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ABSOLUTE CONTINUITY OF THE MAGNETIC SCHRÖDINGER OPERATOR WITH PERIODIC POTENTIAL

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Abstract. We consider the magnetic Schrödinger operator coupled with two different potentials. One of them is a harmonic oscillator and the other is a periodic potential. We give some periodic potential classes for which the operator has purely absolutely continuous spectrum. We also prove that for strong magnetic field or large coupling constant, there are open gaps in the spectrum and we give a lower bound on their number.

1. Introduction

The Hamiltonian of an electron confined to the two dimensional space $\mathbb{R}^2$ under the action of a magnetic field coupled to a harmonic oscillator is given by the following Schrödinger operator P_0 acting on $L^2(\mathbb{R}^2)$ by

$$P_0\varphi = \left((D_x - A_1(x,y))^2 + (D_y - A_2(x,y))^2\right)\varphi + \lambda x^2\varphi, \quad \forall\varphi \in L^2(\mathbb{R}^2). \tag{1}$$

The constant λ is positive and $(D_x, D_y) = \left(-i\dfrac{\partial}{\partial x}, -i\dfrac{\partial}{\partial y}\right)$. The magnetic field is a associated to the magnetic potential $\vec{A} = (A_1, A_2)$. We assume that the magnetic field has a constant intensity $B > 0$ and we make

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a Landau gauge such that $A_2(x,y) = -Bx$ et $A_1(x,y) = 0, \forall (x,y) \in \mathbb{R}^2$. In this work, we are interested in the nature of the spectrum of some periodic perturbations of P_0. For $\varepsilon > 0$, we let $P_\varepsilon = P_0 + \varepsilon V$ where V is a periodic function on $\mathbb{R}^2$. More precisely, let $a = (a_1, a_2) \in \mathbb{R}^2$ and $\Gamma = (\mathbb{Z}a_1, \mathbb{Z}a_2) = \{(k_1 a_1, k_2 a_2), (k_1, k_2) \in \mathbb{Z}^2\}$ the periodicity lattice of V. The spectrum of the unperturbed operator P_0 is $\sigma(P_0) = [\sqrt{B^2 + \lambda}, +\infty[$. It is purely absolutely continuous for all $B > 0$. Our aim is to study the stability of the absolute continuity of the spectrum when we add the periodic potential. The existence of an absolutely continuous component in the spectrum of a given operator is important in the scattering theory and the propagation phenomena for the evolution problem related to it. The absence of singular continuous spectrum for such operators was considered, for example, in [8]. The question of stability of the continuous spectrum was considered in different contexts. In [3], the authors studied the case of a straight waveguide of width L identified to the strip $[-\frac{L}{2}, \frac{L}{2}] \times \mathbb{R}$ and periodic in the unbounded transverse direction. They proved that for L large enough, the absolute continuity of the spectrum is not conserved and that some localized states are created near the Landau levels. These correspond to eigenvalues of finite multiplicities. The coupling between the magnetic field and the potential is made through the parameter L. In [1], the paper deals with a constant magnetic field B and a bounded periodic potential. It was proved that if the absolutely continuous spectrum has a gap for some value B_0 of B then this gap remains open for $|B - B_0|$ small enough. In [4] and [5] the absolute continuity of the spectrum was proved for the magnetic Schrödinger operator with a periodic potential that satisfies some symmetry condition.

In this paper we deal with a magnetic field coupled with two potentials. One potential is periodic and the other is a harmonic oscillator that acts as a confining in a prescribed direction in the space $\mathbb{R}^2$. We study the cases when the periodic potential is weak and when it does not depend on one of variables x and y, so it is confining in the direction of the harmonic oscillator or parallel to it. In all the cases, we prove that at least one component of the spectrum is absolutely continuous. We also give a result on the existence of open gaps in the spectrum and on their number and stability when the magnetic field is varying. Our results may be considered as a generalization of those in [1] and [5] to some different configurations. In [1], the magnetic field is coupled to a bounded potential, while in our context the potential is $\lambda x^2 + \varepsilon V(x,y)$

which does not belong to $L^\infty(\mathbb{R}^2)$ when $\lambda \neq 0$. Moreover, our results on the stability of the absolute continuity of the spectrum can be useful in the study of the scattering. Namely, this means that the scattering is possible in periodic structures that interact with a unbounded potential having a parabolic confining property.

