{1,\Omega} \le C(\|p-\chi_h\|_{1,\Omega}+h\|p-P_hp\|_{0,\Omega}+\|P_hp-p_h\|_{0,\Omega}+|\lambda-\lambda_h|) \tag{9}$$

and

$$\|p-\psi_h\|_{0,\Omega} \le C(h^\gamma\|p-\chi_h\|_{1,\Omega}+h\|p-P_hp\|_{0,\Omega}+\|P_hp-p_h\|_{0,\Omega}+|\lambda-\lambda_h|). \tag{10}$$

Proof. The standard variational formulation of the eigenvalue problem (1) and the equation (8) together leads to

$$a(p - \psi_h, \phi_h) = \int_\Omega (\lambda p - \lambda_h p_h)\rho\phi_h \, d\boldsymbol{x} \qquad \forall \phi_h \in \Psi_h, \tag{11}$$

and thus

$$a(\psi_h - \chi_h, \phi_h) = a(p - \chi_h, \phi_h) - \int_\Omega (\lambda p - \lambda_h p_h)\rho\phi_h \, d\boldsymbol{x}. \tag{12}$$

The second term of the right-hand side can be written as

$$\begin{aligned}
&\int_\Omega (\lambda p - \lambda_h p_h)\rho\phi_h \, d\boldsymbol{x} \\
&= \int_\Omega \lambda(p - P_hp)\rho\phi_h \, d\boldsymbol{x} + \int_\Omega (\lambda - \lambda_h)P_hp\,(\rho\phi_h)\, d\boldsymbol{x} + \int_\Omega \lambda_h(P_hp - p_h)\rho\phi_h \, d\boldsymbol{x} \\
&= \int_\Omega \lambda(p - P_hp)\{\rho\phi_h - P_h(\rho\phi_h)\}\, d\boldsymbol{x} \\
&\qquad + \int_\Omega (\lambda - \lambda_h)P_hp\,(\rho\phi_h)\, d\boldsymbol{x} + \int_\Omega \lambda_h(P_hp - p_h)\rho\phi_h \, d\boldsymbol{x}.
\end{aligned}$$

Now applying Hölder's inequality and the estimate $\|v - P_h v\|_{0,\Omega} \leq Ch|v|_{1,\Omega}$, we obtain

(13)
$$\left| \int_\Omega (\lambda p - \lambda_h p_h) \rho \phi_h \, d\boldsymbol{x} \right| \leq C(h\|p - P_h p\|_{0,\Omega} + |\lambda - \lambda_h| + \|P_h p - p_h\|_{0,\Omega}) \|\phi_h\|_{1,\Omega}.$$

Then, by taking $\phi_h = \psi_h - \chi_h$ in (12), it follows that

$$\|\psi_h - \chi_h\|_{1,\Omega} \leq C(\|p - \chi_h\|_{1,\Omega} + h\|p - P_h p\|_{0,\Omega} + |\lambda - \lambda_h| + \|P_h p - p_h\|_{0,\Omega}),$$

which gives the first result (9) by the triangle inequality.

To derive (10), we use the well-known duality argument. Let $\xi \in H_0^1(\Omega)$ be the solution of

$$a(\phi, \xi) = \int_\Omega (p - \psi_h) \phi \, d\boldsymbol{x} \qquad \forall \phi \in H_0^1(\Omega)$$

such that $\|\xi\|_{1+\gamma,\Omega} \leq C\|p - \psi_h\|_{0,\Omega}$ (see (6)). Then we have

$$\|p - \psi_h\|_{0,\Omega}^2 = a(p - \psi_h, \xi) = a(p - \psi_h, \xi - I_h^1 \xi) + a(p - \psi_h, I_h^1 \xi),$$

where $I_h^1 \xi \in H_0^1(\Omega)$ denotes the standard linear interpolant of ξ. The first term of the right-hand side is easily bounded using the approximation property of I_h^1

$$a(p - \psi_h, \xi - I_h^1 \xi) \leq C\|p - \psi_h\|_{1,\Omega} \|\xi - I_h^1 \xi\|_{1,\Omega} \leq Ch^\gamma \|p - \psi_h\|_{1,\Omega} \|\xi\|_{1+\gamma,\Omega}.$$

