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EXISTENCE OF RANDOM ATTRACTORS FOR STOCHASTIC NON-AUTONOMOUS REACTION-DIFFUSION EQUATION WITH MULTIPLICATIVE NOISE ON $\mathbb{R}^n$

Fadlallah Mustafa Mosa, Qiaozhen Ma*, and Mohamed Y. A. Bakhet

Abstract. In this paper, we are concerned with the existence of random dynamics for stochastic non-autonomous reaction-diffusion equations driven by a Wiener-type multiplicative noise defined on the unbounded domains.

1. Introduction

In this paper, we consider the following stochastic non-autonomous reaction-diffusion equation on $\mathbb{R}^n$ perturbed by a Wiener-type multiplicative noise:

$$\frac{du}{dt} + \lambda u - \Delta u = f(u) + g(t,x) + bu \circ \dot{W}(t),\ t > \tau,\ \tau \in \mathbb{R}, \tag{1.1}$$

$$u(x,\tau) = u_\tau(x)\ ,\ x \in \mathbb{R}^n,\ \tau \in \mathbb{R}, \tag{1.2}$$

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$*$ Corresponding author.

where $-\Delta$ is the Laplacian operator with respect to the variable $x \in \mathbb{R}^n$, $u = (x,t)$ is a real function of $x \in \mathbb{R}^n$ and $t > \tau$, $u_\tau(x) \in L^2(\mathbb{R}^n)$, $g \in L^2_{loc}(\mathbb{R}, L^2(\mathbb{R}^n))$, $\lambda, b > 0, W(t)$ is a Wiener process defined on a standard probability space $(\Omega, \mathcal{F}, P)$, where $\Omega = \{\omega \in C(\mathbb{R},\mathbb{R}) : \omega(0) = 0\}$, and $\mathcal{F}$ is the Borel σ -algebra induced by the compact-open topology of Ω and P is the corresponding Wiener measure on $\mathcal{F}$. The nonlinearity is a smooth function and satisfies the following conditions for some positive constants $\beta_1, \beta_2, \beta_3$ and β_4,

$$f(0) = 0, \tag{1.3}$$

$$f(s)s \leq -\beta_1|s|^p + \beta_2|s|^2,\ \forall s \in \mathbb{R}, \tag{1.4}$$

$$|f(s)| \leq \beta_3|s|^{p-1} + \beta_4|s|,\ \forall s \in \mathbb{R}, \tag{1.5}$$

where $p \geq 2$.
It is well known that the asymptotic behavior of a random dynamical system is presented by a random attractor. The existence of random attractors have been studied by many authors recently, see $[1,3,5,7-8,11,12,14,16,18,20]$ and the reference therein. Notice that the partial differential equations (PDEs) studied in the most of these literatures are all defined on the bounded domains.

In [8], the authors considered the stochastic non-autonomous reaction-diffusion equation with multiplicative noise: $du - \Delta u dt + f(u)dt = g(x,t)dt + bu \circ dW(t)$ in $U \times [\tau, +\infty)$, $\tau \in \mathbb{R}$, U is an open bounded set of $\mathbb{R}^3$ and $f(u) = a_0 + a_1u + a_2u^2 + a_3u^3,\ a_3 > 0, a_i \in \mathbb{R}, i = 0,1,2,3$ and $\|g\|^2 = \sup_{t\in\mathbb{R}} \|g(t,\cdot)\|^2 < \infty$.
In [13], the authors investigated the existence of random exponential attractor for the stochastic non-autonomous reaction-diffusion equation with multiplicative noise: $du + (\lambda u - \Delta u)dt = (f(x,u) + g(x,t))dt + \epsilon u \circ dW$, where $t > \tau$, $\tau \in \mathbb{R}$.

Recently, in the case of unbounded domains, the existence of random attractors was established for the autonomous stochastic reaction-diffusion equation with additive noise in [2], and with multiplicative noise in [6].
However, there is no results on random attractors for stochastic non-autonomous reaction-diffusion equation with multiplicative noise on unbounded domain, while it is our concerned.

This paper is organized as follows. In section 2, we recall some basic concepts and properties for general random dynamics system. In section 3, we define a random cocycle through the solution of Eq.(1.1) and give the attraction domain and the properties of Wiener process. In section 4, we prove the existence of random attractors in $L^2(\mathbb{R}^n)$ by tail estimate method and some compact embedding.

In the sequel, we use $\|\cdot\|$ and $(\cdot,\cdot)$ to denote the norm and inner product of $L^2(\mathbb{R}^n)$, respectively.

2. Preliminaries and abstract results

As mentioned in the introduction, our main purpose is to prove the existence of random attractor. For that matter, first, we will recapitulate basic concepts related to random attractors for stochastic dynamical systems. The reader is referred to [1, 10, 17, 19] for more details.

Let both $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be separable Banach spaces, where X is called the initial space which contains all the initial data, and Y is called a terminate space which contains all solutions of a SPDE. Both X and Y are not necessary embedding in any direction, but we need to impose the following hypothesis on them:

(H_1) if $\{x_n\}_n \subset X\bigcap Y$ such that $x_n \stackrel{\|\cdot\|_X}{\longrightarrow} x$ and $x_n \stackrel{\|\cdot\|_Y}{\longrightarrow} y$ respectively, then

$x = y \in X\bigcap Y$.

Let Q be a nonempty set and $(\Omega, \mathcal{F}, P)$ be a probability space . We assume that there are two groups $\{\sigma_t\}_{t\in\mathbb{R}}$ and $\{\vartheta_t\}_{t\in\mathbb{R}}$ over Q and Ω, respectively. Specifically, the mapping $\sigma : \mathbb{R}\times Q \longrightarrow Q$ satisfies that σ_0 is the identity on Q, and $\sigma_{s+t} = \sigma_s \circ \sigma_t$ for all $s, t \in \mathbb{R}$. Similarly , $\vartheta : \mathbb{R}\times\Omega \longrightarrow \Omega$ is a $(\mathcal{B}(\mathbb{R})\times\mathcal{F}, \mathcal{F})$- measurable mapping such that ϑ_0 is the identity on Ω , $\vartheta_{s+t} = \vartheta_s \circ \vartheta_t$ for all $s, t \in \mathbb{R}$ and $\vartheta_t P = P$ for all $t \in \mathbb{R}$. In particular, we call both $(Q, \{\sigma_t\}_{t\in\mathbb{R}})$ and $(\Omega, \mathcal{F}, P, \{\vartheta_t\}_{t\in\mathbb{R}})$ the parametric dynamical system. Let $\mathbb{R}^+ = \{t \in \mathbb{R} : t \geq 0\}$ and 2^X be the collection of all subsets of X.