The paper is organized as follows. In section 2, we introduce the Floquet reduction method and we make an explicit computation of the spectrum of fibre operators. In section 3, we study two types of the periodic perturbations V. We prove that if the potential is independent of the variable transverse to the harmonic oscillator then, for all $\mu \in \mathbb{R}$, the part of the spectrum in $]-\infty, \mu]$ is absolutely continuous whenever $\|V\|_\infty$ is small enough.

Then we consider the potentials that depend on the transverse variable only. We prove that in this case, the spectrum is absolutely continuous. In section 4, we investigate the more general situation. We demonstrate that for ε small enough, the bottom of the spectrum is absolutely continuous. We also give a result on the number of open gaps in the spectrum of P_ε when the magnetic field is strong or the coupling with the harmonic oscillator is large.

2. Floquet theory

Let V be a real valued function defined on $\mathbb{R}^2$. We assume that V is bounded and periodic with respect to the variable y and without loss of generality, we assume that

$$V(x, y+1) = V(x, y) \text{ for all } (x, y) \in \mathbb{R}^2. \tag{2}$$

For $\varepsilon \geq 0$, we consider the operator $P_\varepsilon = P_0 + \varepsilon V$ where P_0 is the unbounded operator on $L^2(\mathbb{R}^2)$ given by

$$P_0 = -\partial_x^2 + (-i\partial_y + Bx)^2 + \lambda x^2, \quad \text{with } (\partial_x \varphi, \partial_y \varphi) = \left(\frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial y}\right).$$

The magnetic field B is assumed to be constant with an intensity $B > 0$. It is associated to an magnetic potential A using a Landau gauge such that $A(x, y) = (0, -Bx)\ \forall (x, y) \in \mathbb{R}^2$. Using a Fourier transform in the $y-$variable and a decomposition on the harmonic oscillator modes in the variable x, the spectrum of P_0 is known to be purely absolutely continuous.

Proposition 2.1. *For $B > 0$ and $\lambda > 0$, the spectrum of P_0 is* $\sigma(P_0) = [\sqrt{B^2+\lambda}, +\infty[$.

We are interested in the spectrum of P_ε. This latter is an essentially selfadjoint operator on, $\mathcal{C}_0^\infty(\mathbb{R}^2, \mathbb{R})$, the space of compactly supported $C^\infty-$functions on $\mathbb{R}^2$. The periodicity of the potential V allows us to use the Floquet-Bloch reduction. Namely, we denote by $\Gamma^* := 2\pi\mathbb{Z}$ the dual lattice of $\mathbb{Z}$ and we let $\mathbb{E} = [-\pi, \pi]$ the elementary cell of Γ^*. Thus, we can write P_ε as a direct integral over $\mathbb{E}$ in the following way :

$$P_\varepsilon = \int_{\mathbb{E}}^{\oplus} P_\varepsilon(\xi) d\xi$$

where for every $\xi \in \mathbb{E}$, the fibre operator $P_\varepsilon(\xi)$ is acting on $L^2(\mathbb{R} \times [0,1])$ by

(3)
$$\begin{cases} P_\varepsilon(\xi)\varphi(x,y) = \left((-i\partial_y + Bx + \xi)^2 - \partial_x^2 + \lambda x^2 + \varepsilon V(x,y)\right)\varphi(x,y) \\ \varphi(x,0) = \varphi(x,1), \forall x \in \mathbb{R} \\ \dfrac{\partial\varphi}{\partial y}(x,0) = \dfrac{\partial\varphi}{\partial y}(x,1), \forall x \in \mathbb{R} \end{cases}$$

for $\varphi \in L^2(\mathbb{R} \times [0,1])$. Let us remark that for every $\xi \in \mathbb{E}$, we have $P_\varepsilon = P_0(\xi) + \varepsilon V$. Thus when ε is small one can use the perturbation theory techniques to find the spectral properties of P_ε form those of P_0. We first recall the definition of holomorphic families of operators in the sense of Kato. We refer to ([7], Ch. VII) for details and extended properties.

Definition 2.2. A family $(T(\xi))_{\xi\in U}$ of operators defined for ξ in a domain U of the complex plane is said to be a holomorphic family of type **(A)** (or a type **(A)** family) if :

(i) there exists a domain $\mathcal{D}$ independent of ξ such that $\mathcal{D} = D(T(\xi))$, for all $\xi \in U$.
(ii) the mapping $\xi \mapsto T(\xi)u$ is holomorphic on U for every $u \in \mathcal{D}$.

