Korean J. Math. **27** (2019), No. 1, pp. 53–62
https://doi.org/10.11568/kjm.2019.27.1.53

# NON-FINITELY BASED FINITE INVOLUTION SEMIGROUPS WITH FINITELY BASED SEMIGROUP REDUCTS

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Abstract. Recently, an infinite class of finitely based finite involution semigroups with non-finitely based semigroup reducts have been found. In contrast, only one example of the opposite type—non-finitely based finite involution semigroups with finitely based semigroup reducts—has so far been published. In the present article, a sufficient condition is established under which an involution semigroup is non-finitely based. This result is then applied to exhibit several examples of the desired opposite type.

## 1. Introduction

An algebra is *finitely based* if the equations it satisfies are finitely axiomatizable. The question of which algebras are finitely based—the *finite basis problem*—is one of the most prominent research problems in universal algebra. In the 1960s, Perkins [8] published the first examples of non-finitely based finite semigroups. Since then, the finite basis problem for finite semigroups has been intensely investigated. Shortly after, the same problem has also been considered for *involution semigroups* $\langle \mathcal{S}, {}^* \rangle$, that is, semigroups $\mathcal{S}$ endowed with a unary operation $^*$ that satisfy the equations

$$(0) \qquad (x^*)^* \approx x \quad \text{and} \quad (xy)^* \approx y^* x^*.$$

Received June 13, 2018. Revised February 14, 2019. Accepted February 15, 2019.
2010 Mathematics Subject Classification: 20M05.
Key words and phrases: Semigroup, involution, $*$-semigroup, finitely based.

Involution semigroups are also commonly called $*$-*semigroups*. Examples of $*$-semigroups include matrix semigroups $\langle \mathsf{M}_n(\mathfrak{F}), {}^T \rangle$ over any field $\mathfrak{F}$ with transposition ${}^T$ and groups $\langle \mathcal{G}, {}^{-1} \rangle$ with inversion ${}^{-1}$.

The equational theory of a $*$-semigroup $\langle \mathcal{S}, {}^* \rangle$ strictly contains that of its reduct $\mathcal{S}$; in other words, any equation satisfied by $\mathcal{S}$ is also satisfied by $\langle \mathcal{S}, {}^* \rangle$. But in general, a $*$-semigroup $\langle \mathcal{S}, {}^* \rangle$ and its reduct $\mathcal{S}$ need not be simultaneously finitely based. As observed by Volkov [10, Section 2], examples of the following exist:

(X.1) finitely based $*$-semigroup $\langle \mathcal{S}, {}^* \rangle$ with non-finitely based reduct $\mathcal{S}$;
(X.2) non-finitely based $*$-semigroup $\langle \mathcal{S}, {}^* \rangle$ with finitely based reduct $\mathcal{S}$.

These examples have been available since the 1980s but are all infinite. Therefore finite examples are of particular interest, and they were only recently discovered. For finite examples of (X.1), an infinite class was constructed from the $\mathscr{J}$-trivial $*$-semigroup

$$\mathcal{L}_3 = \langle e, f \mid e^2 = e,\ f^2 = f,\ fef = 0 \rangle$$

of order six and finite cyclic groups [5, 6]. In contrast, finite examples of (X.2) seem much rarer since only one—the Rees matrix semigroup

$$M^0\Big(\{1,2,3\}, \mathcal{E}, \{1,2,3\}; \left[\begin{smallmatrix} 0 & e & e \\ e & 0 & e \\ e & e & 0 \end{smallmatrix}\right]\Big)$$

over the trivial group $\mathcal{E} = \{e\}$ with unary operation ${}^*$ given by $0^* = 0$ and $(i, e, j)^* = (j, e, i)$ for all $i, j \in \{1, 2, 3\}$—has so far been found [4].

The main objective of the present article is to demonstrate how other finite examples of (X.2) can be located. After some background material is given in Section 2, a sufficient condition under which a $*$-semigroup is non-finitely based is established in Section 3. This result is then applied to exhibit several finite examples of (X.2).

Results of the present article were first announced at the 1275th meeting of L. N. Shevrin's seminar "Algebraic Systems" at Ural Federal University, Dec 28, 2017.

## 2. Preliminaries

Acquaintance with rudiments of universal algebra is assumed of the reader. Refer to the monograph of Burris and Sankappanavar [1] for more information.