Definition 2.1. A mapping $\varphi : \mathbb{R}^+\times Q\times\Omega\times X \longrightarrow X, (t, q, \omega, x) \mapsto \varphi(t, q, \omega, x)$ is called a random cocycle on X over $(Q, \{\sigma_t\}_{t\in\mathbb{R}})$ and $(\Omega, \mathcal{F}, P, \{\vartheta_t\}_{t\in\mathbb{R}})$, if for all $q \in Q, \omega \in \Omega$ and $s, t \in \mathbb{R}^+$, the following statements are satisfied :

(i) $\varphi(\cdot, q, \cdot, \cdot) : \mathbb{R}^+\times\Omega\times X \longrightarrow X$ is $(\mathcal{B}(\mathbb{R}^+)\times\mathcal{F}\times\mathcal{B}(X), \mathcal{B}(X))$- measurable.

(ii) $\varphi(0, q, \omega, \cdot)$is the identity on X.

(iii) $\varphi(t + s, q, \omega, \cdot) = \varphi(t, \sigma_s q, \vartheta_s\omega, \cdot) \circ \varphi(s, q, \omega, \cdot)$.

A random cocycle φ is said to be continuous in X if the operator $\varphi(t, q, \omega, \cdot)$ is continuous in X for each $q \in Q, \omega \in \Omega$ and $t \in \mathbb{R}^{+}$.
In the sequel the random cocycle φ on X is further assumed to take its values into the terminate space Y in the following sense:
(H_2) for any $t > 0$, $q \in Q$ and $\omega \in \Omega, \varphi(t, q, \omega, \cdot) : X \longrightarrow Y$.

DEFINITION 2.2. A random cocycle φ is said to be $\mathfrak{D}$-omega-limit compact in X if for each $q \in Q, \omega \in \Omega$ and $D \in \mathfrak{D}$,

$$\lim_{T \to \infty} \mathcal{K}_X(\bigcup_{t \geq T} \varphi(t, \sigma_{-t}q, \vartheta_{-t}\omega, D(\sigma_{-t}q, \vartheta_{-t}\omega))) = 0, \tag{2.1}$$

where $\mathcal{K}_X(\cdot)$ is the Kuratowski non-compact measure such that for $A \subset X$,

$\mathcal{K}_X(\mathcal{A}) = \inf\{d > 0 : \mathcal{A}$ has a finite cover by sets of diameter $\leq d\}$.

Similarly φ is called $\mathfrak{D}$-omega-limit compact in Y if and only if (2.1) holds for the measure $\mathcal{K}_Y(\cdot)$.

We after next recollect the existence theorem of random attractor in both the initial and terminate spaces .The interesting novelty is that in order to obtain a random attractor in Y we do not need the absorption and continuity properties for the random cocycle φ in the terminate space Y. We can refer to [9, 15] for the detailed proof.

PROPOSITION 2.3. *Assume that* φ *is* $\mathfrak{D}$*-omega-limit compact in* $L^2(\mathbb{R}^n)$. *Then* φ *is* $\mathfrak{D}$*-omega-limit compact in* $L^r(\mathbb{R}^r)$ *if for every* $\zeta > 0$, *and each* $q \in Q$, $\omega \in \Omega$ *and* $D \in \mathfrak{D}$, *there exist positive constants* $M = M(q, \omega, \zeta, D)$ *and* $T = T(q, \omega, \zeta, D)$ *such that for all* $t \geq T$,

$$\sup_{u_0 \in D(\sigma_{-t}q, \vartheta_{-t}\omega)} \int_{(|\varphi(t)| \geq M)} |\varphi(t)|^r dx \leq \zeta,$$

where $\varphi(t) =: \varphi(t, \sigma_{-t}q, \vartheta_{-t}\omega, u_0)$.

THEOREM 2.4. *Let* φ *be a continuous random cocycle in* X *over* $(Q, \{\sigma_t\}_{t\in\mathbb{R}})$ *and* $(\Omega, \mathcal{F}, P, \{\vartheta_t\}_{t\in\mathbb{R}})$. *Assume that*
(i) φ *has a closed and measurable* $\mathfrak{D}$*-pullpack bounded absorbing set* $K = \{K(q, \omega) : q \in Q, \omega \in \Omega\} \in \mathfrak{D}$ *in* X.

(ii) φ is $\mathfrak{D}$-omega-limit compact in X. Then the random cocycle φ admite a unique $\mathfrak{D}$-random attractor $\mathcal{A} = \{\mathcal{A}(q,\omega) : q \in Q, \omega \in \Omega\} \in \mathfrak{D}$, whose section is structured by

$$\mathcal{A}(q,\omega) = \bigcap_{s\geq 0} \overline{\bigcup_{t\geq s} \varphi(t, \sigma_{-t}q, \vartheta_{-t}\omega, K(\sigma_{-t}q, \vartheta_{-t}\omega))}^{X} . \tag{2.2}$$

If further (H_1) and (H_2) hold and
(iii) φ is $\mathfrak{D}$-omega-limit compact in Y, then $\mathcal{A}$ whose section is defined by (2.2), *is also a unique $\mathfrak{D}$-random attractor for φ in Y in the sense of Definition 2.4, in* [10].

If $X = L^2$ and $Y = L^r(r > 2)$, then (H_1) holds true automatically , see Lemma 4.1 in [17]. Suppose that φ is a random cocycle on L^2 over $(Q, \{\sigma_t\}_{t\in\mathbb{R}})$ and $(\Omega, \mathcal{F}, P, \{\vartheta_t\}_{t\in\mathbb{R}})$ taking its values into L^r . Then we have a way at hand to check the $\mathfrak{D}$-omega-limit compact for φ in $L^r(r > 2)$, which is called asymptotic a priori estimate technique, see [4].