Definition 2.3. Let U be a domain in the complex plane symmetric with respect to the real axis. A type **(A)** holomorphic family $(T(\xi))_{\xi\in U}$ is said to be a self-adjoint family of type **(A)** if $T(\xi)$ is densely defined and $T(\xi)^* = T(\bar{\xi})$ for every $\xi \in U$.
Let us note that when $(T(\xi))_{\xi\in U}$ is a selfadjoint holomorphic family of type **(A)** then $T(\xi)$ is selfadjoint for every $\xi \in U \cap \mathbb{R}$.

PROPOSITION 2.4. *Let $V \in L^\infty(\mathbb{R}^2)$. The family $(P_\varepsilon(\xi))_{\xi\in\mathbb{C}}$ is a selfadjoint holomorphic family of type* **(A)** *on the domain*

$$\mathcal{D} = \mathcal{S}(\mathbb{R})\otimes\left\{u \in L^2([0,1]) |\ u', u'' \in L^2([0,1]), u(0) = u(1), u'(0) = u'(1)\right\}.$$

Here $\mathcal{S}(\mathbb{R})$ is the Schwartz space.

Proof. The potential V is bounded. The multiplication by V is a relatively bounded operator with respect to $P_0(\xi)$. Thus we need only to prove that $P_0(\xi)$ is selfadjoint. For this we write

$$\begin{aligned} P_0(\xi) &= (i\partial_y + Bx)^2 + 2\xi(-i\partial_y + Bx) + \xi^2 + \lambda x^2 - \partial_x^2 \\ &= P_0(0) + 2\xi(-i\partial_y + Bx) + \xi^2. \end{aligned} \tag{4}$$

The operator $P_0(0)$ is selfadjoint on the domain $\mathcal{D}$. We set $S = 2\xi(-i\partial_y + Bx) + \xi^2$ so that $P_0(\xi) = P_0(0) + S$. We claim that S is symmetric and relatively bounded with respect to $P_0(0)$. Let $R_0(z)$ be the resolvent of $P_0(0)$ defined for $z \in \mathbb{C} \setminus (\sigma(P_0(0)))$. We have:

$$\begin{aligned} ||S\varphi||^2 &= ||(2\xi(-i\partial_y + Bx) + \xi^2)\varphi||^2 \\ &\le 4|\xi|^2||(-i\partial_y + Bx)\varphi||^2 + |\xi|^4||\varphi||^2 \\ &\le 4|\xi|^2 \langle \varphi, P_0(0)\varphi\rangle + |\xi|^4||\varphi||^2 \\ &\le 4|\xi|^2 \left(||R_0(z)||||P_0(0)\varphi||^2 + |z| \langle \varphi, R_0(\bar{z})P_0(0)\varphi\rangle\right) + |\xi|^4||\varphi||^2 \\ &\le C(z)||P_0(0)\varphi||^2 + (C(z)|z|^2 + |\xi|^4)||\varphi||^2. \end{aligned}$$

The constant $C(z)$ satisfies $C(z) = O\left(\frac{1}{\Im(z)}\right)$ when $\Im(z) \longrightarrow +\infty$. Taking $|\Im(z)|$ large enough we can choose $C(z) < 1$ and then we deduce that $P_0(\xi)$ has the same domain as $P_0(0)$. Therefore $(P_0(\xi))_{\xi\in\mathbb{C}}$ is a type **(A)** holomorphic family on $\mathbb{C}$. Using Theorem 4.3 in [7], the operator $P_0(\xi)$ is selfadjoint on $\mathcal{D}$. □

Using the fact that the operator of multiplication by εV is relatively bounded with respect to $P_0(\xi)$ we deduce that the essential spectrum of $P_\varepsilon(\xi) = P_0(\xi) + \varepsilon V$ is the same as the essential spectrum of $P_0(\xi)$ for ξ real.

In the following we compute the spectrum of $P_0(\xi)$.