**2.1. Words and terms.** Let $\mathscr{A}$ be an infinite alphabet throughout that excludes the symbol $\mathsf{0}$ and let $\mathscr{A}^* = \{x^* \mid x \in \mathscr{A}\}$ be a disjoint copy of $\mathscr{A}$. Elements of $\mathscr{A} \cup \mathscr{A}^*$ are called *variables*. The *free* $*$*-semigroup* over $\mathscr{A}$ is the free semigroup $(\mathscr{A} \cup \mathscr{A}^*)^+$ with unary operation $^*$ given by $(x^*)^* = x$ for all $x \in \mathscr{A}$ and $(x_1 x_2 \cdots x_m)^* = x_m^* x_{m-1}^* \cdots x_1^*$ for all $x_1, x_2, \ldots, x_m \in \mathscr{A} \cup \mathscr{A}^*$. Elements of $(\mathscr{A} \cup \mathscr{A}^*)^+ \cup \{\varnothing\}$ are called *words* and elements of $\mathscr{A}^+ \cup \{\varnothing\}$ are called *plain words*. A word $\mathbf{u}$ is a *factor* of a word $\mathbf{v}$ if $\mathbf{aub} = \mathbf{v}$ for some $\mathbf{a}, \mathbf{b} \in (\mathscr{A} \cup \mathscr{A}^*)^+ \cup \{\varnothing\}$.

For any word $\mathbf{u} \in (\mathscr{A} \cup \mathscr{A}^*)^+ \cup \{\varnothing\}$ and variables $x_1, x_2, \ldots, x_n \in \mathscr{A}$, let $\mathbf{u}[x_1, x_2, \ldots, x_n]$ be the word obtained from $\mathbf{u}$ by retaining only the variables $x_1, x_1^*, x_2, x_2^*, \ldots, x_n, x_n^*$. For instance, if $\mathbf{u} = xyx^*yz^*x$, then $\mathbf{u}[x] = xx^*x$, $\mathbf{u}[y,z] = y^2z^*$, $\mathbf{u}[x,y,z] = \mathbf{u}$, and $\mathbf{u}[x,z,t] = xx^*z^*x$.

The set $\mathsf{T}(\mathscr{A})$ of *terms* over the alphabet $\mathscr{A}$ is the smallest set containing $\mathscr{A}$ that is closed under concatenation and $^*$. The *subterms* of a term $\mathbf{t}$ are defined as follows: $\mathbf{t}$ is a subterm of $\mathbf{t}$; if $\mathbf{s}_1\mathbf{s}_2$ is a subterm of $\mathbf{t}$ where $\mathbf{s}_1, \mathbf{s}_2 \in \mathsf{T}(\mathscr{A})$, then $\mathbf{s}_1$ and $\mathbf{s}_2$ are subterms of $\mathbf{t}$; if $\mathbf{s}^*$ is a subterm of $\mathbf{t}$ where $\mathbf{s} \in \mathsf{T}(\mathscr{A})$, then $\mathbf{s}$ is a subterm of $\mathbf{t}$. The proper inclusion $(\mathscr{A} \cup \mathscr{A}^*)^+ \cup \{\varnothing\} \subset \mathsf{T}(\mathscr{A})$ holds and the equations (0) can be used to convert any nonempty term $\mathbf{t} \in \mathsf{T}(\mathscr{A})$ into some unique word $\lfloor \mathbf{t} \rfloor \in (\mathscr{A} \cup \mathscr{A}^*)^+$. For instance, $\lfloor x(x^3(yx^*)^*)^*zy^* \rfloor = xy(x^*)^4zy^*$.

Remark 2.1. For any subterm $\mathbf{s}$ of a term $\mathbf{t}$, either $\lfloor \mathbf{s} \rfloor$ or $\lfloor \mathbf{s}^* \rfloor$ is a factor of $\lfloor \mathbf{t} \rfloor$.

**2.2. Equations, deducibility and satisfiability.** An *equation* is an expression $\mathbf{s} \approx \mathbf{t}$ formed by terms $\mathbf{s}, \mathbf{t} \in \mathsf{T}(\mathscr{A}) \backslash \{\varnothing\}$. Specifically, a *word equation* is an equation $\mathbf{u} \approx \mathbf{v}$ formed by words $\mathbf{u}, \mathbf{v} \in (\mathscr{A} \cup \mathscr{A}^*)^+$ and a *plain equation* is an equation $\mathbf{u} \approx \mathbf{v}$ formed by plain words $\mathbf{u}, \mathbf{v} \in \mathscr{A}^+$.

An equation $\mathbf{s} \approx \mathbf{t}$ is *directly deducible* from an equation $\mathbf{p} \approx \mathbf{q}$ if there exists some substitution $\varphi : \mathscr{A} \to \mathsf{T}(\mathscr{A}) \backslash \{\varnothing\}$ such that $\mathbf{p}\varphi$ is a subterm of $\mathbf{s}$, and replacing this particular subterm $\mathbf{p}\varphi$ of $\mathbf{s}$ with $\mathbf{q}\varphi$ results in the term $\mathbf{t}$. An equation $\mathbf{s} \approx \mathbf{t}$ is *deducible* from some set $\Sigma$ of equations if there exists a finite sequence $\mathbf{s} = \mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_r = \mathbf{t}$ of nonempty terms such that each equation $\mathbf{s}_i \approx \mathbf{s}_{i+1}$ is directly deducible from some equation in $\Sigma$.

Remark 2.2. An equation $\mathbf{s} \approx \mathbf{t}$ is deducible from (0) if and only if $\lfloor \mathbf{s} \rfloor = \lfloor \mathbf{t} \rfloor$.

