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SOME WEIGHTED APPROXIMATION PROPERTIES OF NONLINEAR DOUBLE INTEGRAL OPERATORS

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Abstract. In this paper, we present some recent results on weighted pointwise convergence and the rate of pointwise convergence for the family of nonlinear double singular integral operators in the following form:

$$T_\eta (f;x,y) = \iint\limits_{\mathbb{R}^2} K_\eta (t-x, s-y, f(t,s))\, dsdt, \ \ (x,y)\in \mathbb{R}^2, \ \eta \in \Lambda,$$

where the function $f:\mathbb{R}^2 \to \mathbb{R}$ is Lebesgue measurable on $\mathbb{R}^2$ and Λ is a non-empty set of indices. Further, we provide an example to support these theoretical results.

1. Introduction

Approximation by integral operators or the functions having good properties has broad applications in miscellaneous areas of mathematics including Fourier theory, convergence of orthogonal series, theory of differential equations and harmonic analysis theory. Therefore, integral operators have various applications in many academic disciplines such as physics, engineering and medicine. The following integral operator is

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a well-known one:

$$L_n(f;y)=\int_{-\pi}^{\pi} f(s)\,K_n(s,y)\,ds,\quad y\in[-\pi,\pi],\quad n\in\mathbb{N}, \tag{1.1}$$

where $K_n:\mathbb{R}\times\mathbb{R}\to\mathbb{R}_0^+$ denotes a $2\pi-$periodic kernel as a function of s satisfying some conditions similar to well-known approximate identities. In fact, magnetic resonance imaging, face recognition and computer aided geometric design are some of the application areas in which the integral operators are used. There are famous integral operators which are named after their constructers, such as Gauss-Weierstrass, Riesz-Bessel, Jackson-Stechkin, Fejer, Picard and Calderon-Zygmund singular integral operators.

In [17], Taberski, who mentioned the importance of singular integrals in Fourier analysis in his works, handled the pointwise approximation of $2\pi-$periodic functions which are integrable on $\langle -\pi,\pi\rangle$, where $\langle -\pi,\pi\rangle$ is an arbitrary closed, semi-closed or open interval. The mentioned study used a two-parameter family of convolution-type singular integral operators in the following form:

$$L_\lambda(f;y)=\int_{-\pi}^{\pi} f(s)\,K_\lambda(s-y)\,ds,\quad y\in\langle -\pi,\pi\rangle,\quad \lambda\in\Lambda, \tag{1.2}$$

where $K_\lambda:\mathbb{R}\to\mathbb{R}_0^+$ denotes a $2\pi-$periodic kernel function satisfying suitable conditions and Λ is a non-empty set of numbers with accumulation point λ_0.

Taberski [18] advanced his previous analysis by using the double singular integral operators of the form:

$$L_\lambda(f;x,y)=\iint_Q f(t,s)\,K_\lambda(t-x,s-y)\,dsdt,\quad (x,y)\in Q, \tag{1.3}$$

where $Q=\langle -\pi,\pi\rangle\times\langle -\pi,\pi\rangle$ is an arbitrary closed, semi-closed or open rectangular region, $K_\lambda:\mathbb{R}^2\to\mathbb{R}_0^+$ stands for a kernel function comprising appropriate features and $\lambda\in\Lambda$, Λ is a non-empty set of numbers with accumulation point λ_0. Those results were later used by Siudut [14] presenting considerable theorems. It should be noted that, Rydzewska [13] also improved her previous analysis [12] by using the results of [18] and obtained the rate of convergence of the operators of type (1.3) at $\mu-$generalized $d-$point of integrable functions which are $2\pi-$periodic with respect to each variables, separately.

Later on, Musielak [10] used the Lipschitz condition for $K_\lambda : G \times \mathbb{R} \to \mathbb{R}$ for the first time to prove some theorems for nonlinear integral operators of the form:

$$T_\lambda(f;s) = \int_G K_\lambda(t-s, f(t))dt, \quad s \in G, \quad \lambda \in \Lambda, \tag{1.4}$$

where G is a locally compact Abelian group equipped with Haar measure and Λ is a non-empty index set with a topology. Therefore, well-known solution technics are used in nonlinear problems by the aid of indicated Lipschitz condition. Afterwards, Swiderski and Wachnicki [16] investigated the pointwise convergence of two-parameter setting of integrals of type (1.4) to the functions which are integrable on some locally compact Abelian groups with Haar measure.

Nowadays, the usage of nonlinear integral operators in sampling theory is very common. On the other hand, signal and image processing are two main research fields around sampling theory. We recommend the reader to see the book by Bardaro et al. [3] for further studies regarding the convergence of nonlinear integral operators and sampling type operators considered in some function spaces. As concerns the study of integral operators in various settings, the reader may see also [1–6, 8–10, 12–21].

One may also consider the pointwise approximation by singular integral operators in weighted Lebesgue spaces like in usual Lebesgue spaces. Therefore, we shall mention the papers [1], [9] and [19] which are important works on weighted approximation by singular integral operators. Those contain detailed information regarding the characterization of the weight functions with examples.