PROPOSITION 2.5. *Let $\xi \in \mathbb{E}$. The spectrum of $P_0(\xi)$ is pure point and given by*

$$\sigma\left(P_0(\xi)\right) = \left\{\frac{\lambda}{\lambda + B^2}(2\pi n + \xi)^2 + (2m + 1)\sqrt{\lambda + B^2}; n \in \mathbb{Z}, m \in \mathbb{N} \setminus \{0\}\right\}$$

Proof. First we notice that the family $(\varphi_n)_{n\in\mathbb{Z}}$ defined by $\varphi_n(y) = e^{2i\pi ny}$ is an orthonormal basis in $L^2([0,1])$ satisfying the periodic boundary conditions $\varphi_n^{(k)}(0) = \varphi_n^{(k)}(1)$, for $k = 0,1$. Moreover, for all $y \in [0,1], \xi \in \mathbb{E}, n \in \mathbb{Z}$ we have

$$(-i\partial_y + Bx + \xi)^2\varphi_n(y) = (2\pi n + Bx + \xi)^2\varphi_n(y).$$

Next, for $n \in \mathbb{Z}$, the operator

$$T_n := -\partial_x^2 + (2\pi n + Bx + \xi)^2 + \lambda x^2$$

is unitary equivalent to the shifted harmonic oscillator

$$-\partial_s^2 + (\lambda + B^2)s^2 + \frac{\lambda(2\pi n + \xi)^2}{\lambda + B^2}.$$

Hence the spectrum of T_n is

$$\left\{(2m+1)\sqrt{\lambda + B^2} + \frac{\lambda(2\pi n + \xi)^2}{\lambda + B^2}, m \in \mathbb{N} \setminus \{0\}\right\}.$$

Finally, we identify $L^2(\mathbb{R} \times [0,1])$ to $L^2(\mathbb{R}) \otimes L^2([0,1])$ to conclude. $\square$

Remark 2.6. The eigenfunctions of the shifted harmonic oscillator

$$-\partial_s^2 + (\lambda + B^2)s^2 + \frac{\lambda(2\pi n + \xi)^2}{\lambda + B^2}$$

are independent of ξ. This implies that for $\xi \in \mathbb{C}$, the eigenvalues of $P_0(\xi)$ have the same expression as for ξ real and hence $\forall \xi \in \mathbb{C}$, the spectrum of $P_0(\xi)$ is

$$\left\{\frac{\lambda}{\lambda + B^2}(2\pi n + \xi)^2 + (2m+1)\sqrt{\lambda + B^2}; n \in \mathbb{Z}, m \in \mathbb{N} \setminus \{0\}\right\}.$$

For $\xi \in \mathbb{E}$, the spectrum of $P_\varepsilon(\xi)$ is pure point and is made up of a family $\{\lambda_j(\xi), j \in \mathbb{N}\}$ of eigenvalues of finite multiplicities. These are called the Floquet eigenvalues.

From [9] and [12], the $\lambda_j(\xi)$ can be arranged so that for every $j \in \mathbb{N}$, the map $\xi \mapsto \lambda_j(\xi)$ is a branch of an analytic function. The Floquet theory for operators with periodic coefficients yields

$$\Sigma_\varepsilon := \sigma(P_\varepsilon) = \bigcup_{j\in\mathbb{N}} \lambda_j(\mathbb{E}).$$

The continuity of $\xi \mapsto \lambda_j(\xi)$ implies that Σ_ε is the union of closed bands of the real line. Using Theorem 1 of Thomas [12], we know that for a non constant Floquet eigenvalue $\lambda_{j_0}(\xi)$ corresponds a band $\lambda_{j_0}(\mathbb{E})$ in the absolutely continuous spectrum of P_ε.

3. Absolute continuity of the spectrum

We consider in this paragraph the absolute continuity of the spectrum of P_ε for some particular families of the potential V. We shall study the cases of confinant potentials and of transverse ones.

3.1. Case of a confinant potential. We consider here a potential V which is independant of the transverse variable y. This can be considered as a perturbation acting in the same direction as the one of the confining harmonic oscillator λx^2. We prove in this case that the spectrum of P_ε is absolutely continuous.

THEOREM 3.1. *Let* $V : \mathbb{R} \longrightarrow \mathbb{R},\ x \mapsto V(x)$ *a continuous and bounded function. Then, for all* $B > 0$ *and* $\lambda > 0$*, there exists* $\varepsilon_0 > 0$ *such that for* $\varepsilon \in [0, \varepsilon_0]$*, the spectrum of* P_ε *is purely absolutely continuous.*

Proof. Since $V(x, y) = V(x)$ for all $(x, y) \in \mathbb{R}^2$ we may use the Fourier transform with respect to y and hence the operator P_ε will have an integral representation $P_\varepsilon = \displaystyle\int_{\mathbb{R}}^{\oplus} P_\varepsilon(\eta) d\eta$ where $P_\varepsilon(\eta)$ is acting on $L^2(\mathbb{R})$ by

$$P_\varepsilon(\eta) = -\partial_x^2 + (\eta + Bx)^2 + \lambda x^2 + \varepsilon V(x), \quad \forall \eta \in \mathbb{R}. \tag{5}$$