The second term is identical to (11) with $\phi_h = I_h^1 \xi$. Hence it follows by (13) that

$$\begin{aligned} a(p - \psi_h, I_h^1 \xi) &\leq C(h\|p - P_h p\|_{0,\Omega} + |\lambda - \lambda_h| + \|P_h p - p_h\|_{0,\Omega}) \|I_h^1 \xi\|_{1,\Omega} \\ &\leq C(h\|p - P_h p\|_{0,\Omega} + |\lambda - \lambda_h| + \|P_h p - p_h\|_{0,\Omega}) \|\xi\|_{1+\gamma,\Omega}. \end{aligned}$$

The proof is completed by combining the above results. □

As a corollary of Theorem 2, we obtain the following estimates which show that ψ_h indeed improves the convergence rate of the eigenfunction approximation p_h (if the eigenfunctions are smooth enough).

COROLLARY 1. *Under the conditions of Theorem 1, we have for sufficiently small h*

$$\|p - \psi_h\|_{1,\Omega} \leq Ch^{\min\{k+1+\gamma, t\}}, \qquad \|p - \psi_h\|_{0,\Omega} \leq Ch^{\min\{k+1+\gamma, t+\gamma\}}.$$

Proof. Using the standard error estimates for $\|p - \chi_h\|_{1,\Omega}$ and $\|p - P_h p\|_{0,\Omega}$, the error estimate (5) for $|\lambda - \lambda_h|$ and Theorem 1 for $\|P_h p - p_h\|_{0,\Omega}$, we obtain from (9)

$$\|p-\psi_h\|_{1,\Omega} \le Ch^{\min\{k+2,t\}}+Ch^{1+\min\{k+1,t+1\}}+Ch^{\min\{k+1,t\}+\gamma}+Ch^{2\min\{k+1,t\}},$$

which proves the first result. The second result follows similarly from (10). ☐

Eigenvalue improvement: The superconvergent eigenfunction ψ_h can be used to improve the eigenvalue approximation via the Rayleigh quotient (see, e.g., [15])

$$\widehat{\lambda}_h = \frac{a(\psi_h, \psi_h)}{b(\psi_h, \psi_h)}.$$

Estimation of the improved eigenvalue error $\lambda - \widehat{\lambda}_h$ depends on the following identity from [2]

$$\frac{a(\phi,\phi)}{b(\phi,\phi)} - \lambda = \frac{a(p-\phi,p-\phi)}{b(\phi,\phi)} - \lambda\frac{b(p-\phi,p-\phi)}{b(\phi,\phi)}, \tag{14}$$

where (p, λ) is an eigenpair of (1) and ϕ is any nonzero function in $H_0^1(\Omega)$.

THEOREM 3. *Under the conditions of Theorem 1, we have for sufficiently small h*

$$|\lambda - \widehat{\lambda}_h| \le Ch^{2\min\{k+1+\gamma,t\}},$$

which is of higher order than $|\lambda - \lambda_h| = O(h^{2s}) = O(h^{2\min\{k+1,t\}})$ *if* $t > k+1$.

Proof. Taking $\phi = \psi_h$ in (14), we obtain

$$|\lambda - \widehat{\lambda}_h| \le C\|p - \psi_h\|^2_{1,\Omega}.$$

The proof is completed by invoking Corollary 1. ☐

A Posteriori Error Estimates: We can also use ψ_h to define the a posteriori error estimates

$$\eta_{\boldsymbol{u}} = \|\boldsymbol{u}_h - A\nabla\psi_h\|_{0,\Omega}, \qquad \eta_p = \|p_h - \psi_h\|_{0,\Omega}. \tag{15}$$

Again thanks to the superconvergence results of Corollary 1, the following theorem shows that $\eta_{\boldsymbol{u}}$ and η_p are asymptotically exact for the eigenfunction errors $\|\boldsymbol{u} - \boldsymbol{u}_h\|_{0,\Omega}$ and $\|p - p_h\|_{0,\Omega}$, respectively, as the mesh size h tends to zero.