As a continuation of [20,21], the aim of this paper is to investigate the pointwise convergence and the rate of pointwise convergence of nonlinear double singular integral operators in the following form:

$$T_\eta(f;x,y) = \iint_{\mathbb{R}^2} K_\eta(t-x, s-y, f(t,s))\, dsdt, \quad (x,y) \in \mathbb{R}^2, \quad \eta \in \Lambda, \tag{1.5}$$

where Λ is a non-empty set of indices with a topology and $K_\eta : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ is a kernel function. Also, $f \in L_{p,\varphi}(\mathbb{R}^2)$, where the symbol $L_{p,\varphi}(\mathbb{R}^2)$ denotes the space of all measurable functions f for which $\left|\frac{f(t,s)}{\varphi(t,s)}\right|^p$ $(1 \le p < \infty)$ is integrable and $\varphi : \mathbb{R}^2 \to \mathbb{R}^+$ is a locally bounded weight (positive and measurable) function. The norm formula for the space $L_{p,\varphi}(\mathbb{R}^2)$ (see, e.g., [9,15]) is given by

$$\|f\|_{L_{p,\varphi}(\mathbb{R}^2)} = \left(\iint\limits_{\mathbb{R}^2} \left| \frac{f(t,s)}{\varphi(t,s)} \right|^p dsdt \right)^{\frac{1}{p}}, \quad 1 \le p < \infty.$$

The paper is organized as follows: In Section 2, we give some preliminary concepts. In Section 3, the existence of the operators of type (1.5) is explored. In Section 4, main result is presented. In Section 5, the rate of pointwise convergence of the operators of type (1.5) is established. In Section 6, we summarize the results of this paper.

2. Preliminaries

Now, we give main definitions used in the manuscript.

Definition 2.1. Let δ_1 be a fixed positive real number such that $0 < h, k < \delta_1$. A $p-\mu-$generalized Lebesgue point of a locally $p-$integrable function $g : \mathbb{R}^2 \to \mathbb{R}$ *is a point* $(x_0, y_0) \in \mathbb{R}^2$ satisfying

(2.1)
$$\lim_{(h,k)\to(0,0)} \left(\frac{1}{\mu(h,k)} \int_0^h \int_0^k |g(x_0 \pm t, y_0 \pm s) - g(x_0, y_0)|^p \, dsdt \right)^{\frac{1}{p}} = 0, \; 1 \le p < \infty,$$

where $\mu : \mathbb{R}^2 \to \mathbb{R}$ is absolutely continuous in two-dimensional sense on $[0, \delta_1] \times [0, \delta_1]$ (see [13]), increasing with respect to each variables separately on $[0, \delta_1]$ and $\mu(t,s) = 0$ whenever $ts = 0$.

Remark 2.2. The given definition is an updated version of the $\mu-$generalized $d-$ point definition given in [13] depending on the requirements of the new problem (see also [6]).

Definition 2.3. (*Class* A_φ) Let $\Lambda \subset \mathbb{R}_0^+$ be a non-empty set of indices. We allow the symbol η_0 to be either accumulation point of Λ or ∞. Let $\varphi > 0$ be a locally bounded weight function defined on $\mathbb{R}^2$, that is to say φ is bounded on arbitrary bounded subsets of $\mathbb{R}^2$, such that the following inequality:

$$\varphi(t+x,s+y) \leq \varphi(t,s)\varphi(x,y) \tag{2.2}$$

holds for every $(t,s) \in \mathbb{R}^2$ and $(x,y) \in \mathbb{R}^2$.

A family $(K_\eta)_{\eta\in\Lambda}$ consisting of the functions $K_\eta : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ is called *Class* A_φ, if the following conditions hold:

(a) $K_\eta(t,s,0) = 0$ for every $(t,s) \in \mathbb{R}^2$ and for each $\eta \in \Lambda$, and $K_\eta(.,.,u) \in L_1(\mathbb{R}^2)$ for all values of $u \in \mathbb{R}$ and for any $\eta \in \Lambda$.

(b) There exists a family $(L_\eta)_{\eta\in\Lambda}$ consisting of the (globally) integrable functions $L_\eta \geq 0$ defined on $\mathbb{R}^2$ such that the Lipschitz inequality:

$$|K_\eta(t,s,u) - K_\eta(t,s,v)| \leq L_\eta(t,s)\,|u-v|$$

holds for every $(t,s) \in \mathbb{R}^2$, $u,v \in \mathbb{R}$, and for any $\eta \in \Lambda$.

(c) $\lim\limits_{(x,y,\eta)\to(x_0,y_0,\eta_0)} \left| \iint_{\mathbb{R}^2} K_\eta\left(t-x, s-y, \frac{u}{\varphi(x_0,y_0)}\varphi(t,s)\right) dsdt - u \right| = 0$ for every $u \in \mathbb{R}$ and for any $(x_0,y_0) \in \mathbb{R}^2$.

(d) $\lim_{\eta\to\eta_0} \left[\sup_{\xi\leq\sqrt{t^2+s^2}} \left[\varphi(t,s)L_\eta(t,s)\right]\right] = 0$ for every $\xi > 0$.

(e) $\lim_{\eta\to\eta_0} \left[\iint_{\xi\leq\sqrt{t^2+s^2}} \varphi(t,s)L_\eta(t,s)\,dsdt\right] = 0$ for every $\xi > 0$.

(f) $\|\varphi L_\eta\|_{L_1(\mathbb{R}^2)} \leq M < \infty$ for every $\eta \in \Lambda$ (value of M is independent of η).

(g) For a given positive real number δ_0 satisfying $\delta_0 \geq \delta_1 > 0$, L_η is monotonically increasing on $(-\delta_0, 0]$ and monotonically decreasing on $[0,\delta_0)$ with respect to t and similarly, L_η is monotonically increasing on $(-\delta_0, 0]$ and monotonically decreasing on $[0,\delta_0)$ with respect to s, for any $\eta \in \Lambda$. Analogously, L_η is bimonotonically increasing with respect to (t,s) on $[0,\delta_0) \times [0,\delta_0)$ and $(-\delta_0,0] \times (-\delta_0,0]$ and similarly, L_η is bimonotonically decreasing with respect to (t,s) on $[0,\delta_0) \times (-\delta_0,0]$ and $(-\delta_0,0] \times [0,\delta_0)$ for any $\eta \in \Lambda$.

Throughout this paper the kernel function K_η belongs to *Class* A_φ.

