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HOM-LIE-YAMAGUTI SUPERALGEBRAS

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Abstract. (Multiplicative) Hom-Lie-Yamaguti superalgebras are defined as a $\mathbb{Z}_2$-graded generalization of Hom-Lie Yamaguti algebras and also as a twisted generalization of Lie-Yamaguti superalgebras. Hom-Lie-Yamaguti superalgebras generalize also Hom-Lie supertriple systems (and subsequently ternary multiplicative Hom-Nambu superalgebras) and Hom-Lie superalgebras in the same way as Lie-Yamaguti superalgebras generalize Lie supertriple systems and Lie superalgebras. Hom-Lie-Yamaguti superalgebras are obtained from Lie-Yamaguti superalgebras by twisting along superalgebra even endomorphisms. We show that the category of (multiplicative) Hom-Lie-Yamaguti superalgebras is closed under twisting by self-morphisms. Constructions of some examples of Hom-Lie-Yamaguti superalgebras are given. The notion of an *nth* derived (binary) Hom-superalgebras is extended to the one of an *nth* derived binary-ternary Hom-superalgebras and it is shown that the category of Hom-Lie-Yamaguti superalgebras is closed under the process of taking *nth* derived Hom-superalgebras.

1. Introduction

A Lie-Yamaguti superalgebra [12] (LY superalgebra for short) is a triple $(L, *, \{,,\})$ where $L = L_0 \oplus L_1$ is $\mathbb{K}$-vector superspace i.e $\mathbb{Z}_2$-graded vector space, "$*$" a bilinear map (the binary operation on L) and "$\{,,\}$"

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a trilinear map (the ternary operation on L) satisfying $L_i * L_j \subseteq L_{i+j}$ and $\{L_i, L_j, L_k\} \subseteq L_{i+j+k}$ such that

$(SLY1)$ $x * y = -(-1)^{|x||y|} y * x$,
$(SLY2)$ $\{x, y, z\} = -(-1)^{|x||y|}\{y, x, z\}$,
$(SLY3)$ $\{x, y, z\} + (-1)^{|x|(|y|+|z|)}\{y, z, x\} + (-1)^{|z|(|x|+|y|)}\{z, x, y\}$
$+J(x, y, z) = 0$,
$(SLY4)$ $\{x * y, z, u\} + (-1)^{|x|(|y|+|z|)}\{y * z, x, u\}$
$+(-1)^{|z|(|x|+|y|)}\{z * x, y, u\} = 0$,
$(SLY5)$ $\{x, y, u * v\} = \{x, y, u\} * v + (-1)^{|u|(|x|+|y|)} u * \{x, y, v\}$,
$(SLY6)$ $\{x, y, \{u, v, w\}\} = \{\{x, y, u\}, v, w\} +$
$(-1)^{|u|(|x|+|y|)}\{u, \{x, y, v\}, w\} + (-1)^{(|x|+|y|)(|u|+|v|)}\{u, v, \{x, y, w\}\}$,

for all homogeneous elements x, y, z, u, v, w in L and where

(1) $J(x, y, z) = (x*y)*z + (-1)^{|x|(|y|+|z|)}(y*z)*x + (-1)^{|z|(|x|+|y|)}(z*x)*y.$

for all homogeneous $x, y, z \in L$ is called the super-Jacobian of L. Observe that if $x * y = 0$, for all homogeneous elements x, y in L, then a Lie-Yamaguti superalgebra $(L, *, \{,,\})$ reduces to a Lie supertriple system $(L, \{,,\})$ as defined in [17] and if $\{x, y, z\} = 0$ for all homogeneous elements x, y, z in L, then $(L, *, \{,,\})$ is a Lie superalgebra $(L, *)$ [17]. Recall that a Lie supertriple system [17] is a pair $(T, \{,,\})$ where $T = T_0 \oplus T_1$ is a $\mathbb{K}$-vector superspace and "$\{,,\}$" a trilinear map (the ternary operation on T) satisfying $\{T_i, T_j, T_k\} \subseteq T_{i+j+k}$ such that

(i) $\{x, y, z\} = (-1)^{|x||y|}\{y, x, z\}$
(ii) $\{x, y, z\} + (-1)^{|x|(|y|+|z|)}\{y, z, x\} + (-1)^{|z|(|x|+|y|)}\{z, x, y\} = 0$
(iii) $\{x, y, \{u, v, w\}\} = \{\{x, y, u\}, v, v\} + (-1)^{(|x|+|y|)|u|}\{u, \{x, y, v\}, w\}$
$+(-1)^{(|x|+|y|)(|u|+|v|)}\{u, v, \{x, y, w\}\}$

for all homogeneous elements x, y, z in T and a Lie superalgebra [17] is a pair $(A, *)$ where $A = A_0 \oplus A_1$ is a $\mathbb{K}$-vector superspace with "$*$" a bilinear map (the binary operation on A) satisfying $A_i * A_j \subseteq A_{i+j}$ such that

(i) $x * y = (-1)^{|x||y|} y * x$ (super skew-symmetry)
(ii) $J(x, y, z) = 0$ (super Jacobi identity)

for all homogeneous elements x, y, z in A.