3. Non-autonomous stochastic reaction-diffusion equation on $\mathbb{R}^n$

In this section, we will give the transformed version of Eq.(1.1) by using a Wiener process, which is rougher (thus has lower regularity)than the Ornstein-Uhlenbeck process. The attraction domain $\mathfrak{D}$, which is larger than the general one, is defined.

The non-autonomous term $g \in L^2_{loc}(\mathbb{R}, L^2(\mathbb{R}^n))$ satisfies, for any $\tau \in \mathbb{R}$ and some
$0 < \delta_0 < \delta = \lambda - \beta_2$,

$$\int_{-\infty}^{\tau} e^{\delta_0 s} \|g(s,\cdot)\|^2_{L^2(\mathbb{R}^n)} ds < +\infty, \tag{3.1}$$

which implies that for any $\tau \in \mathbb{R}$, $\int_{-\infty}^{0} e^{\delta_0 s} \|g(s+\tau,\cdot)\|^2_{L^2(\mathbb{R}^n)} ds < +\infty$. Throughout this paper, we can write that

$$\lambda - \beta_2 > 0, \text{ where } \beta_2 \text{ is given in (1.4)} \tag{3.2}$$

To model the random noise in Eq.(1.1), we need to define a shift operator $\{\vartheta_t\}_{t\in\mathbb{R}}$ on Ω (where Ω is defined in the introduction) by

$\vartheta_t\omega(s) = \omega(s+t) - \omega(t)$ for any $\omega \in \Omega$, $s, t \in \mathbb{R}$. Then ϑ_t is a measure preserving transformation group on $(\Omega, \mathcal{F}, P)$ such that $(\Omega, \mathcal{F}, P, \{\vartheta_t\}_{t\in\mathbb{R}})$ is ergodic in the sense that for every ϑ_t-invariant set $B \in \mathcal{F}$ we have $P(B) = 1$ or $P(B) = 0$, see [19]. By the law of the iterated logarithm (see [22], [23]), there exists a ϑ_t-invariant set $\tilde{\Omega} \subset \Omega$ of full measure such that for any $\omega \in \tilde{\Omega}$,

$$\frac{\omega(t)}{t} \to 0, \ \ as\ |t| \to +\infty. \tag{3.3}$$

By (3.3) we immediately have the following useful results.

Lemma 3.1. [10] *For every $\delta_0 > 0$ and $\alpha \in \mathbb{R}$, we have*

$$e^{-\delta_0 t+\alpha\omega(-t)} \longrightarrow 0,\ e^{-\delta_0 t+\alpha|\omega(-t)|} \longrightarrow 0,\ for\ each\ \omega \in \tilde{\Omega},$$

as $t \to +\infty$.

In the sequel, all statements are understood to hold on a ϑ_t-invariant set $\tilde{\Omega} \subset \Omega$ of full measure such that (3.3) holds, although for convenience we keep the notation Ω for the set $\tilde{\Omega}$.

We then put $Q = \mathbb{R}$ to model the non-autonomous term. Define a family of shift operator $\{\sigma_t\}_{t\in\mathbb{R}}$ on $\mathbb{R}$ by $\sigma_t(\tau) = t + \tau$ for all $t, \tau \in \mathbb{R}$. Then both $(\mathbb{R}, \{\sigma_t\}_{t\in\mathbb{R}})$ and $(\Omega, \mathcal{F}, P, \{\vartheta_t\}_{t\in\mathbb{R}})$ are parametric dynamical systems. For $\tau \in \mathbb{R}$ and $\omega \in \Omega$, let u satisfy Eq.(1.1) and write

$$v(t, \tau, \omega, v_\tau) = e^{-b\omega(t)}u(t, \tau, \omega, u_\tau). \tag{3.4}$$

Then v solves the following equation

$$\frac{dv}{dt} + \lambda v - \Delta v = e^{-b\omega(t)}f(e^{b\omega(t)}v) + e^{-b\omega(t)}g(t, x), \tag{3.5}$$

with initial value

$$v(x, \tau) = v_\tau(x) = e^{-b\omega(\tau)}u_\tau(x). \tag{3.6}$$

By the Galerkin method we can show that if the nonlinear term satisfies (1.3) - (1.6), and the non-autonomous term satisfies (3.1), then for any $v_\tau(x) \in L^2(\mathbb{R}^n)$, Eq.(3.5) possesses a unique weak solution v which is continuous with respect to the initial value $v_\tau(x)$ in $L^2(\mathbb{R}^n)$ for all $t > 0$. Then formally $u = e^{b\omega(t)}v(t)$ is continuous solution to Eq. (1.1) in $L^2(\mathbb{R}^n)$.

Given $t \in \mathbb{R}^+, \tau \in \mathbb{R}, \omega \in \Omega$ and $u_\tau \in L^2(\mathbb{R}^n)$, define

$$\varphi(t, \tau, \omega, u_\tau) = u(t+\tau, \tau, \vartheta_{-\tau}\omega, u_\tau) = e^{b\omega(t)-b\omega(-\tau)} v(t+\tau, \tau, \vartheta_{-\tau}\omega, v_\tau), \tag{3.7}$$

where $u_\tau = e^{-b\omega(-\tau)} v_\tau$. Then it is easy to show that φ is a continuous random cocycle φ associated with Eq. (1.1) on $L^2(\mathbb{R}^n)$ over $(Q, \{\sigma_t\}_{t\in\mathbb{R}})$ and $(\Omega, \mathcal{F}, P, \{\vartheta_t\}_{t\in\mathbb{R}})$.
For the attraction domain, we consider a family $\mathfrak{D} = \{D(\tau, \omega) \subset L^2(\mathbb{R}^n) : \tau \in \mathbb{R}, \omega \in \Omega\}$ such that for each $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$\lim_{t\to+\infty} e^{-\delta t - 2b\omega(-t)} \|D(\tau - t, \vartheta_{-t}\omega)\|^2 = 0, \tag{3.8}$$

where $\delta = \lambda - \beta_2 > 0$ and $\|D\| = \sup\{\|u\| : u \in D\}$. Denote by $\mathfrak{D}$ the collection of families of nonempty bounded subsets of $L^2(\mathbb{R}^n)$ such that (3.8) hold. Then it is obvious that $\mathfrak{D}$ is inclusion closed.