Remark 2.4. For the motivation of above definition which is used except condition (g) in also [20], we refer the reader to see [4, 9, 19].

REMARK 2.5. If the function $g:\mathbb{R}^2\to\mathbb{R}$ is bimonotonic on $[\alpha_1,\alpha_2]\times[\beta_1,\beta_2]\subset\mathbb{R}^2$ then the equality given as

$$\begin{aligned} V(g;[\alpha_1,\alpha_2]\times[\beta_1,\beta_2]) &= \bigvee_{\alpha_1}^{\alpha_2}\bigvee_{\beta_1}^{\beta_2}(g(t,s)) \\ &= |g(\alpha_1,\beta_1)-g(\alpha_1,\beta_2)-g(\alpha_2,\beta_1)+g(\alpha_2,\beta_2)| \end{aligned}$$

holds [7,18]. Here, V, which is given as $V(g;[\alpha_1,\alpha_2]\times[\beta_1,\beta_2])=\bigvee_{\alpha_1}^{\alpha_2}\bigvee_{\beta_1}^{\beta_2}(g(t,s))$, denotes bivariation of g on $[\alpha_1,\alpha_2]\times[\beta_1,\beta_2]$.

3. Existence of the Operators

This section starts with the following theorem which gives the existence of the operators of type (1.5).

Main result in this work is based on the following theorem.

THEOREM 3.1. *If $f\in L_{p,\varphi}(\mathbb{R}^2)$, then $T_\eta(f)\in L_{p,\varphi}(\mathbb{R}^2)$ and*

$$\|T_\eta f\|_{L_{p,\varphi}(\mathbb{R}^2)}\le\|\varphi L_\eta\|_{L_1(\mathbb{R}^2)}\|f\|_{L_{p,\varphi}(\mathbb{R}^2)},$$

holds for every $\eta\in\Lambda$.

Proof. Let $p=1$. Using the norm formula for the space $L_{1,\varphi}(\mathbb{R}^2)$ (see, e.g., [19]), we have

$$\|T_\eta(f;x,y)\|_{L_{1,\varphi}(\mathbb{R}^2)}=\iint\limits_{\mathbb{R}^2}\frac{1}{\varphi(x,y)}\left|\iint\limits_{\mathbb{R}^2}K_\eta(t-x,s-y,f(t,s))\,dsdt\right|dydx.$$

Using conditions (a) and (b), we have

$$\|T_\eta(f;x,y)\|_{L_{1,\varphi}(\mathbb{R}^2)}\le\iint\limits_{\mathbb{R}^2}\frac{1}{\varphi(x,y)}\left(\iint\limits_{\mathbb{R}^2}|f(t+x,s+y)|\,L_\eta(t,s)\,dsdt\right)dydx.$$

Now, applying inequality (2.2) and Fubini's theorem (see, e.g., [5]) to the last inequality, we obtain

$$
\begin{aligned}
&\|T_\eta(f;x,y)\|_{L_{1,\varphi}(\mathbb{R}^2)} \\
&\leq \iint\limits_{\mathbb{R}^2} \frac{1}{\varphi(x,y)} \left(\iint\limits_{\mathbb{R}^2} |f(t+x,s+y)|\, L_\eta\,(t,s)\, dsdt \right) dydx \\
&= \iint\limits_{\mathbb{R}^2} L_\eta\,(t,s) \left(\iint\limits_{\mathbb{R}^2} \left| \frac{f(t+x,s+y)}{\varphi(t+x,s+y)} \right| \frac{\varphi(x+t,s+y)}{\varphi(x,y)} dydx \right) dsdt \\
&\leq \iint\limits_{\mathbb{R}^2} L_\eta\,(t,s) \left(\iint\limits_{\mathbb{R}^2} \left| \frac{f(t+x,s+y)}{\varphi(t+x,s+y)} \right| \frac{\varphi(x,y)\varphi(t,s)}{\varphi(x,y)} dydx \right) dsdt \\
&= \|\varphi L_\eta\|_{L_1(\mathbb{R}^2)} \|f\|_{L_{1,\varphi}(\mathbb{R}^2)}\,.
\end{aligned}
$$

In view of condition (f), the assertion follows. Thus, the proof is completed for this case.

Let $1 < p < \infty$. Using the norm formula for the space $L_{p,\varphi}\,(\mathbb{R}^2)$ (see, e.g., [15]), we have

$$
\begin{aligned}
&\|T_\eta\,(f;x,y)\|_{L_{p,\varphi}(\mathbb{R}^2)} \\
&= \left(\iint\limits_{\mathbb{R}^2} \frac{1}{[\varphi\,(x,y)]^p} \left| \iint\limits_{\mathbb{R}^2} K_\eta\,(t-x,s-y,f(t,s))\, dsdt \right|^p dydx \right)^{\frac{1}{p}}.
\end{aligned}
$$

Using conditions (a) and (b), we have

$$
\begin{aligned}
&\|T_\eta(f;x,y)\|_{L_{p,\varphi}(\mathbb{R}^2)} \\
&\leq \left(\iint\limits_{\mathbb{R}^2} \frac{1}{[\varphi\,(x,y)]^p} \left(\iint\limits_{\mathbb{R}^2} |f(t+x,s+y)|\, L_\eta\,(t,s)\, dsdt \right)^p dydx \right)^{\frac{1}{p}}.
\end{aligned}
$$

Now, applying inequality (2.2), and generalized Minkowski inequality (see, e.g., [15]) to the last inequality, we obtain the desired result, that is

$$
\|T_\eta\|_{L_{p,\varphi}(\mathbb{R}^2)} \leq \|\varphi L_\eta\|_{L_1(\mathbb{R}^2)} \|f\|_{L_{p,\varphi}(\mathbb{R}^2)}\,.
$$