A $*$-semigroup $\langle \mathcal{S}, {}^* \rangle$ *satisfies* an equation $\mathbf{s} \approx \mathbf{t}$, or $\mathbf{s} \approx \mathbf{t}$ is *satisfied by* $\langle \mathcal{S}, {}^* \rangle$, if for any substitution $\varphi : \mathscr{A} \to \mathcal{S}$, the elements $\mathbf{s}\varphi$ and $\mathbf{t}\varphi$ of $\mathcal{S}$

coincide; in this case, $\mathbf{s} \approx \mathbf{t}$ is also said to be an *equation of* $\langle \mathcal{S}, {}^* \rangle$. An equation is satisfied by a $*$-semigroup $\langle \mathcal{S}, {}^* \rangle$ if and only if it is deducible from the equations of $\langle \mathcal{S}, {}^* \rangle$.

A *unital* $*$*-semigroup* is a $*$-semigroup with a unit element. Any unital $*$-semigroup that satisfies an equation $\mathbf{u} \approx \mathbf{v}$ also satisfies the equation $\mathbf{u}[x_1, x_2, \ldots, x_n] \approx \mathbf{v}[x_1, x_2, \ldots, x_n]$ for any $x_1, x_2, \ldots, x_n \in \mathscr{A}$.

A set $\Sigma$ of equations of a $*$-semigroup $\langle \mathcal{S}, {}^* \rangle$ is an *equational basis* for $\langle \mathcal{S}, {}^* \rangle$ if every equation of $\langle \mathcal{S}, {}^* \rangle$ is deducible from $\Sigma$. A $*$-semigroup is *finitely based* if it has some finite equational basis. *Finitely based* semigroups are similarly defined.

## 3. Main results

Recall that a word $\mathbf{w} \in (\mathscr{A} \cup \mathscr{A}^*)^+$ is an *isoterm* for a $*$-semigroup $\langle \mathcal{S}, {}^* \rangle$ if it cannot be used to form a nontrivial word equation of $\langle \mathcal{S}, {}^* \rangle$; in other words, if $\langle \mathcal{S}, {}^* \rangle$ satisfies some nontrivial word equation $\mathbf{w} \approx \mathbf{v}$, then $\mathbf{w} = \mathbf{v}$.

The following sufficient condition for the non-finite basis property of $*$-semigroups resembles a result of Jackson [2, Lemma 2.3.3] that applies to semigroups; see also Jackson and Sapir [3, Lemma 4.4].

Theorem 3.1. *Let* $\langle \mathcal{S}, {}^* \rangle$ *be any unital* $*$*-semigroup with isoterm* $xyx^*y^*$. *Suppose that for infinitely many integers* $n \geq 1$, *the word*

$$\mathbf{p}_n = x_1y_1x_2y_2 \cdots x_ny_nty_1^*y_2^* \cdots y_n^*x_1^*x_2^* \cdots x_n^*$$

*is not an isoterm for* $\langle \mathcal{S}, {}^* \rangle$. *Then* $\langle \mathcal{S}, {}^* \rangle$ *is non-finitely based.*

For each integer $n \geq 1$, define the word

$$\mathbf{q}_n = y_1y_2 \cdots y_nx_1x_2 \cdots x_ntx_1^*y_1^*x_2^*y_2^* \cdots x_n^*y_n^*.$$

In Subsection 3.1, it is shown that for any unital $*$-semigroup $\langle \mathcal{S}, {}^* \rangle$ with isoterm $xyx^*y^*$, the equation $\mathbf{p}_n \approx \mathbf{q}_n$ is the only nontrivial word equation of $\langle \mathcal{S}, {}^* \rangle$ with $\mathbf{p}_n$ on one side. This result is crucial to the proof of Theorem 3.1 in Subsection 3.2. Finite examples of (X.2) are then exhibited in Subsection 3.3. In Subsection 3.4, it is shown that Theorem 3.1 also holds when the unital $*$-semigroup $\langle \mathcal{S}, {}^* \rangle$ is considered as a $*$*-monoid*, that is, an algebra of type $(2, 0, 1)$.

### 3.1. The equations $\mathbf{p}_n \approx \mathbf{q}_n$.

LEMMA 3.2. *Let* $\langle \mathcal{S}, {}^* \rangle$ *be any unital* $*$*-semigroup for which* $xyx^*y^*$ *is an isoterm. Suppose that* $\langle \mathcal{S}, {}^* \rangle$ *satisfies some nontrivial word equation* $\mathbf{p}_n \approx \mathbf{w}$*. Then* $\mathbf{w} = \mathbf{q}_n$.

The remainder of this subsection is devoted to the proof of Lemma 3.2; in the following, suppose that $\langle \mathcal{S}, {}^* \rangle$ is any unital $*$-semigroup such that

(†) $xyx^*y^*$ is an isoterm for $\langle \mathcal{S}, {}^* \rangle$;
(‡) $\mathbf{p}_n \approx \mathbf{w}$ is a nontrivial word equation of $\langle \mathcal{S}, {}^* \rangle$.