4. Existence of random attractors in $L^2(\mathbb{R}^n)$

In this section, we prove the main results, and we will establish the $\mathfrak{D}$-random attractor in $L^2(\mathbb{R}^n)$ for the random cocycle φ defined in (3.7). The method used here is standard. The key point is to prove the omega-limit compact of solution in $L^2(\mathbb{R}^n)$, which is achieved by the tail estimate method and some compact embedding arguments as in [24].

LEMMA 4.1. *Assume that (1.3)-(1.6) and (3.1)-(3.2) hold. Let* $\tau \in \mathbb{R}, \omega \in \Omega, D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathfrak{D}$ *and* $u_{\tau-t} \in D(\tau - t, \vartheta_{-t}\omega)$. *Then there exists a random constant* $T = T(\tau, \omega, D, b) \geq 1$ *such that for all* $t \geq T$, *the solution* v *of Eq. (3.5) satisfies*

$$\sup_{\varsigma\in[\tau-1,\tau]} \|v(\varsigma, \tau - t, \vartheta_{-\tau}\omega, v_{\tau-t})\|^2 \leq e^{2b\omega(-\tau)} C(\tau, \omega, b), \tag{4.1}$$

$$\int_{\tau-t}^{\tau} e^{\delta_0 s + b(p-2)\omega(s-\tau)} \|v(s, \tau - t, \vartheta_{-\tau}\omega, v_{\tau-t})\|_p^p \leq e^{\delta_0\tau + p\omega(-\tau)} C(\tau, \omega, b), \tag{4.2}$$

where $v_{\tau-t} = e^{-b\omega(-t)+b\omega(-\tau)} u_{\tau-t}$, $\delta = \lambda - \beta_2$ *and*

$$C(\tau, \omega, b) = e^{\delta}(1 + \tfrac{1}{\delta} \textstyle\int_{-\infty}^{0} e^{\delta s - 2b\omega(s)} \|g(s+\tau, \cdot)\|^2 ds) < +\infty.$$

Proof. Let $v = v(t, \tau - t, \vartheta_{-\tau}\omega, v_{\tau-t})$ be the solution of Eq. (3.5) at the sample $\vartheta_{-\tau}\omega$ with the initial value $v_{\tau-t}$. Taking the inner product of Eq.(3.5) with v, we find that

$$\frac{1}{2}\frac{d}{dt}\|v\|^2+\lambda\|v\|^2+\|\nabla v\|^2 = (e^{-b\vartheta_{-\tau}\omega(t)}f(e^{b\vartheta_{-\tau}\omega(t)}v), v)+(e^{-b\vartheta_{-\tau}\omega(t)}g(t,x), v). \tag{4.3}$$

By condition (1.4), we get

$$(e^{-b\vartheta_{-\tau}\omega(t)}f(e^{b\vartheta_{-\tau}\omega(t)}v), v) \leq -\beta_1 e^{b(p-2)\vartheta_{-\tau}\omega(t)}\|v\|_p^p + \beta_2\|v\|^2. \tag{4.4}$$

By the Hölder' inequality and the Young inequality, we have

$$(e^{-b\vartheta_{-\tau}\omega(t)}g(t,x), v) \leq e^{-b\vartheta_{-\tau}\omega(t)}\|g(t,\cdot)\|\cdot\|v\| \leq \frac{1}{2\delta}e^{-2b\vartheta_{-\tau}\omega(t)}\|g(t,\cdot)\|^2+\frac{\delta}{2}\|v\|^2. \tag{4.5}$$

Then inserting (4.4)-(4.5) into (4.3), it leads to

$$\frac{1}{2}\frac{d}{dt}\|v\|^2+\frac{\delta}{2}\|v\|^2+\|\nabla v\|^2+\beta_1 e^{b(p-2)\vartheta_{-\tau}\omega(t)}\|v\|_p^p \leq \frac{1}{2\delta}e^{-2b\vartheta_{-\tau}\omega(t)}\|g(t,\cdot)\|^2, \tag{4.6}$$

where $\delta = \lambda - \beta_2$. Hence, we can rewrite (4.6) as

$$\frac{d}{dt}\|v\|^2 + \delta\|v\|^2 + \beta_1 e^{b(p-2)\vartheta_{-\tau}\omega(t)}\|v\|_p^p \leq \frac{1}{\delta}e^{-2b\vartheta_{-\tau}\omega(t)}\|g(t,\cdot)\|^2. \tag{4.7}$$

By applying the Gronwall's lemma to (4.7) over the interval $[\tau - t, \varsigma]$ with $\varsigma \in [\tau-1, \tau]$, we find that

$$\begin{aligned} &\|v(\varsigma)\|^2 + e^{-\delta\varsigma}\int_{\tau-t}^{\varsigma} e^{\delta s+b(p-2)\vartheta_{-\tau}\omega(s)}\|v(s)\|_p^p ds \\ &\leq e^{\delta}(e^{-\delta t}\|v_{\tau-t}\|^2 + \frac{e^{-\delta\tau}}{\delta}\int_{\tau-t}^{\tau} e^{\delta s-2b\vartheta_{-\tau}\omega(s)}\|g(s,\cdot)\|^2 ds) \\ &\leq e^{\delta}(e^{-\delta t-2b\vartheta_{-\tau}\omega(\tau-t)}\|u_{\tau-t}\|^2+\frac{e^{-\delta\tau}}{\delta}\int_{\tau-t}^{\tau} e^{\delta s-2b\vartheta_{-\tau}\omega(s)}\|g(s,\cdot)\|^2 ds), \end{aligned} \tag{4.8}$$

where we have used $e^{(\tau-\varsigma)\delta} \leq e^{\delta}$ for $\varsigma \in [\tau-1, \tau]$. Using the relation $\vartheta_{-\tau}\omega(s) = \omega(s-\tau) - \omega(-\tau)$, from (4.8) and (3.8) we deduce that for each fixed $\tau \in \mathbb{R}, \omega \in \Omega$ and all $u_{\tau-t} \in D(\tau-t, \vartheta_{-\tau}\omega)$, there exists a