For $\eta \in \mathbb{E}$, the operator $P_\varepsilon(\eta)$ is unitary equivalent to the operator

$$\tilde{P}_\varepsilon(\eta) = -\partial_u^2 + (\lambda + B^2)u^2 + \frac{\lambda}{\lambda + B^2}\eta^2 + \varepsilon V\left(u - \frac{\eta B}{\lambda + B^2}\right).$$

The spectrum of $P_\varepsilon(\eta)$ is a sequence $(\mu_j(\eta))_{j\in\mathbb{N}}$ of eigenvalues which are analytic functions of η. When ε is small enough $\mu_j(\eta)$ approaches the eigenvalue $(2j + 1)\sqrt{\lambda + B^2} + \dfrac{\lambda}{\lambda + B^2}\eta^2$ of the harmonic oscillator $-\partial_u^2 + (\lambda + B^2)u^2 + \frac{\lambda}{\lambda+B^2}\eta^2$. This fact implies that the functions $\eta \mapsto \mu_j(\eta)$ are not constant for every j and this ends the proof. $\square$

3.2. Case of transverse potential. Here we consider a function V independent of the variable x. This is a potential acting in the direction transverse to the one of the harmonic oscillator λx^2. We prove in this case that when (2) is satisfied then for arbitrarily $\varepsilon > 0$ the spectrum is absolutely continuous. Thus we will take $\varepsilon = 1$ and we put $P = P_1, P(\xi) = P_1(\xi)$.

THEOREM 3.2. *Let $V \in L^\infty(\mathbb{R})$ such that $V(y+1) = V(y), \forall y \in \mathbb{R}$. For $\lambda > 0$ and $B > 0$, the spectrum of the operator*

$$P_\varepsilon = -\partial_x^2 + (-i\partial_y + Bx)^2 + \lambda x^2 + V(y)$$

is purely absolutely continuous.

Proof. We need to prove that none of the Floquet eigenvalues λ_j is constant on $\mathbb{E}$. We assume by contradiction that $\xi \mapsto \lambda_{j_0}(\xi)$ is constant on $\mathbb{E}$ for some $j_0 \in \mathbb{N}$. Let λ_0 be this constant. Due to the analyticity of λ_{j_0}, the function $\xi \mapsto \lambda_{j_0}(\xi)$ is constant on $\mathbb{C}$. Using Remark 1. we know that λ_0 is an eigenvalue of $P(\xi) = P_0(\xi) + V(y)$, for all $\xi \in \mathbb{C}$. Hence

$$||(P(\xi)+1)^{-1}|| \geq (\lambda_0 + 1)^{-1}. \tag{6}$$

Furthermore, using again Remark 1. we have, for $\xi \in \mathbb{C}$,

$$||(P_0(\xi)+1)^{-1}||^2 = \sup_{m \in \mathbb{N}^*} \left(\left| (2m+1)\sqrt{\lambda + B^2} + \frac{\lambda(2\pi n + \xi)^2}{\lambda + B^2} + 1 \right|^2 \right)^{-1}. \tag{7}$$

where $|Z|$ is the modulus of the complex number Z.
If we put $r = \mathcal{R}e(\xi)$ and $s = \mathcal{I}m(\xi)$ then for $r \neq 0, s \neq 0$, we see that

$$||(P_0(\xi)+1)^{-1}||^2 \leq \left(\frac{\lambda + B^2}{\lambda} \right)^2 \frac{1}{4r^2 s^2}$$

By taking $r = 1$, we obtain

$$||(P_0(\xi)+1)^{-1}||^2 \leq \left(\frac{\lambda + B^2}{\lambda} \right)^2 . \frac{1}{s^2}, \quad \forall s \in \mathbb{R} \setminus \{0\}. \tag{8}$$

The potential V is bounded, so we can find a constant $C > 0$ such that

$$||V(y)(P_0(\xi)+1)^{-1}|| < 1, \quad \text{for } |s| > C.$$

Now if we apply the resolvent identity to $P_0(\xi)$ and $P(\xi) = P_0(\xi) + V(y)$ we get

$$||(P(\xi)+1)^{-1}|| \leq \frac{||(P_0(\xi)+1)^{-1}||}{1 - ||V(y)(P_0(\xi)+1)^{-1}||}. \tag{9}$$

Using (8) and taking the limit when $|s| \to +\infty$ we end with a contradiction to (6). $\square$

4. Case of a general periodic potential

Let $\varepsilon > 0$ and $V \in L^\infty(\mathbb{R}^2)$ areal valued function satisfying (2). We will prove in this section that the bottom of the spectrum is absolutely continuous for ε small. Then we will study the existence of open gaps in the spectrum.