THEOREM 4. *Under the conditions of Theorem 1, we have for sufficiently small h*

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{0,\Omega} = \eta_{\boldsymbol{u}} + O(h^{\min\{k+1+\gamma,t\}}).$$

Moreover, it holds that

$$\left|\frac{\eta_{\boldsymbol{u}}}{\|\boldsymbol{u} - \boldsymbol{u}_h\|_{0,\Omega}} - 1\right| = O(h^{\min\{\gamma,t-k-1\}}),$$

provided that $t > k+1$ and there exists a constant $C > 0$ such that

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{0,\Omega} \geq Ch^{\min\{k+1,t\}} = Ch^{k+1}. \tag{16}$$

Similarly,

$$\|p - p_h\|_{0,\Omega} = \eta_p + O(h^{\min\{k+1+\gamma,t+\gamma\}})$$

and

$$\left|\frac{\eta_p}{\|p - p_h\|_{0,\Omega}} - 1\right| = O(h^{\min\{\gamma,t+\gamma-k-1\}}),$$

provided that $t + \gamma > k+1$ and there exists a constant $C > 0$ such that

$$\|p - p_h\|_{0,\Omega} \geq Ch^{\min\{k+1,t+\gamma\}} = Ch^{k+1}. \tag{17}$$

Proof. By Corollary 1 we obtain

$$\left|\|\boldsymbol{u}-\boldsymbol{u}_h\|_{0,\Omega}-\eta_{\boldsymbol{u}}\right| \leq \|A\nabla p - A\nabla\psi_h\|_{0,\Omega} \leq C\|p-\psi_h\|_{1,\Omega} \leq Ch^{\min\{k+1+\gamma,t\}},$$

and thus

$$\left|\frac{\eta_{\boldsymbol{u}}}{\|\boldsymbol{u} - \boldsymbol{u}_h\|_{0,\Omega}} - 1\right| \leq C\frac{h^{\min\{k+1+\gamma,t\}}}{h^{k+1}} = Ch^{\min\{\gamma,t-k-1\}}.$$

The same argument applied to the error $\|p - p_h\|_{0,\Omega}$ gives

$$\left|\|p - p_h\|_{0,\Omega} - \eta_p\right| \leq \|p - \psi_h\|_{0,\Omega} \leq Ch^{\min\{k+1+\gamma,t+\gamma\}},$$

and thus

$$\left|\frac{\eta_p}{\|p - p_h\|_{0,\Omega}} - 1\right| \leq C\frac{h^{\min\{k+1+\gamma,t+\gamma\}}}{h^{k+1}} = Ch^{\min\{\gamma,t+\gamma-k-1\}}.$$

This completes the proof. □

REMARK 2. The non-degeneracy conditions (16) and (17) come from the a priori error estimates (5) and (7).

REMARK 3. For the lowest-order element $k = 0$, the a posteriori error estimate $\|p_h - \Pi_h p_h\|_{0,\Omega}$ given in [11] is much easier to compute and is asymptotically exact for $\|p - p_h\|_{0,\Omega}$ under the conditions of Theorem 1. But there is only a residual-based estimator for $\|\boldsymbol{u} - \boldsymbol{u}_h\|_{H(\mathrm{div},\Omega)}$ which is not asymptotically exact.

4. Numerical Results

In this section we report some numerical results for the Laplace eigenvalue problem

$$-\Delta p = \lambda p \quad \text{in } \Omega, \qquad p = 0 \quad \text{on } \partial\Omega, \tag{18}$$

where Ω is the unit square $(0, 1)^2$ in the first example and is the L-shaped domain $(-1, 1) \times (-1, 1) \setminus [0, 1] \times [-1, 0]$ in the second example. This problem is solved by the Raviart–Thomas mixed finite element method (4) of the low orders $k = 0, 1$ with the normalization $\|p_h\|_{0,\Omega} = 1$. The eigenpairs $(\boldsymbol{u}_h, p_h, \lambda_h)$ are computed by the MATLAB command 'eigs'.

4.1. Square domain. For the unit square $\Omega = (0, 1)^2$, it is well known that the eigenvalues of (18) are given by

$$\lambda_{l,m} = (l^2 + m^2)\pi^2 \qquad (l, m = 1, 2, 3, \cdots)$$

with the associated eigenfunctions

$$p_{l,m}(x, y) = \sin(l\pi x)\sin(m\pi y).$$

Note that we have $\gamma = 1$ in the regularity estimate (6).