$T = T(\tau, \omega, D, b) \geq 1$ such that for all $t \geq T$,

$$\begin{aligned}&\|v(\varsigma)\|^2 + e^{-\delta\varsigma - b(p-2)\omega(-\tau)} \int_{\tau-t}^{\varsigma} e^{\delta s + b(p-2)\omega(s-\tau)} \|v(s)\|_p^p ds \\ \leq & e^{\delta + 2b\omega(-\tau)} (1 + \frac{1}{\delta} \int_{-\infty}^{0} e^{\delta s - 2b\omega(s)} \|g(s+\tau, \cdot)\|^2 ds).\end{aligned} \tag{4.9}$$

Because of $\delta > \delta_0$, by lemma 3.1, there is a random constant $\rho(\omega) > 0$ such that

$$0 < e^{(\delta-\delta_0)s - 2b\omega(s)} \leq e^{(\delta-\delta_0)s + 2b|\omega(s)|} \leq \rho(\omega) < +\infty, \ \ s \in (-\infty, 0]. \tag{4.10}$$

Then the integral on the right hand side of (4.9) is bounded by

$$\int_{-\infty}^{0} e^{(\delta-\delta_0)s - 2b\omega(s)} e^{\delta_0 s} \|g(s+\tau, \cdot)\|^2 ds \leq \rho(\omega) \int_{-\infty}^{0} e^{\delta_0 s} \|g(s+\tau, \cdot)\|^2 ds < +\infty,$$

which is finite by (3.1). This completes the proof. □

We present a Gronwall-type lemma which is convenient tool for the subsequential discussion.

LEMMA 4.2. [10] *Let y and h be two locally integrable functions on $\mathbb{R}$ such that $\frac{dy}{dt}$ is also locally integrable and*

$\frac{dy(t)}{dt} + \nu y(t) \leq h(t)$, *for $t \in \mathbb{R}$, and some $\nu \in \mathbb{R}$.*

Then for any $r > 0$ and $\tau \in \mathbb{R}$.

$$y(\tau + r) \leq \frac{1}{r} \int_{\tau}^{\tau+r} e^{\nu(s-\tau-r)} y(s) ds + \int_{\tau}^{\tau+r} e^{\nu(s-\tau-r)} h(s) ds.$$

In particular, if $\nu = 0$ then

$$y(\tau + r) \leq \frac{1}{r} \int_{\tau}^{\tau+r} y(s) ds + \int_{\tau}^{\tau+r} h(s) ds.$$

LEMMA 4.3. *Assume that (1.3)-(1.6) and (3.1)-(3.2) hold. Let $\tau \in \mathbb{R}, \omega \in \Omega, D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathfrak{D}$ and $u_{\tau-t} \in D(\tau - t, \vartheta_{-t}\omega)$. Then there exists a random constant $L(\tau, \omega, b)$ and $T = T(\tau, \omega, D, b) \geq 2$ such that for all $t \geq T$ the solution v of Eq. (3.5) satisfies*

$$\sup_{\varsigma \in [\tau-1, \tau]} \|\nabla v(\varsigma, \tau - t, \vartheta_{-t}\omega, v_{\tau-t})\|^2 \leq L(\tau, \omega, b). \tag{4.11}$$

Proof. Let $v = v(t, \tau - t, \vartheta_{-\tau}\omega, v_{\tau-t})$ be the solution of Eq. (3.5) at the sample $\vartheta_{-\tau}\omega$ with the initial value $v_{\tau-t}$. Multiplying Eq.(3.5) with $|v|^{p-2}v$ and integrating over $\mathbb{R}^n$, noticing that $(-\Delta v, |v|^{p-2}v) > 0$ for any $p \geq 2$, then by using the Hölder' inequality and the Young inequality and condition (1.4) it is easy to calculate that

$$\frac{1}{p}\frac{d}{dt}\|v\|_p^p + \lambda\|v\|_p^p = e^{-b\vartheta_{-\tau}\omega(t)}((f(u), |v|^{p-2}v) + (g(t, \cdot), |v|^{p-2}v))$$

$$\leq -\frac{\beta_1}{2}e^{b(p-2)\vartheta_{-\tau}\omega(t)}\|v\|_{2p-2}^{2p-2} + \beta_2\|v\|_p^p + \frac{1}{2\beta_1}e^{-bp\vartheta_{-\tau}\omega(t)}\|g(t, \cdot)\|^2, \quad (4.12)$$

which obviously gives

$$\frac{d}{dt}\|v\|_p^p+\lambda\|v\|_p^p+\frac{\beta_1}{2}e^{b(p-2)\vartheta_{-\tau}\omega(t)}\|v\|_{2p-2}^{2p-2} \leq \beta_2 p\|v\|_p^p+\frac{p}{2\beta_1}e^{-bp\vartheta_{-\tau}\omega(t)}\|g(t, \cdot)\|^2. \quad (4.13)$$

In (4.13), by using Lemma 4.2, over the interval $[\tau - 2, \varsigma]$ with $\varsigma \in [\tau - 1, \tau]$, we find that

$$\|v(\varsigma)\|_p^p + \frac{\beta_1}{2}e^{-\lambda\varsigma}\int_{\tau-2}^{\varsigma} e^{\lambda s+b(p-2)\vartheta_{-\tau}\omega(s)}\|v(s)\|_{2p-2}^{2p-2}ds$$

$$\leq e^{-\lambda\varsigma}((\frac{1}{\varsigma - \tau + 2}+\beta_2 p)\int_{\tau-2}^{\tau} e^{\lambda s}\|v(s)\|_p^p ds+\frac{p}{2\beta_1}\int_{\tau-2}^{\tau} e^{\lambda s-bp\vartheta_{-\tau}\omega(s)}\|g(s, \cdot)\|^2 ds)$$

$$\leq e^{-\lambda\varsigma}((1 + \beta_2 p)\int_{\tau-2}^{\tau} e^{\lambda s}\|v(s)\|_p^p ds + \frac{p}{2\beta_1}\int_{\tau-2}^{\tau} e^{\lambda s-bp\vartheta_{-\tau}\omega(s)}\|g(s, \cdot)\|^2 ds), \quad (4.14)$$

where we have used $\frac{1}{\varsigma-\tau+2} \leq 1$ for each $\varsigma \in [\tau - 1, \tau]$. Since $\omega(\cdot)$ is continuous on $\mathbb{R}$, using (4.2), there exists a random constant $T \geq 2$ as given in lemma 4.1, such that

$$\sup_{t\geq T}\int_{\tau-2}^{\tau} \|v(s)\|_p^p ds \leq L(\tau, \omega, b), \quad (4.15)$$

where and in the following $0 < L(\tau, \omega, b) < +\infty$ is a random constant independent of t, v, u and may possess different values in different place even in the same line. Associated with $g \in L^2_{loc}(\mathbb{R}, L^2(\mathbb{R}^n))$, it follows from (4.14) and (4.15) that

$$\sup_{t>T}\sup_{\varsigma\in[\tau-1,\tau]} \|v(\varsigma)\|_p^p \leq L(\tau, \omega, b), \quad (4.16)$$

and

$$\sup_{t>T}\int_{\tau-2}^{\tau}\|v(s)\|_{2p-2}^{2p-2}ds \leq L(\tau,\omega,b). \tag{4.17}$$