A Hom-type generalization of a kind of algebras is obtained by a certain twisting of the defining identities by a linear self-map, called the twisting map, in such a way that when the twisting map is the identity map, then one recovers the original kind of algebras. In this scheme, e.g., associative algebras and Leibniz algebras are twisted into Hom-associative algebras and Hom-Leibniz algebras respectively [16] and, likewise, Hom-type analogues of Novikov algebras, alternative algebras, Jordan algebras or Malcev algebras are defined and discussed in [15], [19], [20]. Moreover, Hom-Lie triple systems [21], n-ary Hom-Nambu and Hom-Nambu-Lie algebras [4], are introduced. In the same way, this generalization of binary or ternary algebras has been extended to the one of binary-ternary algebras. Indeed, Hom-Akivis algebras [10], Hom-Lie-Yamaguti algebras [7] and Hom-Bol algebras [5] which generalize Akivis algebras, Lie-Yamaguti algebras and Bol algebras respectively are introduced. One could say that the theory of Hom-algebras originated in [9] (see also [13], [14]) in the study of deformations of Witt and Virasoro algebras (in fact, some q-deformations of Witt and Virasoro algebras have a structure of a Hom-Lie algebra [9]). Some algebraic abstractions of this study are given in [16], [18]. For further more informations on other Hom-type algebras, one may refer to, e.g., [4], [6], [8], [15], [19], [20], [21], [22].

In [1], the authors introduce Hom-associative superalgebras and Hom-Lie superalgebras which generalize associative superalgebras [17] and Lie superalgebras [17] respectively. They provide a way for constructing Hom-Lie superalgebras from Hom-associative superalgebras which extend the fundamental construction of Lie superalgebras from associative superalgebras via supercommutator bracket (see Proposition 1.1. in [17]). Indeed, they show that the supercommutator bracket defined using the multiplication in a Hom-associative superalgebra leads naturally to Hom-Lie superalgebras. In [2] the authors introduce Hom-alternative, Hom-Malcev and Hom-Jordan superalgebras which are generalizations of alternative, Malcev and Jordan superalgebras respectively.

Our present study, extends the Hom-type generalization of binary superalgebras to the one of ternary and binary-ternary superalgebras. More precisely, we introduce a Hom-type generalization of Lie-Yamaguti superalgebras [12], called Hom-Lie-Yamaguti superalgebras and we extend the notion of an *nth* derived (binary) Hom-superalgebras [2] to the one of an *nth* derived ternary or binary-ternary Hom-superalgebras.

The rest of this paper is organized as follows. In Section two, we recall basic definitions in Hom-superalgebras theory and useful results about Hom-associative superalgebras and Hom-Lie superalgebras. We recall the notion of *nth* derived (binary) Hom-superalgebras introduced in [1] and we show that Hom-Lie superalgebras are closed under the process of taking *nth* derived (binary) Hom-superalgebras (see Proposition 2.4). In the third Section, we introduce ternary and binary-ternary Hom-superalgebras. In particular, Hom-Lie-Yamaguti superalgebras which are binary-ternary Hom-superalgebras are defined. These Hom-superalgebras generalize Hom-Lie supertriple systems (and subsequently ternary multiplicative Hom-Nambu superalgebras) and Hom-Lie superalgebras. We provide that any non-Hom-associative Hom-superalgebra is a Hom-supertriple system and we show that the category of Hom-Lie-Yamaguti superalgebras is closed under twisting by even self-morphisms (see Theorem 3.9). Some examples of these Hom-superalgebras are constructed using Theorem 3.9 via Corollary 3.10. In the last Section, we extend the notion of *nth* derived (binary) Hom-superalgebras to the case of ternary and binary-ternary Hom-superalgebras. We show that the category of Hom-Lie-Yamaguti superalgebras is closed under the process of taking *nth* derived Hom-superalgebras. Throughout this paper, all vectors superspace are assumed to be over a field $\mathbb{K}$ of characteristic 0. Elements like x, y, z, u, v, w should be homogeneous unless otherwise state and for a given $\mathbb{K}$-vector superspace A the set of all homogeneous elements is denoted by $\mathcal{H}(A)$.