4.1. The bottom of the spectrum.

THEOREM 4.1. *Soit $V : \mathbb{R}^2 \to \mathbb{R}$ a bounded function. We assume that (2) is true. Then for any $\mu > 0$ there exists $\varepsilon_\mu > 0$ such that for $0 < \varepsilon \leq \varepsilon_\mu$ the spectrum of P_ε in $]-\infty, \mu]$ is purely absolutely continuous.*

Proof. Let $\mu > 0$. For $\xi \in \mathbb{T} = \mathbb{E}$, there exists a finite number N of eigenvalues of $P_0(\xi)$ in the interval $]-\infty, \mu]$. We denote by $E_j(\xi), j = 1, ..., N$ these eigenvalues.
From one side, the relation $P_\varepsilon(\xi) = P_0(\xi) + \varepsilon V$, implies that $|\lambda_j(\xi) - E_j(\xi)| \leq \varepsilon ||V||_\infty$, for $\xi \in \mathbb{E}$. Here we recall that $\lambda_j(\xi)$ are the eigenvalues of P_ξ.
Indeed we have

$$\sigma(P_0) \cap]-\infty, \mu] = \bigcup_{j=1}^{N} E_j(\mathbb{E}).$$

Let us denote by $[a_j, b_j] = E_j(\mathbb{E}), j = 1, ..., N$ the N bands generated by the E_j. From another side, we already know that the spectrum of P_0 is absolutely continuous so none of the eigenvalues E_j is constant on $\mathbb{E}$. Thus $a_j < b_j$ pour tout $j = 1, ..., N$. For $\varepsilon ||V||_\infty < \dfrac{1}{2} \min_{1 \leq j \leq N} (b_j - a_j)$, the eigenvalue $\xi \mapsto \lambda_j(\xi)$ is not constant. We conclude bye using the argument of Thomas [12] that the band of the spectrum generated by λ_j is an absolutely continuous part of the spectrum. $\square$

4.2. Gap opening. Since we have proved in the previous subsection the absolute continuity of the bottom of the spectrum, an interesting question is about the existence of possible open gaps between the bands of the spectrum.

Definition 4.2. An open gap in the spectrum of P_ε, is an interval $[a, b]$ such that $a < b$ and $[a, b] \cap \sigma(P_\varepsilon) = \emptyset$.

The main result in the what follows is the existence of open gaps for P_ε and to prove this we start by writing P_ε as an infinite matrix in some suitable orthonormal basis then we will compare it to its diagonal terms.

Pour $\xi \in \mathbb{E}$, let $P_\varepsilon(\xi)$ be the operator

$$P_\varepsilon(\xi) = -\partial_x^2 + (-i\partial_y + Bx)^2 + \lambda x^2 + \varepsilon V(x, y)$$

acting on $L^2(\mathbb{R} \times \mathbb{E})$ under the following Bloch boundary conditions:

$$(10) \quad \begin{cases} \varphi(x, \pi^-) &= e^{2i\pi\xi}\varphi(x, -\pi^+) \\ \dfrac{\partial\varphi}{\partial y}(x, \pi^-) &= e^{2i\pi\xi}\dfrac{\partial\varphi}{\partial y}(x, -\pi^+); \end{cases} \quad \forall x \in \mathbb{R}, \varphi \in L^2(\mathbb{R} \times \mathbb{E}).$$

We already know that for $\mu > 0$ fixed, the spectrum of $P_\varepsilon = \displaystyle\int_{\mathbb{E}} P_\varepsilon(\xi)d\xi$ in $]-\infty, \mu]$ is purely absolutely continuous when ε is small enough. Our approach in proving that open gaps exist is to expand P_ε with respect to the orthonormal basis spanned by the normalized eigenfunctions of the harmonic oscillator.