Hence, the assertion follows from condition (f). Thus the proof is completed. □

4. Convergence at Characteristic Points

THEOREM 4.1. *If* $(x_0, y_0) \in \mathbb{R}^2$ *is a common* $p-\mu-$*generalized Lebesgue point of functions* $f \in L_{p,\varphi}(\mathbb{R}^2)$ $(1 \le p < \infty)$ *and* φ, *then*

$$\lim_{(x,y,\eta)\to(x_0,y_0,\eta_0)} |T_\eta(f;x,y) - f(x_0,y_0)| = 0$$

on any set Z *on which the function* $\Delta(x,y,\eta,\delta)$ *defined by*

$$\begin{aligned}\Delta(x,y,\eta,\delta) \ : \ &= \int_{x_0-\delta}^{x_0+\delta}\int_{y_0-\delta}^{y_0+\delta} L_\eta(t-x, s-y)\,|d_t d_s \mu(|t-x_0|, |s-y_0|)| \\ &+2\int_{x_0-\delta}^{x_0+\delta} L_\eta(t-x,0)\,|d_t\mu(|t-x_0|, |y-y_0|)| \\ &+2\int_{y_0-\delta}^{y_0+\delta} L_\eta(0,s-y)\,|d_s\mu(|x-x_0|, |s-y_0|)| \\ &+4L_\eta(0,0)\,\mu(|x-x_0|, |y-y_0|),\end{aligned}$$

where $0 < \delta < \delta_1$, *is bounded as* (x,y,η) *tends to* (x_0,y_0,η_0).

Here, $|d_t d_s \mu(|t-x_0|, |s-y_0|)|$, $|d_t\mu(|t-x_0|, |y-y_0|)|$ and $|d_s\mu(|x-x_0|, |s-y_0|)|$ denote Lebesgue-Stieltjes measures.

Proof. The proof of theorem will be given for the case $1 < p < \infty$. The proof for the case $p = 1$ is similar and it is skipped.

Now, set $I(x,y,\eta) := \left|T_\eta\left(f;x,y\right) - f\left(x_0,y_0\right)\right|$. Using condition (c), we obtain

$$\begin{aligned}
I(x,y,\eta) &= \left|\iint\limits_{\mathbb{R}^2} K_\eta\left(t-x,s-y,f(t,s)\right)dsdt - f(x_0,y_0)\right| \\
&= \left|\iint\limits_{\mathbb{R}^2} K_\eta\left(t-x,s-y,f(t,s)\right)dsdt\right. \\
&\quad - \iint\limits_{\mathbb{R}^2} K_\eta\left(t-x,s-y,\frac{f(x_0,y_0)}{\varphi(x_0,y_0)}\varphi(t,s)\right)dsdt \\
&\quad \left. + \iint\limits_{\mathbb{R}^2} K_\eta\left(t-x,s-y,\frac{f(x_0,y_0)}{\varphi(x_0,y_0)}\varphi(t,s)\right)dsdt - f(x_0,y_0)\right|.
\end{aligned}$$

Using condition (b), it is easy to see that the following inequality holds:

$$\begin{aligned}
I(x,y,\eta) &\le \iint\limits_{\mathbb{R}^2} \left|\frac{f\left(t,s\right)}{\varphi(t,s)} - \frac{f\left(x_0,y_0\right)}{\varphi\left(x_0,y_0\right)}\right| \varphi(t,s) L_\eta\left(t-x,s-y\right)dsdt \\
&\quad + \left|\iint\limits_{\mathbb{R}^2} K_\eta\left(t-x,s-y,\frac{f(x_0,y_0)}{\varphi(x_0,y_0)}\varphi(t,s)\right)dsdt - f(x_0,y_0)\right|.
\end{aligned}$$

Since the inequality $(m+n)^p \le 2^p(m^p+n^p)$ holds provided that m and n are positive numbers (see, e.g., [11]), we have

$$\begin{aligned}
\left[I(x,y,\eta)\right]^p &\le 2^p\left(\iint\limits_{\mathbb{R}^2} \left|\frac{f\left(t,s\right)}{\varphi\left(t,s\right)} - \frac{f\left(x_0,y_0\right)}{\varphi\left(x_0,y_0\right)}\right| \varphi\left(t,s\right) L_\eta\left(t-x,s-y\right)dsdt\right)^p \\
&\quad + 2^p\left|\iint\limits_{\mathbb{R}^2} K_\eta\left(t-x,s-y,\frac{f(x_0,y_0)}{\varphi(x_0,y_0)}\varphi(t,s)\right)dsdt - f(x_0,y_0)\right|^p.
\end{aligned}$$

Now, applying Hölder's inequality (see, e.g., [11]), to the first integral of the resulting inequality, we have

$$[I(x,y,\eta)]^p$$
$$\leq 2^p \rho(x,y,\eta) \iint\limits_{\mathbb{R}^2} \left| \frac{f(t,s)}{\varphi(t,s)} - \frac{f(x_0,y_0)}{\varphi(x_0,y_0)} \right|^p \varphi(t,s) L_\eta(t-x,s-y)\, dsdt$$
$$+ 2^p \left| \iint\limits_{\mathbb{R}^2} K_\eta \left(t-x, s-y, \frac{f(x_0,y_0)}{\varphi(x_0,y_0)} \varphi(t,s) \right) dsdt - f(x_0,y_0) \right|^p,$$

where

$$\rho(x,y,\eta) = \left(\iint\limits_{\mathbb{R}^2} \varphi(t,s) L_\eta(t-x,s-y)\, dsdt \right)^{\frac{p}{q}}.$$