Since the factor $xyx^*$ of $xyx^*y^*$ is an isoterm for $\langle \mathcal{S}, {}^* \rangle$, the words $\mathbf{p}_n[x_i, t] = x_i t x_i^*$ and $\mathbf{p}_n[y_i, t] = y_i t y_i^*$ are also isoterms for $\langle \mathcal{S}, {}^* \rangle$. It follows that the equations $\mathbf{p}_n[x_i, t] \approx \mathbf{w}[x_i, t]$ and $\mathbf{p}_n[y_i, t] \approx \mathbf{w}[y_i, t]$ are trivial. Therefore $\mathbf{w} = \mathbf{a}t\mathbf{b}$, where

(a) $\mathbf{a}$ is a product of $x_1, y_1, x_2, y_2, \ldots, x_n, y_n$ in some order and
(b) $\mathbf{b}$ is a product of $x_1^*, y_1^*, x_2^*, y_2^*, \ldots, x_n^*, y_n^*$ in some order.

Similarly, for $i < j$, since the words

$$\mathbf{p}_n[x_i, x_j] = x_i x_j x_i^* x_j^*, \ \mathbf{p}_n[y_i, y_j] = y_i y_j y_i^* y_j^*, \text{ and } \mathbf{p}_n[y_i, x_j] = y_i x_j y_i^* x_j^*$$

are isoterms for $\langle \mathcal{S}, {}^* \rangle$, the equations

$$\mathbf{p}_n[x_i, x_j] \approx \mathbf{w}[x_i, x_j], \ \mathbf{p}_n[y_i, y_j] \approx \mathbf{w}[y_i, y_j], \text{ and } \mathbf{p}_n[y_i, x_j] \approx \mathbf{w}[y_i, x_j]$$

are trivial. Therefore by (a) and (b),

(c) $\mathbf{a}[x_1, x_2, \ldots, x_n] = x_1 x_2 \cdots x_n$ and $\mathbf{a}[y_1, y_2, \ldots, y_n] = y_1 y_2 \cdots y_n$,
(d) $\mathbf{b}[x_1^*, x_2^*, \ldots, x_n^*] = x_1^* x_2^* \cdots x_n^*$ and $\mathbf{b}[y_1^*, y_2^*, \ldots, y_n^*] = y_1^* y_2^* \cdots y_n^*$,
(e) $\mathbf{a}[y_i, x_{i+1}] = y_i x_{i+1}$ for all $i \in \{1, 2, \ldots, n-1\}$, and
(f) $\mathbf{b}[y_i^*, x_{i+1}^*] = y_i^* x_{i+1}^*$ for all $i \in \{1, 2, \ldots, n-1\}$.

LEMMA 3.3. $\mathbf{a} \neq x_1 y_1 x_2 y_2 \cdots x_n y_n$.

*Proof.* Suppose that $\mathbf{a} = x_1 y_1 x_2 y_2 \cdots x_n y_n$. Then it follows from (b) that either $\mathbf{b}[x_1^*, y_n^*] = x_1^* y_n^*$ or $\mathbf{b}[x_1^*, y_n^*] = y_n^* x_1^*$. If $\mathbf{b}[x_1^*, y_n^*] = x_1^* y_n^*$, then the equation $\mathbf{p}_n[x_1, y_n] \approx \mathbf{w}[x_1, y_n]$ is $x_1 y_n y_n^* x_1^* \approx x_1 y_n x_1^* y_n^*$, whence (†) is contradicted. Therefore $\mathbf{b}[x_1^*, y_n^*] = y_n^* x_1^*$ is the only possibility, so that $\mathbf{b} = y_1^* y_2^* \cdots y_n^* x_1^* x_2^* \cdots x_n^*$ by (b) and (d). It follows that $\mathbf{w} = \mathbf{p}_n$, whence (‡) is contradicted. □

LEMMA 3.4. $\mathbf{a}[x_1, y_n] = y_n x_1$.

*Proof.* By (a), for any $i, j \in \{1, 2, \ldots, n\}$, either $\mathbf{a}[x_i, y_j] = x_i y_j$ or $\mathbf{a}[x_i, y_j] = y_j x_i$. If $\mathbf{a}[x_i, y_j] = x_i y_j$ whenever $1 \le i \le j \le n$, then it follows from (a) and (e) that $\mathbf{a} = x_1 y_1 x_2 y_2 \cdots x_n y_n$, whence Lemma 3.3 is contradicted. Hence there exist $k, \ell \in \{1, 2, \ldots, n\}$ with $k \le \ell$ such that $\mathbf{a}[x_k, y_\ell] = y_\ell x_k$. There are two cases depending on the value of $k$.