Let $p \in \mathbb{N}$, and

$$u_p(x) = \left(\frac{1}{2^p p! \sqrt{\pi}}\right)^{1/2} e^{-\frac{\sqrt{\lambda+B^2}}{2}x^2} H_p\left(\sqrt{\lambda + B^2}x\right), \forall x \in \mathbb{R}$$

where H_p is the Hermite polynomial of order p defined by

$$H_p(x) = (-1)^p e^{-x^2} \frac{d^p}{dx^p}\left(e^{-x^2}\right), \quad \forall p \in \mathbb{N}, x \in \mathbb{R}.$$

Using classical formulae about special functions or the orthogonal polynomilas as in [13] or [10] the following result is well known to be true.

Lemma 4.3. *The family $\mathcal{B} = \{u_p,\ p \in \mathbb{N}\}$ is an orthonormal basis in $L^2(\mathbb{R})$.*

To simplify the notations we put $\rho = \sqrt{\lambda + B^2}$. Let $p, q \in \mathbb{N}$ and $\xi \in \mathbb{E}$. We identify $L^2(\mathbb{R} \times \mathbb{E})$ to $L^2(\mathbb{R}) \otimes L^2(\mathbb{E})$ in a naturel way and we develop the operator $P_\varepsilon(\xi)$ in the basis $\mathcal{B}$.

Let $u \in L^2(\mathbb{R}) \otimes L^2(\mathbb{E})$ such that $u(x, y) = \displaystyle\sum_{p=0}^{+\infty} \alpha_p u_p(x) \otimes v(y)$, where $v \in L^2(\mathbb{E})$ satisfies the conditions (10). Then $P_\varepsilon(\xi)$ is acting on u as an infinite matrix with coefficients $m_{p,q} = \langle P_\varepsilon(\xi)u_p v, u_q v\rangle_{L^2}$.

PROPOSITION 4.4. *The operator $P_\varepsilon(\xi)$ is of the form $P_\varepsilon(\xi) = \mathcal{D}(\xi) + \mathcal{T}(\xi)$ with*

(i) $\mathcal{D}(\xi) = (d_{pq})_{p,q\in\mathbb{N}}$ is an infinite diagonal matrix with operator valued coefficients acting on $L^2(\mathbb{E})$ are given by

$$d_{pp} = -\partial_y^2 + \varepsilon\int_{\mathbb{R}} V(\frac{x}{\rho^{1/2}}, y)|u_p(x)|^2 dx + (2p+1)\rho, \quad \forall p \in \mathbb{N}. \tag{11}$$

(ii) $\mathcal{T}(\xi) = (t_{pq})_{p,q\in\mathbb{N}}$ is an infinite matrix with operator valued coefficients acting on $L^2(\mathbb{E})$ and such that $t_{pp} = 0, \forall p \in \mathbb{N}$ and

$$\begin{cases} t_{pq} = \varepsilon\displaystyle\int_{\mathbb{R}} V(\frac{x}{\rho^{1/2}}, y)u_p(x)u_q(x)dx \;\; \text{pour } q \notin \{p-1, p, p+1\}, \\ t_{pp+1} = -\left(\frac{2(n+1)}{\rho}\right)^{1/2} iB\partial_y + \varepsilon\displaystyle\int_{\mathbb{R}} V(\frac{x}{\rho^{1/2}}, y)u_p(x)u_{p+1}(x)dx \\ t_{q-1q} = \left(\frac{2n}{\rho}\right)^{1/2} iB\partial_y + \varepsilon\displaystyle\int_{\mathbb{R}} V(\frac{x}{\rho^{1/2}}, y)u_{q-1}(x)u_q(x)dx. \end{cases} \tag{12}$$

Proof. Let $v \in L^2(\mathbb{E})$ and $p \in \mathbb{N}$. Then

$$\begin{aligned} P_\varepsilon(\xi)u_p(x)v(y) = &\left[\left(-\partial_x^2 + \rho^2x^2 + \varepsilon V(x,y)u_p(x)\right)u_p(x)\right]v(y) \\ &+ \left[-\partial_y^2 v(y) - 2iBx\partial_y v(y)\right]u_p(x). \end{aligned}$$

For $q \in \mathbb{N}$, let us compute the scalar product with respect to x

$$m_{pq} = \langle P_\varepsilon(\xi)u_p v, u_q v\rangle_{L^2(\mathbb{R})}.$$

If $p = q$, we know that $(-\partial_x^2 + \rho^2x^2)u_p = (2p+1)\rho u_p$. The orthogonality of $\mathcal{B}$ yields

$$h_{pp} = (2p+1)\rho - \partial_y^2 v - \left(2iB < xu_p, u_p >_{L^2(\mathbb{R})}\right)\partial_y v.$$