Let $(x_0,y_0) \in \mathbb{R}^2$ be a common $p-\mu-$generalized Lebesgue point of the functions $f \in L_{p,\varphi}(\mathbb{R}^2)$ and φ. In view of one of the limit relations (2.1), for every $\varepsilon > 0$, there exists $\delta > 0$, we have the following inequality:

$$\int\limits_{x_0}^{x_0+h} \int\limits_{y_0-k}^{y_0} \left| \frac{f(t,s)}{\varphi(t,s)} - \frac{f(x_0,y_0)}{\varphi(x_0,y_0)} \right|^p dsdt < \varepsilon^p \mu(h,k) \tag{4.1}$$

for every h and k satisfying $0 < h, k \leq \delta < \delta_1$. Let $0 < x_0 - x < \frac{\delta}{2}$ and $0 < y_0 - y < \frac{\delta}{2}$. Moreover, the following inequality holds:

$$[I(x,y,\eta)]^p$$
$$\leq 2^p \rho(x,y,\eta) \iint\limits_{\mathbb{R}^2 \setminus B_\delta} \left| \frac{f(t,s)}{\varphi(t,s)} - \frac{f(x_0,y_0)}{\varphi(x_0,y_0)} \right|^p \varphi(t,s) L_\eta(t-x,s-y)\, dsdt$$
$$+ 2^p \rho(x,y,\eta) \iint\limits_{B_\delta} \left| \frac{f(t,s)}{\varphi(t,s)} - \frac{f(x_0,y_0)}{\varphi(x_0,y_0)} \right|^p \varphi(t,s) L_\eta(t-x,s-y)\, dsdt$$
$$+ 2^p \left| \iint\limits_{\mathbb{R}^2} K_\eta \left(t-x, s-y, \frac{f(x_0,y_0)}{\varphi(x_0,y_0)} \varphi(t,s) \right) dsdt - f(x_0,y_0) \right|^p,$$

where $B_\delta := \left\{ (t,s) \in \mathbb{R}^2 : (t-x_0)^2 + (s-y_0)^2 < \delta^2 \right\}$.

Now, applying the inequality given by $(m+n)^p \leq 2^p(m^p+n^p)$ once more to the first term of the above inequality, we obtain

$$\begin{aligned}
&[I(x,y,\eta)]^p \\
&\leq 2^{2p}\rho(x,y,\eta)\left|\frac{f(x_0,y_0)}{\varphi(x_0,y_0)}\right|^p \iint\limits_{\mathbb{R}^2\backslash B_\delta} \varphi(t,s)\, L_\eta(t-x,s-y)\, dsdt \\
&+ 2^{2p}\rho(x,y,\eta)\iint\limits_{\mathbb{R}^2\backslash B_\delta}\left|\frac{f(t,s)}{\varphi(t,s)}\right|^p \varphi(t,s)\, L_\eta(t-x,s-y)\, dsdt \\
&+ 2^{p}\rho(x,y,\eta)\iint\limits_{B_\delta}\left|\frac{f(t,s)}{\varphi(t,s)}-\frac{f(x_0,y_0)}{\varphi(x_0,y_0)}\right|^p \varphi(t,s)\, L_\eta(t-x,s-y)\, dsdt \\
&+ 2^p\left|\iint\limits_{\mathbb{R}^2} K_\eta\left(t-x,s-y,\frac{f(x_0,y_0)}{\varphi(x_0,y_0)}\varphi(t,s)\right) dsdt - f(x_0,y_0)\right|^p .
\end{aligned}$$

Now, we may define the following set as follows:

$$N_\delta = \left\{(x,y)\in\mathbb{R}^2 : (x-x_0)^2+(y-y_0)^2 < \frac{\delta^2}{4}\right\}.$$

Making comparison between the sets B_δ and N_δ gives the relation such that $\mathbb{R}^2\backslash B_\delta \subseteq \mathbb{R}^2\backslash A_\delta$, where

$$A_\delta = \left\{(t,s)\in B_\delta : (t-x)^2+(s-y)^2 < \frac{\delta^2}{4},\ \ (x,y)\in N_\delta\right\}.$$

In view of definition of A_δ and inequality (2.2), the following inequality holds:

$$\begin{aligned}
&[I(x,y,\eta)]^p \\
&\leq 2^{2p}\varphi(x,y)\,\rho(x,y,\eta)\left|\frac{f(x_0,y_0)}{\varphi(x_0,y_0)}\right|^p \iint\limits_{\mathbb{R}^2\backslash A_\delta} \varphi(t-x,s-y)\, L_\eta(t-x,s-y)\, dsdt \\
&\quad + 2^{2p}\varphi(x,y)\,\rho(x,y,\eta)\sup_{(t,s)\in\mathbb{R}^2\backslash A_\delta}\left[\varphi(t-x,s-y)\, L_\eta(t-x,s-y)\right]\|f\|^p_{L_{p,\varphi}(\mathbb{R}^2)}
\end{aligned}$$

$$+ 2^p\rho(x,y,\eta)\iint\limits_{B_\delta}\left|\frac{f(t,s)}{\varphi(t,s)}-\frac{f(x_0,y_0)}{\varphi(x_0,y_0)}\right|^p\varphi(t,s)\,L_\eta(t-x,s-y)\,dsdt$$

$$+ 2^p\left|\iint\limits_{\mathbb{R}^2}K_\eta\left(t-x,s-y,\frac{f(x_0,y_0)}{\varphi(x_0,y_0)}\varphi(t,s)\right)dsdt-f(x_0,y_0)\right|^p.$$

