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MAGNIFYING ELEMENTS IN A SEMIGROUP OF TRANSFORMATIONS PRESERVING EQUIVALENCE RELATION

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Abstract. Let X be a nonempty set, ρ be an equivalence on X, $T(X)$ be the semigroup of all transformations from X into itself, and $T_\rho(X) = \{f \in T(X) \mid (x, y) \in \rho \text{ implies } ((x)f, (y)f) \in \rho\}$. In this paper, we investigate some necessary and sufficient conditions for elements in $T_\rho(X)$ to be left or right magnifying.

1. Introduction and Preliminary

In recent years, there has been an increasing interest in studying magnifying elements or magnifiers of semigroups. It started in 1963 when Ljapin introduced in [9] the notions of left and right magnifying elements of a semigroup. An element a of a semigroup S is said to be left [right] magnifying if there exists a proper subset M of S such that $S = aM$ [$S = Ma$]. Migliorini published many interesting results in [12, 13]. Among others, he introduced the notion of minimal subset relative to magnifying elements and verified the theorem of reduction in [12]. Catino and Migliorini [2] mainly dealt with the existence of

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strong magnifying elements in a semigroup and the existence of magnifying elements in simple and bisimple semigroups. They also concluded the necessary and sufficient conditions for any bisimple semigroup to contain left and right magnifying elements. Later, Gutan showed that there exist semigroups containing both strong and nonstrong magnifying elements in [6] and that every semigroup containing magnifying elements is factorizable in [7].

Let X be a nonempty set. The full transformation semigroup on X is the set $T(X)$ of all transformations from X into itself together with the function composition. Magill, Jr. examined in [11] magnifying elements in some particular transformation semigroups. Moreover, he applied his results to the semigroup of all linear transformations over a vector space and the semigroup of all continuous selfmaps of a topological space. Araujo and Konieczny considered in [1] the semigroup $T(X, \rho, R)$, where ρ is an equivalence relation on X and R is a cross-section of the partition X/ρ, and regular, abundant, inverse, and completely regular elements in this semigroup. In [14], Yonthanthum investigated regular elements of the variant semigroups of transformations preserving double direction equivalences. Recently, Chinram and Baupradist showed in [3] and [4] the necessary and sufficient conditions for elements in some generalized transformation semigroups to be left or right magnifying. Furthermore, Chinram, Petkaew and Baupradist studied in [5] left and right magnifying elements in generalized semigroups of transformations by using partitions of a set. Also, Kaewnoi, Petapirak and Chinram [8] have shown the necessary and sufficient conditions for elements in partial transformation semigroups which preserve an equivalence relation. Meanwhile, Luangchaisri, Changphas and Phanlert characterized in [10] left and right magnifying elements of a partial transformation semigroup.

In this paper, we use the notation $(x)f$ and $(x)fg$ instead of $f(x)$ and $(g \circ f)(x)$ for $f, g \in T(X)$ and $x \in X$, and denote the image of f by $ran\, f$. For an equivalence relation ρ on X, the equivalence class of $x \in X$ determined by ρ is denoted by $[x]_\rho$. It is well-known that

$$T_\rho(X) = \{f \in T(X) \mid (x, y) \in \rho \text{ implies } ((x)f, (y)f) \in \rho\}$$

is a subsemigroup of $T(X)$. In addition, if $\rho = X \times X$, then $T_\rho(X) = T(X)$.

In this paper, we study some necessary and sufficient conditions for elements in $T_\rho(X)$ to be right or left magnifying.

2. Right magnifying elements

LEMMA 2.1. *If f is a right magnifying element in $T_\rho(X)$, then f is onto.*

Proof. By assumption, there exists a proper subset M of $T_\rho(X)$ such that $Mf = T_\rho(X)$. Since the identity map id_X belongs to $T_\rho(X)$, there exists $h \in M$ such that $hf = id_X$. This implies that f is onto. □

LEMMA 2.2. *Let f be a right magnifying element in $T_\rho(X)$. For any $(x, y) \in \rho$, there exists $(a, b) \in \rho$ such that $x = (a)f, y = (b)f$.*

Proof. By assumption, there exists a proper subset M of $T_\rho(X)$ such that $Mf = T_\rho(X)$. Since $id_X \in T_\rho(X)$, there exists $h \in M$ such that $hf = id_X$. Let $x, y \in X$ be such that $(x, y) \in \rho$. Then $(x)hf = (x)id_X = x$ and $(y)hf = (y)id_X = y$. Since $h \in T_\rho(X)$, we have $((x)h, (y)h) \in \rho$ as well. Choose $a = (x)h$ and $b = (y)h$. Therefore, we obtain that $(a, b) \in \rho$ and $x = (a)f$, $y = (b)f$. □