Case 1: $k = 1$, so that $\mathbf{a}[x_1, y_\ell] = y_\ell x_1$. It suffices to further assume that $\ell < n$, since the lemma vacuously holds if $\ell = n$. By (b), either $\mathbf{b}[x_1, y_\ell] = x_1^* y_\ell^*$ or $\mathbf{b}[x_1, y_\ell] = y_\ell^* x_1^*$. If $\mathbf{b}[x_1, y_\ell] = y_\ell^* x_1^*$, then the equation $\mathbf{p}_n[x_1, y_\ell] \approx \mathbf{w}[x_1, y_\ell]$ is $x_1 y_\ell y_\ell^* x_1^* \approx y_\ell x_1 y_\ell^* x_1^*$, whence (†) is contradicted. Thus $\mathbf{b}[x_1, y_\ell] = x_1^* y_\ell^*$ is the only possibility. Now if $\mathbf{a}[x_1, y_m] = x_1 y_m$ for some $m > \ell$, then since $\mathbf{b}[x_1, y_\ell, y_m] = x_1^* y_\ell^* y_m^*$ by (b) and (d), the equation $\mathbf{p}_n[x_1, y_m] \approx \mathbf{w}[x_1, y_m]$ is $x_1 y_m y_m^* x_1^* \approx x_1 y_m x_1^* y_m^*$, whence (†) is again contradicted. Therefore $m$ does not exist, so that $\mathbf{a}[x_1, y_i] = y_i x_1$ for all $i > \ell$. In particular, $\mathbf{a}[x_1, y_n] = y_n x_1$.

Case 2: $k > 1$. By (a), either $\mathbf{a}[x_1, y_\ell] = x_1 y_\ell$ or $\mathbf{a}[x_1, y_\ell] = y_\ell x_1$. Suppose that $\mathbf{a}[x_1, y_\ell] = x_1 y_\ell$. Then there are two subcases to consider: $\mathbf{b}[x_1, y_\ell] = x_1^* y_\ell^*$ and $\mathbf{b}[x_1, y_\ell] = y_\ell^* x_1^*$. In the former subcase, the equation $\mathbf{p}_n[x_1, y_\ell] \approx \mathbf{w}[x_1, y_\ell]$ is $x_1 y_\ell y_\ell^* x_1^* \approx x_1 y_\ell x_1^* y_\ell^*$; in the latter subcase, since $\mathbf{b}[x_1, x_k, y_\ell] = y_\ell^* x_1^* x_k^*$ by (d), the equation $\mathbf{p}_n[x_k, y_\ell] \approx \mathbf{w}[x_k, y_\ell]$ is $x_k y_\ell y_\ell^* x_k^* \approx y_\ell x_k y_\ell^* x_k^*$. Since either subcase implies that (†) is contradicted, it is impossible for the assumption $\mathbf{a}[x_1, y_\ell] = x_1 y_\ell$ to hold. It follows that $\mathbf{a}[x_1, y_\ell] = y_\ell x_1$, whence the result holds by Case 1. □

Hence $\mathbf{a} = y_1 y_2 \cdots y_n x_1 x_2 \cdots x_n$ by (a), (c), and Lemma 3.4. Now (b) implies that for each $i$, either $\mathbf{b}[x_i, y_i] = x_i^* y_i^*$ or $\mathbf{b}[x_i, y_i] = y_i^* x_i^*$. But if $\mathbf{b}[x_i, y_i] = y_i^* x_i^*$ for some $i$, then the equation $\mathbf{p}_n[x_i, y_i] \approx \mathbf{w}[x_i, y_i]$ is $x_i y_i y_i^* x_i^* \approx y_i x_i y_i^* x_i^*$, whence (†) is contradicted. Thus $\mathbf{b}[x_i, y_i] = x_i^* y_i^*$ for all $i$, so that $\mathbf{b} = x_1^* y_1^* x_2^* y_2^* \cdots x_n^* y_n^*$ by (b) and (f). It follows that $\mathbf{w} = \mathbf{q}_n$, so that the proof of Lemma 3.2 is complete.

### 3.2. Proof of Theorem 3.1.

For each integer $n \ge 1$, define the set

$$\mathscr{Z}_n = \{z_1, z_1^*, z_2, z_2^*, \ldots, z_n, z_n^*\}.$$

Lemma 3.5. *Let $\langle \mathcal{S}, {}^* \rangle$ be any unital $*$-semigroup with isoterm $xyx^*y^*$ that satisfies an equation $\mathbf{u} \approx \mathbf{v}$ with $\mathbf{u}, \mathbf{v} \in \mathscr{Z}_n^+$, and let $\mathbf{s} \approx \mathbf{t}$ be any equation that is directly deducible from $\mathbf{u} \approx \mathbf{v}$. Suppose that $\lfloor \mathbf{s} \rfloor = \mathbf{p}_n$. Then $\lfloor \mathbf{t} \rfloor = \mathbf{p}_n$.*