Rearranging and rewriting the last inequality, we obtain

$$\begin{aligned}
&[I(x,y,\eta)]^p\\
&\le 2^{2p}\varphi(x,y)\,\rho(x,y,\eta)\left|\frac{f(x_0,y_0)}{\varphi(x_0,y_0)}\right|^p\iint\limits_{\frac{\delta}{2}\le\sqrt{u^2+v^2}}\varphi(u,v)\,L_\eta(u,v)\,dvdu\\
&\quad+2^{2p}\varphi(x,y)\,\rho(x,y,\eta)\sup_{\frac{\delta}{2}\le\sqrt{u^2+v^2}}[\varphi(u,v)\,L_\eta(u,v)]\,\|f\|^p_{L_{p,\varphi}(\mathbb{R}^2)}\\
&\quad+2^p\rho(x,y,\eta)\iint\limits_{B_\delta}\left|\frac{f(t,s)}{\varphi(t,s)}-\frac{f(x_0,y_0)}{\varphi(x_0,y_0)}\right|^p\varphi(t,s)\,L_\eta(t-x,s-y)\,dsdt\\
&\quad+2^p\left|\iint\limits_{\mathbb{R}^2}K_\eta\left(t-x,s-y,\frac{f(x_0,y_0)}{\varphi(x_0,y_0)}\varphi(t,s)\right)dsdt-f(x_0,y_0)\right|^p\\
&=I_1(x,y,\eta)+I_2(x,y,\eta)+\rho(x,y,\eta)2^pI_3(x,y,\eta)+I_4(x,y,\eta).
\end{aligned}$$

In view of inequality (2.2) and change of variables, the following inequality holds:

$$\begin{aligned}
\rho(x,y,\eta)&=\left(\iint\limits_{\mathbb{R}^2}\varphi(t,s)\,L_\eta(t-x,s-y)\,dsdt\right)^{\frac{p}{q}}\\
&\le(\varphi(x,y))^{\frac{p}{q}}\left(\iint\limits_{\mathbb{R}^2}\varphi(t-x,s-y)\,L_\eta(t-x,s-y)\,dsdt\right)^{\frac{p}{q}}\\
&=(\varphi(x,y))^{\frac{p}{q}}\left(\iint\limits_{\mathbb{R}^2}\varphi(u,v)\,L_\eta(u,v)\,dvdu\right)^{\frac{p}{q}}.
\end{aligned}$$

Thus, boundedness of the term $\varphi(x,y)\,\rho(x,y,\eta)$ as (x,y,η) tends to (x_0,y_0,η_0) follows from condition (f) and boundedness of $\varphi(x,y)$ around (x_0,y_0). On the other hand, $I_4(x,y,\eta)\to 0$ as $(x,y,\eta)\to(x_0,y_0,\eta_0)$ by condition (c). Lastly, $I_1(x,y,\eta)\to 0$ and $I_2(x,y,\eta)\to 0$ as $(x,y,\eta)\to(x_0,y_0,\eta_0)$ by conditions (e) and (d), respectively. Also, for similar technic showing how the terms $I_1(x,y,\eta)$ and $I_2(x,y,\eta)$ tend to zero as $(x,y,\eta)\to(x_0,y_0,\eta_0)$, we refer the reader to [14].

Now, we may write the following inequality for the integral $I_3(x,y,\eta)$:

$$I_3(x,y,\eta)=\iint\limits_{B_\delta}\left|\frac{f(t,s)}{\varphi(t,s)}-\frac{f(x_0,y_0)}{\varphi(x_0,y_0)}\right|^p\varphi(t,s)\,L_\eta(t-x,s-y)\,dsdt$$

$$\le\sup_{(t,s)\in Q_\delta}\varphi(t,s)\iint\limits_{Q_\delta}\left|\frac{f(t,s)}{\varphi(t,s)}-\frac{f(x_0,y_0)}{\varphi(x_0,y_0)}\right|^p L_\eta(t-x,s-y)\,dsdt$$
$$=\sup_{(t,s)\in Q_\delta}\varphi(t,s)\,I_{31}(x,y,\eta),$$

where $Q_\delta=(x_0-\delta,x_0+\delta)\times(y_0-\delta,y_0+\delta)$.

It is easy to see that the following equality holds:

$$I_{31}=\left\{\int\limits_{x_0}^{x_0+\delta}\int\limits_{y_0-\delta}^{y_0}+\int\limits_{x_0-\delta}^{x_0}\int\limits_{y_0-\delta}^{y_0}\right\}\left|\frac{f(t,s)}{\varphi(t,s)}-\frac{f(x_0,y_0)}{\varphi(x_0,y_0)}\right|^p L_\eta(t-x,s-y)\,dsdt$$

$$+\left\{\int\limits_{x_0-\delta}^{x_0}\int\limits_{y_0}^{y_0+\delta}+\int\limits_{x_0}^{x_0+\delta}\int\limits_{y_0}^{y_0+\delta}\right\}\left|\frac{f(t,s)}{\varphi(t,s)}-\frac{f(x_0,y_0)}{\varphi(x_0,y_0)}\right|^p L_\eta(t-x,s-y)\,dsdt$$
$$=\ I_{311}+I_{312}+I_{313}+I_{314}.$$

Now, we consider I_{311}.