2. Some basics on superalgebras

We recall some basic facts about Hom-superalgebras, including Hom-associative and Hom-Lie superalgebras [1]. Also, we recall the notion of (binary) *nth* derived Hom-superalgebras and we show that the category of Hom-Lie superalgebras is closed under the process of taking *nth* derived Hom-superalgebras [1].

Definition 2.1. (i) Let $f : (A, *, \alpha) \to (A', *', \alpha')$ be a linear map, where $A = A_0 \oplus A_1$ and $A' = A'_0 \oplus A'_1$ are $\mathbf{Z}_2$-graded vector spaces ($\mathbb{K}$-vectors superspaces). The map f is called an even (resp. odd) map if $f(A_i) \subset A'_i$ (resp. $f(A_i) \subset A'_{i+1}$), for $i = 0, 1$.

(ii) A multiplicative n**-ary Hom-superalgebra** is a triple $(A, \{, \cdots, \}, \alpha)$ in which $A = A_0 \oplus A_1$ is a $\mathbb{K}$-vectors superspace,

$\{,\cdots,\}: A^n \to A$ is an n-linear map such that $\{A_{i_1}, A_{i_2}, \cdots, A_{i_n}\} \subseteq A_{i_1+i_2+\cdots+i_n}$ and $\alpha: A \to A$ is an even linear map such that $\alpha(\{x_1, x_2, \cdots, x_n\}) = \{\alpha(x_1), \alpha(x_2), \cdots, \alpha(x_n)\}$ (multiplicativity).

We will be interested in binary ($n = 2$), ternary ($n = 3$) and binary-ternary Hom-superalgebras (i.e. Hom-superalgebras with binary and ternary operations). For convenience, throughout this paper we will only consider multiplicative (binary, ternary or binary-ternary) Hom-superalgebras.

Definition 2.2. Let $(A, *, \alpha)$ be a Hom-superalgebra.

(*i*) The Hom-associator of A [2] is the trilinear map $as_\alpha : A \times A \times A \to A$ defined as

$$as_\alpha = * \circ (* \otimes \alpha - \alpha \otimes *). \tag{2}$$

In terms of homogeneous elements, the map as_α is given by

$$as_\alpha(x, y, z) = (x * y) * \alpha(z) - \alpha(x) * (y * z).$$

(*ii*) A Hom-associative superalgebra [1] is a multiplicative Hom-superalgebra $(A, *, \alpha)$ such that

$$as_\alpha(x, y, z) = 0, \;\; \forall\; x, y, z \in \mathcal{H}(A).$$

(*iii*) The Hom-super-Jacobian of A [2] is the trilinear map $J_\alpha : A \times A \times A \to A$ defined as $J_\alpha(x, y, z) := (x * y) * \alpha(z) + (-1)^{|x|(|y|+|z|)}(y * z) * \alpha(x) + (-1)^{|z|(|x|+|y|)}(z * x) * \alpha(y)$ for all $x, y, z \in \mathcal{H}(A)$. If $\alpha = Id$, then the Hom-super-Jacobian reduces to the super Jacobian (see (1).

(*iv*) A Hom-Lie superalgebra [1] is a multiplicative Hom-superalgebra $(A, *, \alpha)$ such that

$$x * y = -(-1)^{|x||y|} y * x \;\; \text{(super skew-symmetry)}$$
$$J_\alpha(x, y, z) = 0 \;\; \text{(Hom-super-Jacobi identity)} \tag{3}$$

for all $x, y, z \in \mathcal{H}(A)$. If $\alpha = Id$, a Hom-Lie superalgebra reduces to a usual Lie superalgebra.

In the following, we recall the notion of (binary) *nth* derived Hom-superalgebras.

Definition 2.3. [2] Let $(A, *, \alpha)$ be a Hom-superalgebra and $n \geq 0$ be an integer. The Hom-superalgebra $A^{(n)}$ defined by

$$A^{(n)} := (A, *^{(n)}, \alpha^{2^n}), \text{ where } x *^{(n)} y := \alpha^{2^n - 1}(x * y), \forall\; x, y \in \mathcal{H}(A)$$

is called the *nth derived Hom-superalgebra* of A.

For simplicity of exposition, $*^{(n)}$ is written as $*^{(n)} = \alpha^{2^n-1} \circ *$. Then notes that $A^{(0)} = (A, *, \alpha), A^{(1)} = (A, *^{(1)} = \alpha \circ *, \alpha^2)$ and $A^{(n+1)} = (A^{(n)})^{(1)}$.