LEMMA 2.3. *If $f \in T_\rho(X)$ is bijective, then f is not right magnifying.*

Proof. By Lemma 2.2, $f^{-1} \in T_\rho(X)$. Suppose to the contrary that f is right magnifying. Then there exists a proper subset M of $T_\rho(X)$ such that $Mf = T_\rho(X)$. This implies $Mf = T_\rho(X)f$. Then $M = Mff^{-1} = T_\rho(X)ff^{-1} = T_\rho(X)$, a contradiction. Hence f is not right magnifying. □

LEMMA 2.4. *If $f \in T_\rho(X)$ is onto but not one-to-one and for any $(x, y) \in \rho$, there exists $(a, b) \in \rho$ such that $x = (a)f, y = (b)f$, then f is right magnifying.*

Proof. Let $M = \{h \in T_\rho(X) \mid h \text{ is not onto}\}$. Then $M \neq T_\rho(X)$. Let g be any function in $T_\rho(X)$. Since f is onto, for each $x \in X$ there exists $a_x \in X$ such that $(a_x)f = (x)g$ (if $(x_1)g = (x_2)g$, we must choose $a_{x_1} = a_{x_2}$ and if $((x)g, (y)g) \in \rho$, we must choose $(a_x, a_y) \in \rho$). Define $h \in T(X)$ by $(x)h = a_x$ for all $x \in X$. To show that $h \in T_\rho(X)$, let $(x, y) \in \rho$. Thus $((x)g, (y)g) \in \rho$. By assumption, we obtain $(a_x, a_y) \in \rho$, which implies that $((x)h, (y)h) \in \rho$. Since f is not one-to-one, h is not onto. Hence $h \in M$. For all $x \in X$, we have

$$(x)hf = ((x)h)f = (a_x)f = (x)g.$$

Then $hf = g$, hence $Mf = T_\rho(X)$. Therefore, f is right magnifying.

□

Example 2.1. Let $X = \mathbb{N}$. Define a relation ρ on X by

$$(x, y) \in \rho \text{ if and only if } \lfloor \frac{x}{2} \rfloor = \lfloor \frac{y}{2} \rfloor.$$

Clearly, ρ is an equivalence relation on X and $X/\rho = \{\{1\}\} \cup \{\{x, x+1\} \mid x \in 2X\} = \{\{1\}, \{2,3\}, \{4,5\}, \ldots\}$. Let $f \in T_\rho(X)$ be defined by $(1)f = 1, (2)f = 2, (3)f = 3$ and $(x)f = x - 2$ for all positive integers $x > 3$, that is,

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ 1 & 2 & 3 & 2 & 3 & 4 & 5 & 6 & \cdots \end{pmatrix}.$$

Then f is onto but not one-to-one and for any $(x, y) \in \rho$, there exists $(a, b) \in \rho$ such that $x = (a)f, y = (b)f$. Let $M = \{h \in T_\rho(X) \mid h$ is not onto$\}$ and $g \in T_\rho(X)$ be any function. By Lemma 2.4, there exists $h \in M$ such that $hf = g$.

For example, let $g \in T_\rho(X)$ be defined by $(x)g = \lfloor \frac{x+2}{2} \rfloor$ for all $x \in X$, that is,

$$g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ 1 & 2 & 2 & 3 & 3 & 4 & 4 & 5 & \cdots \end{pmatrix}.$$

Define a function $h : X \to X$ by $(1)h = 1$ and $(2x)h = (2x+1)h = x+3$ for all $x \in X$, that is,

$$h = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ 1 & 4 & 4 & 5 & 5 & 6 & 6 & 7 & \cdots \end{pmatrix}.$$

So $h \in M$ and we have

$$\begin{aligned} hf &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ 1 & 4 & 4 & 5 & 5 & 6 & 6 & 7 & \cdots \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ 1 & 2 & 3 & 2 & 3 & 4 & 5 & 6 & \cdots \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ 1 & 2 & 2 & 3 & 3 & 4 & 4 & 5 & \cdots \end{pmatrix} = g. \end{aligned}$$

Example 2.2. Let $X = \mathbb{N}$. Define a relation ρ on X by

$$(x, y) \in \rho \text{ if and only if } x \equiv y \pmod 2.$$

Clearly, ρ is an equivalence relation on X and $X/\rho = \{\{x \in X \mid x$ is odd$\}, \{x \in X \mid x$ is even$\}\}$. Let $f \in T_\rho(X)$ be defined by $(1)f = 1, (2)f = 2$ and $(x)f = x - 2$ for all positive integer $x > 2$, that is,

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ 1 & 2 & 1 & 2 & 3 & 4 & 5 & 6 & \cdots \end{pmatrix}.$$

Then f is onto but not one-to-one and for any $(x, y) \in \rho$, there exists $(a, b) \in \rho$ such that $x = (a)f, y = (b)f$. Let $M = \{h \in T_\rho(X) \mid h$ is not onto$\}$ and $g \in T_\rho(X)$ be any function. By Lemma 2.4, there exists $h \in M$ such that $hf = g$.