Taking the inner product of Eq. (3.5) with $-\Delta v$ in $L^2(\mathbb{R}^n)$, we find that

$$\frac{1}{2}\frac{d}{dt}\|\nabla v\|^2+\lambda\|\nabla v\|^2+\|\Delta v\|^2 = e^{-b\vartheta_{-\tau}\omega(t)}(f(u),-\Delta v)+e^{-b\vartheta_{-\tau}\omega(t)}(g(t,\cdot),-\Delta v) \tag{4.18}$$

By the Hölder' inequality and the Young inequality, we have

$$e^{-b\vartheta_{-\tau}\omega(t)}(g(t,\cdot),-\Delta v) \leq \frac{1}{2}e^{-b\vartheta_{-\tau}\omega(t)}\|g(t,\cdot)\|^2+\frac{1}{2}\|\Delta v\|^2, \tag{4.19}$$

and

$$e^{-b\vartheta_{-\tau}\omega(t)}(f(u),-\Delta v) \leq \frac{1}{2}\|e^{-b\vartheta_{-\tau}\omega(t)}f(u)\|^2+\frac{1}{2}\|\Delta v\|^2. \tag{4.20}$$

Then inserting (4.19)-(4.20) into (4.18), it leads to

$$\frac{d}{dt}\|\nabla v\|^2 \leq \|e^{-b\vartheta_{-\tau}\omega(t)}f(u)\|^2+\|e^{-b\vartheta_{-\tau}\omega(t)}g(t,\cdot)\|^2. \tag{4.21}$$

Then by (1.5), we get

$$\frac{d}{dt}\|\nabla v\|^2 \leq \beta_4\|v\|^2+L(\tau,\omega,b)(\|v\|_{2p-2}^{2p-2}+\|g(t,\cdot)\|^2),\ t\in[\tau-2,\tau]. \tag{4.22}$$

We apply Lemma 4.2 (the case $\nu=0$) to (4.22) over the interval $[\tau-1,\varsigma]$ with $\varsigma\in[\tau-1,\tau]$we get

$$\|\nabla v(\varsigma)\|^2 \leq L(\tau,\omega,b)\int_{\tau-1}^{\tau}(\|\nabla v(s)\|^2+\|v(s)\|_{2p-2}^{2p-2}+\|g(s,\cdot)\|^2)ds+\beta_4\int_{\tau-1}^{\tau}\|v(s)\|^2ds. \tag{4.23}$$

Then combining (4.23) with (4.1) and (4.17), we deduce that there exists a random constant $T\geq 2$ such that

$$\sup_{\varsigma\in[\tau-1,\tau]}\|\nabla v(\varsigma,\tau-t,\vartheta_{-t}\omega,v_{\tau-t})\|^2 \leq L(\tau,\omega,b),\ for\ t\geq T. \tag{4.24}$$

This completes the proof . □

Lemma 4.4. *Assume that (1.3)-(1.6) and (3.1)-(3.2) hold. Let $\tau\in\mathbb{R},\omega\in\Omega$, and $D=\{D(\tau,\omega):\tau\in\mathbb{R},\omega\in\Omega\}\in\mathfrak{D}$ and $u_{\tau-t}\in D(\tau-t,\vartheta_{-t}\omega)$. Then for any $\zeta>0$, there exist random constants*

$R = R(\tau, \omega, \zeta, b)$ and $T = T(\tau, \omega, \zeta, D, b) \geq 2$ such that the solution u of Eq. (1.1) satisfies

$$\sup_{t \geq T} \int_{Q_R^c} |u(\tau, \tau - t, \vartheta_{-\tau}\omega, u_{\tau - t})|^2 dx \leq \zeta,$$

where $Q_R^c = \mathbb{R}^n - Q_R$ and Q_R is the ball of $\mathbb{R}^n$ centred zero with radius R.

Proof. We first need to define a smooth function $\sigma(\cdot)$ from $\mathbb{R}^+$ into $[0, 1]$ such that $\sigma(\cdot) = 0$ on $[0, 1]$ and $\sigma(\cdot) = 1$ on $[2, +\infty)$, which evidently implies that there is a positive constant c such that the $|\sigma'(s)| \leq c$ for all $s \geq 0$. For convenience, we write $\sigma_\kappa = \sigma(\frac{|x|^2}{\kappa^2})$.
Multiplying Eq. (3.5) with $\sigma_\kappa v$ and integrating over $\mathbb{R}^n$, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n} \sigma_\kappa |v|^2 dx + \lambda \int_{\mathbb{R}^n} \sigma_\kappa |v|^2 dx \\ &= \int_{\mathbb{R}^n} (\Delta v)\sigma_\kappa v dx + e^{-b\omega(t)}(\int_{\mathbb{R}^n} \sigma_\kappa f(u) v dx + \int_{\mathbb{R}^n} \sigma_\kappa g v dx), \end{aligned} \tag{4.25}$$