As the eigenfunctions are smooth, we consider a sequence of uniform criss-cross meshes constructed by partitioning Ω into equal squares of width $h = 1/2^n$ and then dividing every square into four equal triangles by its two diagonals (see Fig. 1 for $n = 1, 2, 3$).

The numerical results are listed in Tables 1–2 and 3–4 for the RT_0 and RT_1 mixed finite element approximations of the first eigenvalue $\lambda = 2\pi^2$ and its corresponding eigenfunction $p(x, y) = 2\sin(\pi x)\sin(\pi y)$ with $\|p\|_{0,\Omega} = 1$. The numerical convergence order and the effectivity index for $\boldsymbol{u} - \boldsymbol{u}_h$ are computed by

$$\text{Order} = \log_2 \frac{\|\boldsymbol{u} - \boldsymbol{u}_{2h}\|_{0,\Omega}}{\|\boldsymbol{u} - \boldsymbol{u}_h\|_{0,\Omega}}, \qquad I_{\boldsymbol{u}} = \frac{\eta_{\boldsymbol{u}}}{\|\boldsymbol{u} - \boldsymbol{u}_h\|_{0,\Omega}},$$

and similarly for $p - p_h$ et al., where $\eta_{\boldsymbol{u}}$ and η_p are defined by (15).

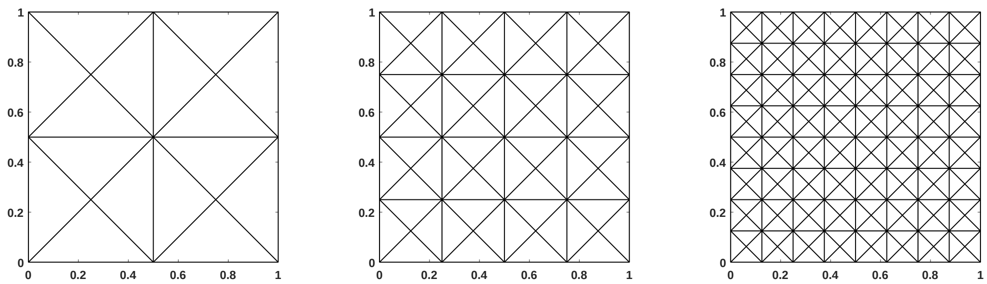

FIGURE 1. Uniform criss-cross meshes on $\Omega = (0,1)^2$ with the mesh size $h = 1/2^n$ $(n = 1,2,3)$

It is clearly seen that the convergence orders are in excellent agreement with the theoretical results of the preceding section. Moreover, it is evident that the a posteriori error estimates $\eta_{\boldsymbol{u}}$ and η_p are asymptotically exact for the L^2 errors of the eigenfunctions (see Theorem 4) and the improved eigenvalue approximation $\widehat{\lambda}_h$ exhibits the two-order higher superconvergence than λ_h (see Theorem 3). It is also worthwhile to observe that the convergence order of $\|p - \psi_h\|_{0,\Omega}$ is not better than that of $|p - \psi_h|_{1,\Omega}$ for the RT_0 element, as predicted by Corollary 1, but it is one-order higher for the RT_1 element.

$1/h$	$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{0,\Omega}$	Order	$I_{\boldsymbol{u}}$	$\|p - p_h\|_{0,\Omega}$	Order	I_p	$\|\lambda - \lambda_h\|$	Order
4	1.001e+0	—	0.9844	1.845e–1	—	0.9862	3.407e–1	—
8	5.028e–1	0.993	0.9962	9.248e–2	0.996	0.9967	8.470e–2	2.008
16	2.517e–1	0.998	0.9990	4.627e–2	0.999	0.9992	2.115e–2	2.002
32	1.259e–1	0.999	0.9998	2.314e–2	1.000	0.9998	5.285e–3	2.001
64	6.296e–2	1.000	0.9999	1.157e–2	1.000	1.0000	1.321e–3	2.000

TABLE 1. Errors and effectivity indices for the RT_0 eigenpair $(\boldsymbol{u}_h, p_h, \lambda_h)$ on $\Omega = (0,1)^2$

4.2. L-shaped domain. For $\Omega = (-1,1)\times(-1,1)\setminus[0,1]\times[-1,0]$, it is known that the eigenfunction $(\boldsymbol{u}, p)$ corresponding to the first eigenvalue λ has a singularity at the origin and belongs to the space $H^t(\Omega)^2 \times H^{1+t}(\Omega)$ with $t = \frac{2}{3} - \varepsilon$ for any $\varepsilon > 0$.