We define the new function such that

$$F(t,s):=\int\limits_{x_0}^{t}\int\limits_{s}^{y_0}\left|\frac{f(u,v)}{\varphi(u,v)}-\frac{f(x_0,y_0)}{\varphi(x_0,y_0)}\right|^p dvdu$$

for all t and s satisfying $0<t-x_0\le\delta$ and $0<y_0-s\le\delta$, in view of (4.1) one may easily observe that

$$|F(t,s)|\le\varepsilon^p\mu(t-x_0,y_0-s). \tag{4.2}$$

The Lebesgue-Stieltjes integral form of the integral I_{311} is as follows:

$$|I_{311}| = \left| \int_{x_0}^{x_0+\delta} \int_{y_0-\delta}^{y_0} L_\eta (t-x, s-y)\, d_t d_s \left[-F(t,s)\right] \right|.$$

Therefore, applying bivariate integration by parts method (see, e.g., [18]) to the integral I_{311}, we have

$$\begin{aligned} |I_{311}| \leq & \int_{x_0}^{x_0+\delta} \int_{y_0-\delta}^{y_0} |F(t,s)| \left| d_t d_s \left[L_\eta (t-x, s-y) \right] \right| \\ & + \int_{x_0}^{x_0+\delta} |F(t, y_0-\delta)| \left| d_t \left[L_\eta (t-x, y_0-\delta-y) \right] \right| \\ & + \int_{y_0-\delta}^{y_0} |F(x_0+\delta, s)| \left| d_s \left[L_\eta (x_0+\delta-x, s-y) \right] \right| \\ & + |F(x_0+\delta, y_0-\delta)| \, L_\eta (x_0+\delta-x, y_0-\delta-y). \end{aligned}$$

In view of inequality (4.2), we obtain

$$\begin{aligned} |I_{311}| \leq & \ \varepsilon^p \int_{x_0}^{x_0+\delta} \int_{y_0-\delta}^{y_0} \mu(t-x_0, y_0-s) \left| d_t d_s \left[L_\eta (t-x, s-y) \right] \right| \\ & + \varepsilon^p \int_{x_0}^{x_0+\delta} \mu(t-x_0, \delta) \left| d_t \left[L_\eta (t-x, y_0-\delta-y) \right] \right| \\ & + \varepsilon^p \int_{y_0-\delta}^{y_0} \mu_2(\delta, y_0-s) \left| d_s \left[L_\eta (x_0+\delta-x, s-y) \right] \right| \\ & + \varepsilon^p \mu(\delta,\delta) L_\eta (x_0+\delta-x, y_0-\delta-y). \end{aligned}$$

Let us give the bivariation and single variations as follows:

$$A_1(t,s) := \begin{cases} \bigvee_{t}^{x_0+\delta-x} \bigvee_{y_0-\delta-y}^{s} L_\eta(u,v), & \begin{array}{l} x_0-x \leq t < x_0+\delta-x, \\ y_0-\delta-y < s \leq y_0-y, \end{array} \\ 0, & \textit{otherwise}, \end{cases}$$

$$A_2(t) := \begin{cases} \bigvee_{t}^{x_0+\delta-x} L_\eta(u, y_0-\delta-y), & x_0 - x \le t < x_0+\delta-x, \\ 0, & otherwise, \end{cases}$$

$$A_3(s) := \begin{cases} \bigvee_{y_0-\delta-y}^{s} L_\eta(x_0+\delta-x, v), & y_0-\delta-y < s \le y_0-y, \\ 0, & otherwise. \end{cases}$$

Taking above variations and Remark 2.5 into account and applying bivariate integration by parts to last inequality for I_{311}, we have

$$|I_{311}| \le \left| -\varepsilon^p \int_{x_0-x}^{x_0-x+\delta} \int_{y_0-y-\delta}^{y_0-y} [A_1(t,s) + A_2(t) + A_3(s) + \right.$$
$$\left. + L_\eta(x_0+\delta-x, y_0-\delta-y)] \, d_t d_s \mu(t-x_0+x, y_0-s-y) \right|.$$

For the similar situation, the reader may see also [13, 18].

Clearly, using Remark 2.5 and condition (g), the following inequality holds for I_{311} :

$$|I_{311}| \le \varepsilon^p \int_{x_0}^{x_0+\delta} \int_{y_0-\delta}^{y_0} L_\eta(t-x, s-y) \left| d_t d_s \mu(|t-x_0|, |s-y_0|) \right|$$
$$+2\varepsilon^p \int_{x_0}^{x_0+\delta} L_\eta(t-x, 0) \, d_t \mu(|t-x_0|, |y-y_0|).$$

Evaluating the integrals I_{312}, I_{313}, I_{314} with the same method and combining the respective inequalities, we obtain

$$I_{31} \le \varepsilon^p \Delta(x, y, \eta, \delta).$$

The remaining part of the proof is obvious by the hyphothesis. Hence, I_{31} tends to 0 as (x, y, η) tends to (x_0, y_0, η_0). Note that, if we reverse each of the assumptions what we supposed at the beginning of the proof, such as $0 < x - x_0 < \frac{\delta}{2}$ and $0 < y_0 - y < \frac{\delta}{2}$, then we obtain the same inequality. Thus, the proof is completed. □

5. Rate of Pointwise Convergence

Theorem 5.1. *Suppose that the hypotheses of Theorem 4.1 are satisfied and the following conditions hold there:*

(i) $\Delta(x,y,\eta,\delta)$ *tends to* 0 *as* (x,y,η) *tends to* (x_0,y_0,η_0) *for some* $\delta\in(0,\delta_1)$.

(ii) For every $\xi>0$, *we have* $\sup\limits_{\xi\leq\sqrt{t^2+s^2}}[\varphi(t,s)L_\eta(t,s)]=o(\Delta(x,y,\eta,\delta))$ *as* (x,y,η) *tends to* (x_0,y_0,η_0).

(iii) For every $u\in\mathbb{R}$, *we have* $\left|\iint_{\mathbb{R}^2}K_\eta\left(t-x,s-y,\frac{u}{\varphi(x_0,y_0)}\varphi(t,s)\right)dsdt-u\right|^p$ $=o(\Delta(x,y,\eta,\delta))$ *as* (x,y,η) *tends to* (x_0,y_0,η_0).

(iv) For every $\xi>0$, *we have* $\iint_{\xi\leq\sqrt{t^2+s^2}}\varphi(t,s)L_\eta(t,s)\,dsdt=o(\Delta(x,y,\eta,\delta))$ *as* (x,y,η) *tends to* (x_0,y_0,η_0).