For example, let $g \in T_\rho(X)$ be defined by $(x)g = x + 2$ for all $x \in X$, that is,

$$g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & \cdots \end{pmatrix}.$$

Define a function $h : X \to X$ by $(x)h = x + 4$ for all $x \in X$, that is,

$$h = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & \cdots \end{pmatrix}.$$

So $h \in M$ and we have

$$\begin{aligned} hf &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & \cdots \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ 1 & 2 & 1 & 2 & 3 & 4 & 5 & 6 & \cdots \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & \cdots \end{pmatrix} = g. \end{aligned}$$

THEOREM 2.5. *f is right magnifying in $T_\rho(X)$ if and only if f is onto but not one-to-one and for any $(x, y) \in \rho$, there exists $(a, b) \in \rho$ such that $x = (a)f, y = (b)f$.*

Proof. This follows by Lemmas 2.1 - 2.4. □

If $\rho = X \times X$, then the following corollary holds.

COROLLARY 2.6. *Let $f \in T(X)$. Then f is right magnifying in $T(X)$ if and only if f is onto but not one-to-one.*

Proof. This follows by Theorem 2.5. □

3. Left magnifying elements

LEMMA 3.1. *If f is a left magnifying element in $T_\rho(X)$, then f is one-to-one.*

Proof. By assumption, there exists a proper subset M of $T_\rho(X)$ such that $fM = T_\rho(X)$. Since $id_X \in T_\rho(X)$. Therefore there exists $h \in M$ such that $fh = id_X$. This implies that f is one-to-one. □

LEMMA 3.2. *Let f be a left magnifying element in $T_\rho(X)$. For any $x, y \in X$, if $((x)f, (y)f) \in \rho$, then $(x, y) \in \rho$.*

Proof. By assumption, there exists a proper subset M of $T_\rho(X)$ such that $fM = T_\rho(X)$. Since $id_X \in T_\rho(X)$, there exists $h \in M$ such that $fh = id_X$. Let $x, y \in X$ be such that $((x)f, (y)f) \in \rho$. Then we obtain $(x)fh = x$ and $(y)fh = y$, and $(x, y) = (((x)f)h, ((y)f)h) \in \rho$ because $h \in T_\rho(X)$. $\square$

LEMMA 3.3. *If $f \in T_\rho(X)$ is bijective, then f is not left magnifying.*

Proof. By Lemma 3.2, $f^{-1} \in T_\rho(X)$. Suppose to the contrary that f is left magnifying. Then there exists a proper subset M of $T_\rho(X)$ such that $fM = T_\rho(X)$. This implies that $fM = fT_\rho(X)$. Then $M = f^{-1}fM = f^{-1}fT_\rho(X) = T_\rho(X)$, a contradiction. Hence f is not left magnifying. $\square$

LEMMA 3.4. *If f is one-to-one but not onto and for any $x, y \in X$, $((x)f, (y)f) \in \rho$ implies that $(x, y) \in \rho$, then f is a left magnifying element in $T_\rho(X)$.*

Proof. **Case 1:** $|[x]_\rho| = 1$ for all $x \in X$.
Let $x_0 \in X$ and $M = \{h \in T_\rho(X) \mid (x)h = x_0 \text{ for all } x \notin ran\, f\}$. Claim that $fM = T_\rho(X)$. Let $g \in T_\rho(X)$. Define $h \in T_\rho(X)$ for $x \in X$ by

$$(x)h = \begin{cases} (x')g & \text{if } x \in ran\, f \text{ and } (x')f = x, \\ x_0 & \text{if } x \notin ran\, f. \end{cases}$$

Then $h \in M$, and for each $x \in X$ we obtain

$$(x)fh = ((x)f)h = (x)g.$$

This shows that $fh = g$, which implies $fM = T_\rho(X)$. Hence f is a left magnifying element in $T_\rho(X)$.