To approximate the first eigenpairs $(\boldsymbol{u}, p, \lambda)$, we successively apply adaptive mesh refinement based on the longest-edge bisection starting

$1/h$	$\lvert p-\psi_h\rvert_{1,\Omega}$	Order	$\lVert p-\psi_h\rVert_{0,\Omega}$	Order	$\widehat{\lambda}_h-\lambda$	Order
4	1.819e–1	—	3.420e–2	—	1.071e–2	—
8	4.531e–2	2.005	8.560e–3	1.998	6.167e–4	4.119
16	1.131e–2	2.002	2.141e–3	1.999	3.766e–5	4.034
32	2.828e–3	2.000	5.354e–4	2.000	2.339e–6	4.009
64	7.069e–4	2.000	1.339e–4	2.000	1.460e–7	4.002

TABLE 2. Errors for the P_2-postprocessed eigenpair $(\psi_h, \widehat{\lambda}_h)$ on $\Omega = (0,1)^2$

$1/h$	$\lVert \boldsymbol{u}-\boldsymbol{u}_h\rVert_{0,\Omega}$	Order	$I_{\boldsymbol{u}}$	$\lVert p-p_h\rVert_{0,\Omega}$	Order	I_p	$\lvert\lambda-\lambda_h\rvert$	Order
2	3.023e–1	—	0.9714	7.028e–2	—	0.9445	6.145e–3	—
4	7.499e–2	2.011	0.9927	1.773e–2	1.987	0.9856	5.803e–4	3.405
8	1.874e–2	2.000	0.9982	4.447e–3	1.995	0.9963	3.897e–5	3.896
16	4.686e–3	2.000	0.9995	1.113e–3	1.999	0.9991	2.477e–6	3.976
32	1.172e–3	2.000	0.9999	2.782e–4	2.000	0.9998	1.554e–7	3.994

TABLE 3. Errors and effectivity indices for the RT_1 eigenpair $(\boldsymbol{u}_h, p_h, \lambda_h)$ on $\Omega = (0,1)^2$

$1/h$	$\lvert p-\psi_h\rvert_{1,\Omega}$	Order	$\lVert p-\psi_h\rVert_{0,\Omega}$	Order	$\widehat{\lambda}_h-\lambda$	Order
2	7.690e–2	—	5.365e–3	—	5.369e–3	—
4	9.892e–3	2.959	3.268e–4	4.037	9.577e–5	5.809
8	1.254e–3	2.980	2.051e–5	3.994	1.563e–6	5.937
16	1.573e–4	2.995	1.283e–6	3.998	2.471e–8	5.983
32	1.968e–5	2.999	8.024e–8	3.999	3.913e–10	5.980

TABLE 4. Errors for the P_3-postprocessed eigenpair $(\psi_h, \widehat{\lambda}_h)$ on $\Omega = (0,1)^2$

with the initial mesh displayed in the left of Fig. 2. In light of (15), the local error estimator is defined as

$$\eta_K^2 = \lVert \boldsymbol{u}_h - \nabla\psi_h\rVert_{0,K}^2 + \lVert p_h - \psi_h\rVert_{0,K}^2 \qquad \forall K \in \mathcal{T}_h,$$

and the element K is marked for refinement if $\eta_K \geq \frac{1}{2}\max_{K' \in \mathcal{T}_h} \eta_{K'}$. Then more elements of $\mathcal{T}_h$ are marked for refinement to avoid hanging nodes.

Fig. 2 displays some adaptively refined meshes generated by the above process. We see that the local error estimator η_K is able to detect the singularity of the eigenfunction around which the mesh refinement is concentrated.