*Proof.* By assumption, there is a substitution $\varphi : \mathscr{A} \to \mathsf{T}(\mathscr{A}) \backslash \{\varnothing\}$ such that $\mathbf{u}\varphi$ is a subterm of $\mathbf{s}$, and replacing this particular subterm $\mathbf{u}\varphi$

of $\mathbf{s}$ with $\mathbf{v}\varphi$ results in $\mathbf{t}$. Then by Remark 2.1, either $\lfloor \mathbf{u}\varphi \rfloor$ or $\lfloor (\mathbf{u}\varphi)^* \rfloor$ is a factor of $\lfloor \mathbf{s} \rfloor$. It suffices to consider the former case since the latter is similar. Hence there exist words $\mathbf{a}, \mathbf{b} \in (\mathscr{A} \cup \mathscr{A}^*)^+ \cup \{\varnothing\}$ such that $\lfloor \mathbf{s} \rfloor = \mathbf{a}\lfloor \mathbf{u}\varphi \rfloor\mathbf{b}$ and $\lfloor \mathbf{t} \rfloor = \mathbf{a}\lfloor \mathbf{v}\varphi \rfloor\mathbf{b}$. Since $\langle \mathcal{S}, {}^* \rangle$ satisfies the equation $\lfloor \mathbf{s} \rfloor \approx \lfloor \mathbf{t} \rfloor$ with $\lfloor \mathbf{s} \rfloor = \mathbf{p}_n$, it follows from Lemma 3.2 that $\lfloor \mathbf{t} \rfloor$ is either $\mathbf{p}_n$ or $\mathbf{q}_n$. The proof is complete if $\lfloor \mathbf{t} \rfloor = \mathbf{p}_n$, thus assume that $\lfloor \mathbf{t} \rfloor = \mathbf{q}_n$.

Now since the words $\mathbf{p}_n$ and $\mathbf{q}_n$ do not share the same first and last variables, the words $\mathbf{a}$ and $\mathbf{b}$ have to be empty, that is, $\mathbf{p}_n = \lfloor \mathbf{s} \rfloor = \lfloor \mathbf{u}\varphi \rfloor$ and $\mathbf{q}_n = \lfloor \mathbf{t} \rfloor = \lfloor \mathbf{v}\varphi \rfloor$. Given that the word $\mathbf{p}_n$ involves more than $2n$ distinct variables, the factors $\lfloor z_1\varphi \rfloor, \lfloor z_1^*\varphi \rfloor, \lfloor z_2\varphi \rfloor, \lfloor z_2^*\varphi \rfloor, \ldots, \lfloor z_n\varphi \rfloor, \lfloor z_n^*\varphi \rfloor$ of $\mathbf{p}_n$ cannot all be single variables. Hence there exists some variable $x \in \{z_1, z_1^*, z_2, z_2^*, \ldots, z_n, z_n^*\}$ such that $\lfloor x\varphi \rfloor$ is a factor of $\mathbf{p}_n$ of length at least two. It follows that either $\lfloor x\varphi \rfloor$ or $\lfloor x^*\varphi \rfloor$ is a factor of $\mathbf{q}_n$, but this is impossible by simple inspection. □

Let $\langle \mathcal{S}, {}^* \rangle$ be any $*$-semigroup that satisfies the assumptions in Theorem 3.1. Seeking a contradiction, suppose that $\langle \mathcal{S}, {}^* \rangle$ is finitely based. Then there exists a finite set $\Sigma$ of word equations of $\langle \mathcal{S}, {}^* \rangle$ such that all equations of $\langle \mathcal{S}, {}^* \rangle$ are deducible from $(0) \cup \Sigma$. Generality is not lost by assuming that each equation in $\Sigma$ is formed by some pair of words in $\mathscr{Z}_m^+$. By assumption, there exists some $n \geq m$ such that the word $\mathbf{p}_n$ is not an isoterm for $\langle \mathcal{S}, {}^* \rangle$. Then there exists some word $\mathbf{t} \neq \mathbf{p}_n$ such that $\langle \mathcal{S}, {}^* \rangle$ satisfies the equation $\mathbf{p}_n \approx \mathbf{t}$. It follows that $\mathbf{p}_n \approx \mathbf{t}$ is deducible from $(0) \cup \Sigma$, whence there exists a sequence $\mathbf{p}_n = \mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_r = \mathbf{t}$ of terms such that each equation $\mathbf{s}_i \approx \mathbf{s}_{i+1}$ is directly deducible from some equation $\mathbf{u}_i \approx \mathbf{v}_i$ from $(0) \cup \Sigma$. The equality $\mathbf{p}_n = \lfloor \mathbf{s}_1 \rfloor$ holds vacuously. Suppose that $\mathbf{p}_n = \lfloor \mathbf{s}_i \rfloor$ for some $i \geq 1$. Then there are two cases depending on whether the equation $\mathbf{u}_i \approx \mathbf{v}_i$ is from (0) or $\Sigma$. If $\mathbf{u}_i \approx \mathbf{v}_i$ is from (0), then $\lfloor \mathbf{s}_i \rfloor = \lfloor \mathbf{s}_{i+1} \rfloor$ as observed in Remark 2.2, whence $\mathbf{p}_n = \lfloor \mathbf{s}_{i+1} \rfloor$. If $\mathbf{u}_i \approx \mathbf{v}_i$ is from $\Sigma$, then since $\mathbf{u}_i, \mathbf{v}_i \in \mathscr{Z}_m^+ \subseteq \mathscr{Z}_n^+$, it follows from Lemma 3.5 that $\mathbf{p}_n = \lfloor \mathbf{s}_{i+1} \rfloor$. Therefore $\mathbf{p}_n = \lfloor \mathbf{s}_{i+1} \rfloor$ in any case, whence $\mathbf{p}_n = \lfloor \mathbf{s}_1 \rfloor = \lfloor \mathbf{s}_2 \rfloor = \cdots = \lfloor \mathbf{s}_r \rfloor$ by induction. The contradiction $\mathbf{p}_n = \lfloor \mathbf{s}_r \rfloor = \lfloor \mathbf{t} \rfloor = \mathbf{t}$ follows. Theorem 3.1 is thus established.