where

$$\begin{aligned} \int_{\mathbb{R}^n} (\Delta v)\sigma_\kappa v dx &= -\int_{\mathbb{R}^n} |\nabla v|^2 \sigma_\kappa dx - \int_{\mathbb{R}^n} v\sigma'_\kappa \frac{2x}{\kappa^2}(\nabla v) dx \\ &\leq -\int_{\mathbb{R}^n} |\nabla v|^2 \sigma_\kappa dx - \int_{\kappa \leq |x| \leq \sqrt{2}\kappa} v\sigma'_\kappa \frac{2x}{\kappa^2}(\nabla v) dx \\ &\leq -\int_{\mathbb{R}^n} |\nabla v|^2 \sigma_\kappa dx + \frac{2\sqrt{2}}{\kappa}\int_{\kappa \leq |x| \leq \sqrt{2}\kappa} |v| \cdot |\sigma'_\kappa| \cdot |\nabla v| dx \\ &\leq -\int_{\mathbb{R}^n} |\nabla v|^2 \sigma_\kappa dx + \frac{2\sqrt{2}\, c}{\kappa}\int_{\mathbb{R}^n} |v| \cdot |\nabla v| dx \\ &\leq -\int_{\mathbb{R}^n} |\nabla v|^2 \sigma_\kappa dx + \frac{C_0}{\kappa}(\|v\|^2 + \|\nabla v\|^2), \end{aligned} \tag{4.26}$$

where C_0 is a positive constant.
By condition (1.4), we get

$$e^{-b\omega(t)}\int_{\mathbb{R}^n} \sigma_\kappa f(u) v dx \leq -\beta_1 e^{b(p-2)\omega(t)}\int_{\mathbb{R}^n} \sigma_\kappa |v|^p dx + \beta_2 \int_{\mathbb{R}^n} \sigma_\kappa |v|^2 dx. \tag{4.27}$$

For the last term in (4.25), we obtain

$$e^{-b\omega(t)}\int_{\mathbb{R}^n}\sigma_\kappa gvdx \le \frac{\delta}{2}\int_{\mathbb{R}^n}\sigma_\kappa|v|^2dx+\frac{1}{2\delta}e^{-2b\omega(t)}\int_{\mathbb{R}^n}\sigma_\kappa|g(t,\cdot)|^2dx. \tag{4.28}$$

Then inserting (4.26) - (4.28) into (4.25), it leads to

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}^n}\sigma_\kappa|v|^2dx+\delta\int_{\mathbb{R}^n}\sigma_\kappa|v|^2dx+\beta_1e^{b(p-2)\omega(t)}\int_{\mathbb{R}^n}\sigma_\kappa|v|^pdx\\ &\le\frac{C_0}{\kappa}(\|v\|^2+\|\nabla v\|^2)+\frac{1}{\delta}e^{-2b\omega(t)}\int_{|x|\ge\kappa}|g(t,\cdot)|^2dx,\end{aligned} \tag{4.29}$$

where $\delta=\lambda-\beta_2$. By using the Gronwall's lemma to (4.29) over the interval $[\tau-t,\tau]$ to show that

$$\begin{aligned}&\int_{\mathbb{R}^n}\sigma_\kappa|v(\tau)|^2dx+\beta_1\int_{\tau-t}^{\tau}e^{\delta(s-\tau)+b(p-2)\vartheta_{-\tau}\omega(s)}\int_{\mathbb{R}^n}\sigma_\kappa|v(s)|^pdxds\\ &\le e^{-\delta t}\int_{\mathbb{R}^n}\sigma_\kappa|v_{\tau-t}|^2dx+\frac{C_0}{\kappa}\int_{\tau-t}^{\tau}e^{\delta(s-\tau)}(\|v(s)\|^2+\|\nabla v(s)\|^2)ds\\ &\qquad+\frac{1}{\delta}\int_{\tau-t}^{\tau}e^{\delta(s-\tau)-2b\vartheta_{-\tau}\omega(s)}\int_{|x|\ge\kappa}|g(s,\cdot)|^2dxds.\end{aligned} \tag{4.30}$$

We estimate every term on the right hand side of (4.30). First, by (3.8) for t large enough, we have

$$e^{-\delta t}\int_{\mathbb{R}^n}\sigma_\kappa|v_{\tau-t}|^2dx\le e^{-\delta t-2b\vartheta_{-\tau}\omega(\tau-t)}\|u_{\tau-t}\|^2\le\frac{\zeta}{3}. \tag{4.31}$$

From (4.1) and (4.11), for κ and t large enough , it follows that

$$\frac{C_0}{\kappa}\int_{\tau-t}^{\tau}e^{\delta(s-\tau)}(\|v\|^2+\|\nabla v\|^2)ds\le\frac{K(\tau,\omega,b)}{\kappa}\le\frac{\zeta}{3}. \tag{4.32}$$

For the last term, we have

$$\begin{aligned}&\frac{1}{\delta}\int_{\tau-t}^{\tau}e^{\delta(s-\tau)-2b\vartheta_{-\tau}\omega(s)}\int_{|x|\ge\kappa}|g(s,\cdot)|^2dxds\\ =&\frac{e^{2b\omega(-\tau)}}{\delta}\int_{-\infty}^{0}e^{\delta s-2b\omega(s)}\int_{|x|\ge\kappa}|g(s+\tau,\cdot)|^2dxds\end{aligned}$$

(by (4.10)) $\le \frac{e^{2b\omega(-\tau)}\rho(\omega)}{\delta}\int_{-\infty}^{0}e^{\delta_0 s}\int_{|x|\ge\kappa}|g(s+\tau,\cdot)|^2dxds$

(by (3.1)) $\le\frac{\zeta}{3}$. (4.33)

For κ large enough. Then it follows from (4.30) to (4.33) that for t and κ large enough ,

$$\int_{|x|\geq\sqrt{2\kappa}} |v(\tau)|^2 dx = \int_{\mathbb{R}^n} \sigma_\kappa |v(\tau)|^2 dx \leq \zeta.$$

Which and along with (3.4) imply the desired result. □

THEOREM 4.5. *Suppose that (1.3)-(1.6) and (3.1)-(3.2)hold. Let $\mathfrak{D}$ be defined in (3.8). Then the random cocycle φ associated with Eq. (1.1) possesses a unique $\mathfrak{D}$-random attractor $\mathcal{A} = \{\mathcal{A}(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ in the initial space $L^2(\mathbb{R}^n)$.*

Proof. We use the method of [10] to prove our main results. Given $\tau \in \mathbb{R}, \omega \in \Omega$ and $D \in \mathfrak{D}$, define

$$G_T(\tau,\omega) =: \bigcup_{t\geq T} \varphi(t, \tau - t, \vartheta_{-t}\omega, D(\tau - t, \vartheta_{-t}\omega)).$$