Fig. 3 plots the errors for the eigenvalue approximation λ_h and the postprocessed eigenvalue approximation $\widehat{\lambda}_h$ in terms of the number of unknowns N. Since the exact value of λ is not known, we take the numerical value $\lambda \approx 9.6397238440219$ used in [11] as the exact one. It is observed that $\widehat{\lambda}_h$ exhibits the almost one-order higher superconvergence than λ_h and the errors decay at nearly optimal orders, where the convergence order is computed in powers of $1/N$ for adaptively refined meshes.

To check the asymptotic exactness of the a posteriori error estimates $\eta_{\boldsymbol{u}}$ and η_p, we use the following equality (cf. Lemma 4 of [8])

$$\lambda - \lambda_h = \|\boldsymbol{u} - \boldsymbol{u}_h\|_{0,\Omega}^2 - \lambda_h \|p - p_h\|_{0,\Omega}^2$$

which holds for the Laplace eigenvalue problem (18) with $\|p\|_{0,\Omega} = \|p_h\|_{0,\Omega} = 1$. This leads us to define the effectivity index for the eigenvalue error by

$$I_\lambda = \frac{\eta_{\boldsymbol{u}}^2 - \lambda_h \eta_p^2}{\lambda - \lambda_h},$$

The results are plotted in Fig. 4 which shows that I_λ eventually converges to 1. This indicates that $\eta_{\boldsymbol{u}}$ and η_p are asymptotically exact for the L^2 errors of the eigenfunctions.

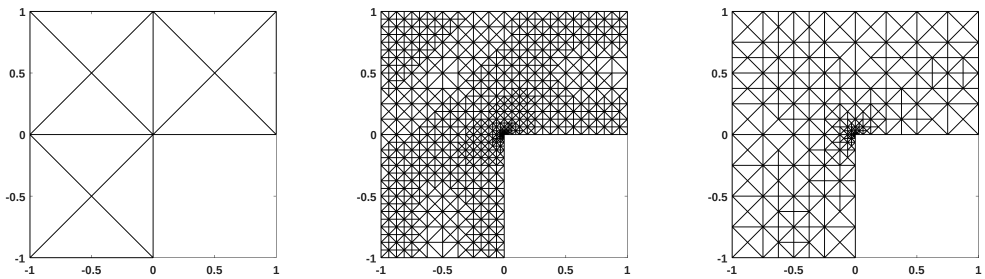

FIGURE 2. Initial (left) and adaptively refined meshes for RT_0 (middle) and RT_1 (right)

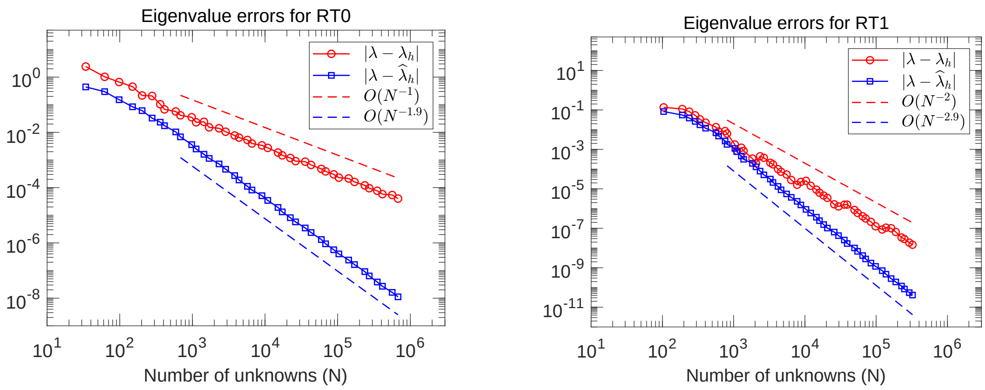

FIGURE 3. Eigenvalue errors for RT_0 (left) and RT_1 (right)