Case 2: $|[x]_\rho| > 1$ for some $x \in X$.
For each $x \in X$, let $a_x \in [x]_\rho$ (if $(x, y) \in \rho$, we must choose $a_x = a_y$). Let $I = \{a_x \mid x \in X\}$. Then $I \neq X$. Let $M = \{h \in T_\rho(X) \mid (x)h \in I \text{ for all } x \notin ran\, f\}$. Next, we will show that $fM = T_\rho(X)$. Let $g \in T_\rho(X)$ and define $h \in T_\rho(X)$ for $x \in X$ by

$$(x)h = \begin{cases} (x')g & \text{if } x \in ran\, f \text{ and } (x')f = x, \\ a_{(x')g} & \text{if } x \notin ran\, f \text{ and } \exists x' \in X \text{ such that } (x, (x')f) \in \rho, \\ a_x & \text{otherwise.} \end{cases}$$

Then $h \in M$, and for each $x \in X$ we obtain

$$(x)fh = ((x)f)h = (x)g.$$

This shows that $fh = g$, which implies $fM = T_\rho(X)$. Hence f is a left magnifying element in $T_\rho(X)$. $\square$

EXAMPLE 3.1. Let $X = \mathbb{N}$. Define a relation ρ on X by

$$(x, y) \in \rho \text{ if and only if } x = y.$$

Let $f \in T_\rho(X)$ be defined by $(x)f = 2x$ for all positive integer x, that is,

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ 2 & 4 & 6 & 8 & 10 & 12 & 14 & 16 & \cdots \end{pmatrix}.$$

Then f is one-to-one but not onto, and for any $x, y \in X$, $((x)f, (y)f) \in \rho$ implies $(x, y) \in \rho$. Let $M = \{h \in T_\rho(X) \mid (2x+1)h = 2 \text{ for all } x \in X\}$ and $g \in T_\rho(X)$ be any function. By Lemma 3.4, there exists $h \in M$ such that $fh = g$.

For example, let $g \in T_\rho(X)$ be defined by $(x)g = 4x$ for all $x \in X$, that is,

$$g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ 4 & 8 & 12 & 16 & 20 & 24 & 28 & 32 & \cdots \end{pmatrix}.$$

Define a function $h \in T_\rho(X)$ by $(2x)h = 4x$ and $(2x-1)h = 2$ for all $x \in X$, that is,

$$h = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ 2 & 4 & 2 & 8 & 2 & 12 & 2 & 16 & \cdots \end{pmatrix}.$$

So $h \in M$ and we have

$$\begin{aligned} fh &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ 2 & 4 & 6 & 8 & 10 & 12 & 14 & 16 & \cdots \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ 2 & 4 & 2 & 8 & 2 & 12 & 2 & 16 & \cdots \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ 4 & 8 & 12 & 16 & 20 & 24 & 28 & 32 & \cdots \end{pmatrix} = g. \end{aligned}$$

EXAMPLE 3.2. Let $X = \mathbb{N}$. Define a relation ρ on X by

$$(x, y) \in \rho \text{ if and only if } x \equiv y \pmod 2.$$

Clearly, ρ is an equivalence relation on X and $X/\rho = \{\{x \in X \mid x \text{ is odd}\}, \{x \in X \mid x \text{ is even}\}\}$. Let $f \in T_\rho(X)$ be defined by $(x)f = x + 2$ for all positive integer x, that is,

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & \cdots \end{pmatrix}.$$

Then f is one-to-one but not onto, and for any $x, y \in X$, $((x)f, (y)f) \in \rho$ implies $(x, y) \in \rho$. Let $M = \{h \in T_\rho(X) \mid (1)h, (2)h \in \{1, 2\}\}$ and

$g \in T_\rho(X)$ be any function. By Lemma 3.4, there exists $h \in M$ such that $fh = g$.

For example, let $g \in T_\rho(X)$ be defined by $(x)g = x + 1$ for all $x \in X$, that is,

$$g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \cdots \end{pmatrix}.$$

Define a function $h \in T_\rho(X)$ by $(1)h = 2, (2)h = 1$ and $(x)h = x - 1$ if $x > 2$, that is,

$$h = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ 2 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \cdots \end{pmatrix}.$$

So $h \in M$ and we have

$$\begin{aligned} fh &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & \cdots \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ 2 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \cdots \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \cdots \end{pmatrix} = g. \end{aligned}$$

Theorem 3.5. *f is left magnifying in $T_\rho(X)$ if and only if f is one-to-one but not onto and for any $x, y \in X$, $((x)f, (y)f) \in \rho$ implies $(x, y) \in \rho$.*

Proof. This follows by Lemmas 3.1 - 3.4. □

If $\rho = X \times X$, then the following corollary holds.

Corollary 3.6. *Let $f \in T(X)$. Then f is a left magnifying element in $T(X)$ if and only if f is one-to-one but not onto.*

Proof. This follows by Theorem 3.5. □

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