**3.3. Finite examples of (X.2).** For any set $\mathscr{W} \subseteq (\mathscr{A} \cup \mathscr{A}^*)^+ \cup \{\varnothing\}$ of words, let $\mathscr{W}_{\preccurlyeq}$ denote the set of factors of every word in $\mathscr{W}$, and let $\mathsf{Rq}\,\mathscr{W}$ denote the Rees quotient of $(\mathscr{A} \cup \mathscr{A}^*)^+ \cup \{\varnothing\}$ over the ideal of all words that are not in $\mathscr{W}_{\preccurlyeq} \cup \mathscr{W}_{\preccurlyeq}^*$. Equivalently, $\mathsf{Rq}\,\mathscr{W}$ can be treated as the semigroup that consists of elements from $\mathscr{W}_{\preccurlyeq} \cup \mathscr{W}_{\preccurlyeq}^*$, together with a zero element $\mathsf{0}$, with binary operation $\cdot$ given by

$$\mathbf{u}\cdot\mathbf{v} = \begin{cases} \mathbf{uv} & \text{if } \mathbf{uv} \in \mathscr{W}_{\preccurlyeq} \cup \mathscr{W}_{\preccurlyeq}^{*}, \\ \mathsf{0} & \text{otherwise.} \end{cases}$$

The empty word $\varnothing$ is the unit element of $\mathsf{Rq}\,\mathscr{W}$. Under the unary operation $^{*}$ on the free $*$-semigroup $(\mathscr{A} \cup \mathscr{A}^{*})^{+}$ introduced in Subsection 2.1, the semigroup $\mathsf{Rq}\,\mathscr{W}$ is a unital $*$-semigroup. For example, the semigroup $\mathsf{Rq}\{xyx^{*}\}$ contains twelve elements:

$$\begin{aligned} \mathsf{Rq}\{xyx^{*}\} &= \{xyx^{*}\}_{\preccurlyeq} \cup \{xyx^{*}\}_{\preccurlyeq}^{*} \cup \{\mathsf{0}\} \\ &= \{\varnothing,\, x,\, y,\, x^{*},\, y^{*},\, xy,\, yx^{*},\, xy^{*},\, y^{*}x^{*},\, xyx^{*},\, xy^{*}x^{*},\, \mathsf{0}\}. \end{aligned}$$

A word $\mathbf{w} \in (\mathscr{A} \cup \mathscr{A}^{*})^{+}$ is said to be *pseudo-simple* if for each variable $x \in \mathscr{A}$, the number of occurrences of $x$ in $\mathbf{w}$ and the number of occurrences of $x^{*}$ in $\mathbf{w}$ are at most one. In other words, $\mathbf{w}$ is pseudo-simple if $\mathbf{w}[x] \in \{\varnothing, x, x^{*}, xx^{*}, x^{*}x\}$ for each $x \in \mathscr{A}$. For instance, $x^{*}zxyz^{*}$ is pseudo-simple but $x^{*}yxy$ is not. For any set $\mathscr{W}$ of pseudo-simple words, the semigroup $\mathsf{Rq}\,\mathscr{W}$ satisfies the equation $xyx \approx x^{2}y$ and so is finitely based by Pollák and Volkov [9, Proposition C]. Therefore addressing the following problem is one direction in finding finite examples of $(\mathtt{X}.2)$.

Problem 3.6. *Locate finite sets $\mathscr{W}$ of pseudo-simple words for which the $*$-semigroup $\langle \mathsf{Rq}\,\mathscr{W}, {}^{*}\rangle$ is non-finitely based.*

Proposition 3.7. *The $*$-semigroup $\langle \mathsf{Rq}\{xyx^{*}y^{*}\}, {}^{*}\rangle$ is a finite example of* $(\mathtt{X}.2)$.