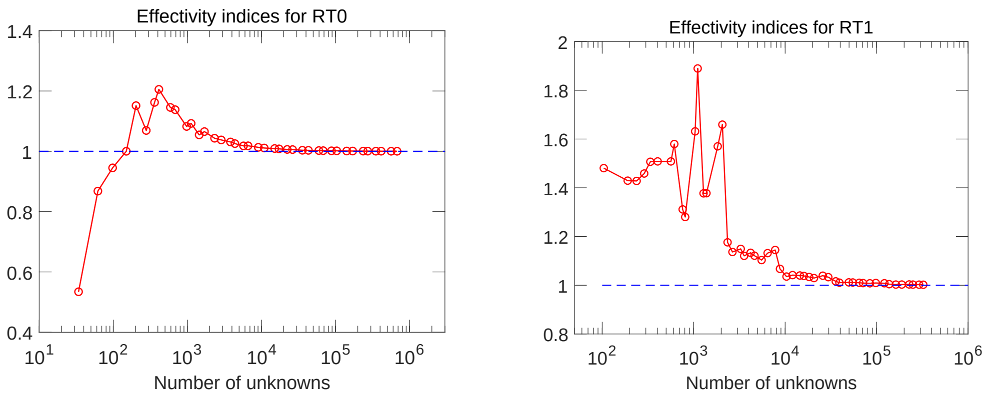

FIGURE 4. Effectivity indices for RT_0 (left) and RT_1 (right)

References

[1] A. Alonso., A. D. Russo, and V. Vampa, *A posteriori error estimates in finite element acoustic analysis*, J. Comput. Appl. Math. **117** (2000), 105–119.

[2] I. Babuška and J. Osborn, *Eigenvalue Problems, in Handbook of Numerical Analysis II, Finite Element Methods (Part 1)*, edited by P.G. Lions and P.G. Ciarlet, North-Holland, Amsterdam, 1991, 641–787.

[3] D. Boffi, *Finite element approximation of eigenvalue problems*, Acta Numer. **19** (2010), 1–120.

[4] D. Boffi, F. Brezzi, and L. Gastaldi, *On the convergence of eigenvalues for mixed formulations*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. **25** (1997), 131–154.

[5] D. Boffi, F. Brezzi, and L. Gastaldi, *On the problem of spurious eigenvalues in the approximation of linear elliptic problems in mixed form*, Math. Comp. **69** (2000), 121–140.
[6] F. Brezzi and M. Fortin, *Mixed and hybrid finite element methods*, Springer–Verlag, New York, 1991.
[7] H. Chen, S. Jia, and H. Xie, *Postprocessing and higher order convergence for the mixed finite element approximations of the eigenvalue problem*, Appl. Numer. Math. **61** (2011), 615–629.
[8] R. Durán, L. Gastaldi, and C. Padra, *A posteriori error estimators for mixed approximations of eigenvalue problems*, Math. Models Methods Appl. Sci. **9** (1999), 1165–1178.
[9] F. Gardini, *Mixed approximation of eigenvalue problems: a superconvergence result*, ESAIM: M2AN **43** (2009), 853–865.
[10] P. Grisvard, *Elliptic Problems in Non-Smooth Domains, Monographs and Studies in Mathematics 24*, Pitman, Boston, 1985.
[11] S. Jia, H. Chen, and H. Xie, *A posteriori error estimator for eigenvalue problems by mixed finite element method*, Sci. China Math. **56** (2013), 887–900.
[12] J. Douglas Jr. and J. E. Roberts, *Global estimates for mixed methods for second order elliptic equations*, Math. Comp. **44** (1985), 39–52.
[13] Q. Lin and H. Xie, *A superconvergence result for mixed finite element approximations of the eigenvalue problem*, ESAIM: M2AN **46** (2012), 797–812.
[14] B. Mercier, J. Osborn, J. Rappaz, and P. A. Raviart, *Eigenvalue approximation by mixed and hybrid methods*, Math. Comp. **36** (1981), 427–453.
[15] A. Naga and Z. Zhang, *Function value recovery and its application in eigenvalue problems*, SIAM J. Numer. Anal. **50** (2012), 272–286.
[16] M. R. Racheva and A. B. Andreev, *Superconvergence postprocessing for eigenvalues*, Comp. Methods Appl. Math. **2** (2002), 171–185.
[17] J. Xu and A. Zhou, *A two-grid discretization scheme for eigenvalue problems*, Math. Comp. **70** (2001), 17–25.

Kwang-Yeon Kim
Department of Mathematics
Kangwon National University
Chun-Cheon 24341, Korea
E-mail: `eulerkim@kangwon.ac.kr`