Moreover, the Hermite polynomials are known to satisfy the following functional relation (voir [10]),

$$xH_p(x) = \sqrt{p+1}H_{p+1}(x) + \sqrt{p}H_{p-1}(x), \forall x \in \mathbb{R}, p \geq 1. \tag{13}$$

Using $H_0(x) = 1$, we find $< xu_p, u_p >_{L^2(\mathbb{R})} = 0, \forall p \in \mathbb{N}$.
Now if $q \neq p$, then

$$m_{pq} =< \varepsilon V(.,y)u_p, uq >_{L^2(\mathbb{R})} - \left(2iB < xu_p, u_q >_{L^2(\mathbb{R})}\right)\partial_y v.$$

We use again the relation (13) and we find (11) and (12) then we conclude. □

We let $\mathcal{D} = \displaystyle\int_{\mathbb{E}}^{\oplus} \mathcal{D}(\xi)d\xi$ and $\mathcal{T} = \displaystyle\int_{\mathbb{E}}^{\oplus} \mathcal{T}(\xi)d\xi$.

Proposition 4.5. *Let $\rho = \sqrt{\lambda + B^2}$. The operator family $(\mathcal{D}(\xi))_\rho$ converges in the strong resolvent sense to $P_\varepsilon(\xi)$ as $\rho \to +\infty$ uniformly with respect to ξ in $\mathbb{E}$.*

Proof. We need to prove that $r(\xi) := \| (P_\varepsilon(\xi)+i)^{-1} - (\mathcal{D}(\xi)+i)^{-1}\|$ goes to 0 when $\rho \to +\infty$.
by the resolvent identity we have

$$r(\xi) = \| (P_\varepsilon(\xi)+i)^{-1}\, \mathcal{T}(\xi)(\mathcal{D}(\xi)+i)^{-1}\|. \tag{14}$$

We claim that there exists $C > 0$ such that

$$\|\mathcal{T}(\xi)(\mathcal{D}(\xi)+i)^{-1}\| \leq \frac{C}{\rho}$$

uniformly on $\mathbb{E}$. For this to be justified, we remark first that

$$\|(\mathcal{D}(\xi)+i)^{-1}\| = O\left(\frac{1}{\rho}\right).$$

Moreover, if we put $\widetilde{V} = \bigoplus_{p \neq q \in \mathbb{N}} t_{pq}$ then

$$\|\widetilde{V}(\mathcal{D}(\xi)+i)^{-1}\| = O\left(\frac{1}{\rho}\right).$$

It remains to study the off-diagonal terms of $\mathcal{T}$ and those of the form t_{pp+1} without the potential term. Let $p \in \mathbb{N}$ and

$$V_{pp} = \int_{\mathbb{R}} V(\frac{x}{\rho^{1/2}}, y)|u_p(x)|^2 dx.$$

Then

$$\begin{aligned}
&\left\|(-\,(2(p+1)\rho)^{1/2}\, iB\partial_y)\,(d_{pp}+i)^{-1}\right\| \leq \\
&\quad \frac{B}{\rho}\left\|\big(-\partial_y^2 + (2p+1)\rho\big)\,(-\partial_y^2 + \varepsilon V_{pp} + (2p+3)\rho + i)^{-1}\right\| \\
&\qquad \leq \frac{B}{\rho}\left(1 + \varepsilon\left\|V_{pp}(-\partial_y^2 + \varepsilon V_{pp} + (2p+3)\rho + i)^{-1}\right\|\right) = O(\frac{1}{\rho}).
\end{aligned}$$

□

The convergence in the strong resolvent sense implies the convergence of the spectra of $(\mathcal{D}(\xi))_\rho$ to the spectrum of $P(\xi)$ when ρ tends to infinity. When the potential V is not constant the operator $d_{00}(\xi)$ given by (11) has open gaps in its spectrum ([2], [11]). Hence for ρ large enough, these gaps remains open in the spectrum of $P(\xi)$. We end this discussion by the following result.

THEOREM 4.6. *Let $V \in L^\infty(\mathbb{R}^2)$ satisfying (2) and non constant. Let $\mu > 0$ and ε_0 such that $\sigma(P_\varepsilon) \cap] - \infty, \mu]$ is absolutely continuous for $0 < \varepsilon \leq \varepsilon_0$. There exists $M > 0$ such that for $max(B, \lambda) \geq M$, the number of open gaps in the spectrum of P_ε is non zero and is bigger than the number of open gaps in the spectrum of d_{00}.*

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