*Proof.* Since the word $xyx^{*}y^{*}$ is pseudo-simple, it suffices to show that the $*$-semigroup $\langle \mathsf{Rq}\{xyx^{*}y^{*}\}, {}^{*}\rangle$ is non-finitely based. In the following, it is shown that $\langle \mathsf{Rq}\{xyx^{*}y^{*}\}, {}^{*}\rangle$ satisfies the equation $\mathbf{p}_n \approx \mathbf{q}_n$ for each $n \geq 1$, so that the words $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \ldots$ are not isoterms. Since $xyx^{*}y^{*}$ is an isoterm for $\langle \mathsf{Rq}\{xyx^{*}y^{*}\}, {}^{*}\rangle$, this unital $*$-semigroup is non-finitely based by Theorem 3.1.

Let $\varphi : \mathscr{A} \to \mathsf{Rq}\{xyx^{*}y^{*}\}$ be any substitution. It is clear that if either $x_i\varphi = \varnothing$ for all $i \in \{1, 2, \ldots, n\}$ or $y_i\varphi = \varnothing$ for all $i \in \{1, 2, \ldots, n\}$, then $\mathbf{p}_n\varphi = \mathbf{q}_n\varphi$. Hence assume the existence of some $k, \ell \in \{1, 2, \ldots, n\}$ such that $x_k\varphi \neq \varnothing \neq y_\ell\varphi$. If $k \leq \ell$, then

$$\begin{aligned} \mathbf{p}_n\varphi &= \cdots x_k\varphi \cdots y_\ell\varphi \cdots y_\ell^{*}\varphi \cdots x_k^{*}\varphi \cdots = \mathsf{0} \\ \text{and}\quad \mathbf{q}_n\varphi &= \cdots y_\ell\varphi \cdots x_k\varphi \cdots x_k^{*}\varphi \cdots y_\ell^{*}\varphi \cdots = \mathsf{0}; \end{aligned}$$

if $k > \ell$, then $\mathbf{p}_n[x_k, y_\ell] = y_\ell x_k y_\ell^* x_k^* = \mathbf{q}_n[x_k, y_\ell]$, so that

$$(\mathbf{p}_n\varphi, \mathbf{q}_n\varphi) = \begin{cases} (xyx^*y^*, xyx^*y^*) & \text{if } x_k\varphi = y,\ y_\ell\varphi = x, \text{ and} \\ & z\varphi = \varnothing \text{ for all } z \in \mathscr{A}\backslash\{x_k, y_\ell\}, \\ (0, 0) & \text{otherwise.} \end{cases}$$

Therefore $\mathbf{p}_n\varphi = \mathbf{q}_n\varphi$ in any case. □

COROLLARY 3.8. *Given any unital $*$-semigroup $\langle \mathcal{S}, {}^* \rangle$ that satisfies the equations $xyx \approx x^2y$ and $\mathbf{p}_n \approx \mathbf{q}_n$ for all $n \geq 1$, the direct product $\langle \mathsf{Rq}\{xyx^*y^*\}, {}^* \rangle \times \langle \mathcal{S}, {}^* \rangle$ is a finite example of* (X.2).

For instance, for any integer $m \geq 0$ and any commutative unital $*$-semigroup $\langle \mathcal{S}, {}^* \rangle$, the direct product

$$\langle \mathsf{Rq}\{xyx^*y^*\}, {}^* \rangle \times \langle \mathsf{Rq}\{x_1x_1^*x_2x_2^* \cdots x_mx_m^*\}, {}^* \rangle \times \langle \mathcal{S}, {}^* \rangle$$

is an example of (X.2).

**3.4. Algebras of type $(2, 0, 1)$.** Theorem 3.1 is concerned with establishing the non-finite basis property of a unital $*$-semigroup $\langle \mathcal{S}, {}^* \rangle$ as an algebra of type $(2, 1)$. It turns out that this result also holds when $\langle \mathcal{S}, {}^* \rangle$ is considered as an algebra of type $(2, 0, 1)$.

THEOREM 3.9. *Any unital $*$-semigroup $\langle \mathcal{S}, {}^* \rangle$ that satisfies the hypotheses of Theorem 3.1 is non-finitely based as an algebra of type $(2, 0, 1)$.*

*Proof.* The main arguments in establishing Theorem 3.1 are Lemmas 3.2–3.5. Lemmas 3.2–3.4 do not involve equational deductions and so are valid for $\langle \mathcal{S}, {}^* \rangle$ as an algebra of type $(2, 1)$ or $(2, 0, 1)$. As for Lemma 3.5, the only change in its proof that is required for the result to hold for $\langle \mathcal{S}, {}^* \rangle$ as an algebra of type $(2, 0, 1)$ is to assume that $\varphi$ is a substitution from the variables in $\mathscr{A}$ to the set $\mathsf{T}(\mathscr{A})$ of all terms including the empty term $\varnothing$. The rest of the proof is then easily seen to follow through. □

REMARK 3.10. The $*$-semigroup $\langle \mathsf{Rq}\{xyx^*y^*\}, {}^* \rangle$ plays a crucial role in the construction of several algebras of type $(2, 0, 1)$ that generate varieties with extreme and contrasting properties [7].

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