Then, at each common $p-\mu-$*generalized Lebesgue point of the functions* $f\in L_{p,\varphi}(\mathbb{R}^2)$ $(1\leq p<\infty)$ *and* φ, *we have*

$$|T_\eta(f;x,y)-f(x_0,y_0)|^p=o(\Delta(x,y,\eta,\delta))$$

as (x,y,η) *tends to* (x_0,y_0,η_0).

Proof. By the hypotheses of Theorem 4.1, we may write

$$\begin{aligned}
&|T_\eta(f;x,y)-f(x_0,y_0)|^p\\
&\leq 2^{2p}\varphi(x,y)\,\rho(x,y,\eta)\left|\frac{f(x_0,y_0)}{\varphi(x_0,y_0)}\right|^p\iint\limits_{\frac{\delta}{2}\leq\sqrt{u^2+v^2}}\varphi(u,v)\,L_\eta(u,v)\,dvdu\\
&\quad+2^{2p}\varphi(x,y)\,\rho(x,y,\eta)\sup_{\frac{\delta}{2}\leq\sqrt{u^2+v^2}}[\varphi(u,v)\,L_\eta(u,v)]\,\|f\|^p_{L_{p,\varphi}(\mathbb{R}^2)}\\
&\quad+2^p\varepsilon^p\rho(x,y,\eta)\sup_{(t,s)\in Q_\delta}\varphi(t,s)\,\Delta(x,y,\eta,\delta)\\
&\quad+2^p\left|\iint\limits_{\mathbb{R}^2}K_\eta\left(t-x,s-y,\frac{f(x_0,y_0)}{\varphi(x_0,y_0)}\varphi(t,s)\right)dsdt-f(x_0,y_0)\right|^p.
\end{aligned}$$

Here, Q_δ denotes the set defined in the proof of Theorem 4.1. From (i)-(iv), and using *Class* A_φ conditions, the assertion follows. Thus, the proof is completed. □

EXAMPLE 5.2. Define the kernel function such that

$$K_\eta(t,s,u)=\begin{cases}\frac{\eta u}{2}+\sin\frac{\eta u}{2}, & if \quad (t,s)\in\left[\frac{-1}{\sqrt{2\eta}},\frac{1}{\sqrt{2\eta}}\right]\times\left[\frac{-1}{\sqrt{2\eta}},\frac{1}{\sqrt{2\eta}}\right],\\ 0, & if \quad (t,s)\in\mathbb{R}^2\backslash\left[\frac{-1}{\sqrt{2\eta}},\frac{1}{\sqrt{2\eta}}\right]\times\left[\frac{-1}{\sqrt{2\eta}},\frac{1}{\sqrt{2\eta}}\right],\end{cases}$$

where $\eta\in\mathbb{N}$ and $\eta_0=\infty$. Let $u,v\in\mathbb{R}$. This kernel is the two dimensinal analogue of the kernel given in [16].

Observe that $|K_\eta(t,s,u) - K_\eta(t,s,v)| \leq \eta\,|u-v|$ for $(t,s) \in \left[\frac{-1}{\sqrt{2\eta}}, \frac{1}{\sqrt{2\eta}}\right] \times \left[\frac{-1}{\sqrt{2\eta}}, \frac{1}{\sqrt{2\eta}}\right]$ and $|K_\eta(t,s,u) - K_\eta(t,s,v)| = 0$ otherwise. Hence, we have

$$L_\eta(t,s) = \begin{cases} \eta, & if \quad (t,s) \in \left[\frac{-1}{\sqrt{2\eta}}, \frac{1}{\sqrt{2\eta}}\right] \times \left[\frac{-1}{\sqrt{2\eta}}, \frac{1}{\sqrt{2\eta}}\right], \\ 0, & if \quad (t,s) \in \mathbb{R}^2 \backslash \left[\frac{-1}{\sqrt{2\eta}}, \frac{1}{\sqrt{2\eta}}\right] \times \left[\frac{-1}{\sqrt{2\eta}}, \frac{1}{\sqrt{2\eta}}\right]. \end{cases}$$

It is easy to see that the conditions of class A_φ are satisfied.

Now, let $\varphi : \mathbb{R}^2 \to \mathbb{R}^+$ be given by $\varphi(t,s) = (1+|t|)(1+|s|)$. Let us find the rate of convergence according to hypotheses of Theorem 5.1.

For the simplicity take $(x_0, y_0) = (1,1)$, $\mu(t,s) = ts$ and $p = 1$. Denote

$$\begin{aligned} \Delta_1(x,y,\eta,\delta) &= \int_{x_0-\delta}^{x_0+\delta}\int_{y_0-\delta}^{y_0+\delta} L_\eta\,(t-x, s-y)\,|d_t d_s \mu(|t-x_0|, |s-y_0|)| \\ &= \int_{1-\delta}^{1+\delta}\int_{1-\delta}^{1+\delta} \eta ds dt. \end{aligned}$$

Let $\lim\limits_{\eta\to\infty} 4\eta\delta^2 = 0$. Consequently, if we choose $\delta > 0$ such that $\delta = o\left(\frac{1}{\eta}\right)$, then $\lim\limits_{\eta\to\infty} \Delta_1(x,y,\eta,\delta) = 0$. The evaluations of the remaining terms of $\Delta(x,y,\eta,\delta)$ yield the same conclusion. Thus, we have

$$|T_\eta\,(f;x,y) - f\,(x_0,y_0)| = o(\frac{1}{\eta}).$$

6. Conclusion

In this paper, the pointwise convergence of the convolution type nonlinear double integral operators depending on three parameters is investigated. The main result is presented as Theorem 4.1. By using main result, the rate of pointwise convergence of the indicated operators is investigated.

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