Let $\zeta > 0$. From (4.1), there exist $T_1 = T_1(\tau,\omega,D,b)$ and a ball $B_{L^2(\mathbb{R}^n)}(0, C(\tau,\omega,b))$ centred at zero with radium $C(\tau,\omega,b)$ such that

$$G_{T_1}(\tau,\omega) \subset B_{L^2(\mathbb{R}^n)}(0, C(\tau,\omega,b)).$$

We can use the compact sobolev embedding due to Lemma 4.3 in the bounded domain, and thus for any $\zeta > 0$ there exist a finite $\zeta/4 - net$ in $L^2(Q_R)$ covering $G_{T_1}(\tau,\omega)|_{Q_R}$. Therefore

$$\kappa_{L^2}(G_{T_1}(\tau,\omega)|_{Q_R}) \leq \frac{\zeta}{2},$$

where $\kappa_{L^2}(\cdot)$ is non-compact measure in $L^2(Q_R)$. On the other hand, by Lemma 4.4, there exist $T_2 = T_2(\tau,\omega,\zeta,D,b)$ and $R(\tau,\omega,\zeta,b)$ such that

$$G_{T_2}(\tau,\omega)|_{Q_R^c} \subset B_{L^2(\mathbb{R}^n)}(0, \frac{\zeta}{4}).$$

$T = max\{T_1, T_2\}$. Then by additive property of non-compact measure, we have

$$\begin{aligned}\kappa_{L^2}(G_T(\tau,\omega)) &\leq \kappa_{L^2}(G_{T_1}(\tau,\omega)|_{Q_R}) + \kappa_{L^2}(G_{T_2}(\tau,\omega)|_{Q_R^c}) \\ &\leq \frac{\zeta}{2} + \kappa(B_{L^2(\mathbb{R}^n)}(0, \frac{\zeta}{4})) \leq \frac{\zeta}{2} + \frac{\zeta}{2} = \zeta. \qquad (4.34)\end{aligned}$$

Thus by the arbitrariness of ζ, φ is omega-limit compact in $L^2(\mathbb{R}^n)$.

Considering $v(\tau,\tau-t,\vartheta_{-\tau}\omega,v_{\tau-t}) = e^{b\omega(-\tau)}u(\tau,\tau-t,\vartheta_{-\tau}\omega,u_{\tau-t})$ in (3.4). Then by (4.1) it follows that

$$\|u(\tau,\tau-t,\vartheta_{-\tau}\omega,u_{\tau-t})\|^2 \le C(\tau,\omega,b) =: e^{\delta}(1+\frac{1}{\delta}\int_{-\infty}^{0} e^{\delta s-2b\omega(s)}\|g(s+\tau,\cdot)\|^2 ds). \tag{4.35}$$

Notice that

$$\frac{C(\tau-t,\vartheta_{-\tau}\omega,\omega,b)}{e^{\delta}} = 1+\frac{1}{\delta}\int_{-\infty}^{0} e^{\delta s-2b\vartheta_{-\tau}\omega(s)}\|g(s+\tau-t,\cdot)\|^2 ds$$

$$= 1+\frac{e^{2b\omega(-t)}}{\delta}\int_{-\infty}^{0} e^{\delta s-2b\omega(s-t)}\|g(s+\tau-t,\cdot)\|^2 ds$$

$$= 1+\frac{e^{2b\omega(-t)+\delta t}}{\delta}\int_{-\infty}^{-t} e^{\delta s-2b\omega(s)}\|g(s+\tau,\cdot)\|^2 ds. \tag{4.36}$$

Then by (4.10), into (4.36) we get

$$\frac{C(\tau-t,\vartheta_{-\tau}\omega,\omega,b)}{e^{\delta}} \le 1+\frac{\rho(\omega)(e^{2b\omega(-t)+\delta t})}{\delta}\int_{-\infty}^{-t} e^{\delta_0(s)}\|g(s+\tau,\cdot)\|^2 ds. \tag{4.37}$$

Lemma 3.1, implies that $\lim_{t\longrightarrow+\infty} e^{-\delta t-2b\omega(-t)} = 0$, whence by (3.1) and (4.37) we conclude that for any $\tau\in\mathbb{R}$ and $\omega\in\Omega$,

$$\lim_{t\longrightarrow+\infty} e^{-\delta t-2b\omega(-t)}C(\tau-t,\vartheta_{-\tau}\omega,\omega,b) = 0\ .$$

Thus from (3.8) we have

$$K = \{K(\tau,\omega) = \{u\in L^2(\mathbb{R}^n) : \|u\|^2 \le C(\tau,\omega,b)\} : \tau\in\mathbb{R},\omega\in\Omega\} \in \mathfrak{D}. \tag{4.38}$$

The measurability of the absorbing set $K(\tau,\omega)$ follows from the measurability of the variable $C(\tau,\omega,b)$. Therefore by the first part of Theorem 2.4, the random cocycle φ of Eq.(1.1) possesses a unique compact random attractor $\mathcal{A} = \{\mathcal{A}(\tau,\omega) : \tau\in\mathbb{R},\omega\in\Omega\}$ in $L^2(\mathbb{R}^n)$. This completes the proof. □

Remark 4.6. In this paper, we study the multiplicative case for stochastic non-autonomous reaction-diffusion equations on unbounded domains. The additive noises are considered in the forthcoming article.

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Fadlallah Mustafa Mosa
[1] College of Mathematics and Statistics
Northwest Normal University
Lanzhou, Gansu 730070, China

[2] Department of Mathematics and physics, Faculty of Education
University of Kassala
Kassala, Sudan
E-mail: `fadlmushan@hotmail.com`

Qiaozhen Ma
College of Mathematics and Statistics
Northwest Normal University
Lanzhou, Gansu 730070, China
E-mail: `maqzh@nwnu.edu.cn`

Mohamed Y. A. Bakhet
[1] College of Mathematics and Statistics
Northwest Normal University
Lanzhou, Gansu 730070, China

[2] Department of Mathematics, College of Education
Rumbek University of Science and Technology
Rumbek, South Sudan
E-mail: `myabakhet@hotmail.com`