Now, we prove the following:

Proposition 2.4. *Let $(A, [,], \alpha)$ be a Hom-Lie superalgebra. Then the nth derived Hom-superalgebra $A^{(n)} = (A, [,]^{(n)} = \alpha^{2^n-1} \circ [,], \alpha^{2^n})$ is also a Hom-Lie superalgebra for each $n \geq 0$.*

Proof. Observe that $[x, y]^{(n)} = -(-1)^{|x||y|}[y, x]^{(n)}$. Indeed, $[x, y]^{(n)} = \alpha^{2^n-1}([x, y]) = \alpha^{2^n-1}(-(-1)^{|x||y|}[y, x]) = -(-1)^{|x||y|}\alpha^{2^n-1}([y, x]) = -(-1)^{|x||y|}[y, x]^{(n)}$. Then, the super antisymmetry of $[x, y]^{(n)}$ holds in $A^{(n)}$. Next, we have

$$\begin{aligned}
& [[x, y]^{(n)}, \alpha^{2^n}(z)]^{(n)} + (-1)^{|x|(|y|+|z|)}[[y, z]^{(n)}, \alpha^{2^n}(x)]^{(n)} \\
& +(-1)^{|x|(|y|+|z|)}[[z, x]^{(n)}, \alpha^{2^n}(y)]^{(n)} \\
= \; & \alpha^{2^n-1}([\alpha^{2^n-1}([x, y]), \alpha^{2^n}(z)]) + (-1)^{|x|(|y|+|z|)}\alpha^{2^n-1}([\alpha^{2^n-1}([y, z]), \alpha^{2^n}(x)]) \\
+ \; & (-1)^{|z|(|x|+|y|)}\alpha^{2^n-1}([\alpha^{2^n-1}([z, x]), \alpha^{2^n}(y)]) \\
= \; & \alpha^{2(2^n-1)}([[x, y], \alpha^2(z)] + (-1)^{|x|(|y|+|z|)}[[y, z], \alpha^2(x)] \\
& +(-1)^{|z|(|x|+|y|)}[[z, x], \alpha^2(y)]) = \alpha^{2(2^n-1)}(0) \quad \text{(by (3))} \\
& = 0.
\end{aligned}$$

and so the Hom-superJacobi identity (3) holds in $A^{(n)}$. Thus, we conclude that $A^{(n)}$ is a (multiplicative) Hom-Lie superalgebra. □

3. Ternary and binary-ternary Hom-superalgebras

In this section, we introduce Hom-Lie supertriple systems and consequently ternary multiplicative Hom-Nambu superalgebras. Hom-Lie-Yamaguti superalgebras which are binary-ternary Hom-superalgebras are also defined and they reduce to Lie-Yamaguti superalgebras [12] when the twisting map is the identity map. Note also that Hom-Lie-Yamaguti superalgebras generalize Hom-Lie supertriple systems (and subsequently ternary multiplicative Hom-Nambu superalgebras) and Hom-Lie superalgebras. We finish this section by giving some examples of Hom-Lie-Yamaguti superalgebras constructed by using Theorem 3.9 via Corollary 3.10.

3.1. Definitions and Theorem of construction. In this subsection, we give some definitions of ternary Hom-superalgebras and we show that any non-Hom-associative Hom-superalgebra has a natural Hom-supertriple system structure. In Theorem 3.9, we show that Hom-Lie-Yamaguti superalgebras are closed under twisting by self-morphisms.

Definition 3.1. A ternary Hom-Nambu superalgebra is a multiplicative ternary Hom-superalgebra $(T, \{,,\}, \alpha)$ satisfying

$$\begin{aligned}\{\alpha(x), \alpha(y), \{u, v, w\}\} &= \{\{x, y, u\}, \alpha(v), \alpha(w)\}\\ &+ (-1)^{|u|(|x|+|y|)}\{\alpha(u), \{x, y, v\}, \alpha(w)\}\\ &+ (-1)^{(|x|+|y|)(|u|+|v|)}\{\alpha(u), \alpha(v), \{x, y, w\}\} \qquad (4)\end{aligned}$$

for all $u, v, w, x, y, z \in \mathcal{H}(T)$. The condition (4) is called the *ternary Hom-super Nambu identity.*

Definition 3.2. A Hom-supertriple system is a multiplicative ternary Hom-superalgebra $(T, \{,,\}, \alpha)$ such that

(*i*) $\{x, y, z\} = -(-1)^{|x||y|}\{y, x, z\}$,
(*ii*) $\{x, y, z\} + (-1)^{|x|(|y|+|z|)}\{y, z, x\} + (-1)^{|z|(|x|+|y|)}\{z, x, y\} = 0$ for all $x, y, z \in \mathcal{H}(T)$.

Remark 3.3. Our definition here is motivated by the concern of giving a Hom-type analogue of the relationships between nonassociative superalgebras and supertriple systems (see Proposition 3.4).

Proposition 3.4. *Any non-Hom-associative superalgebra is a Hom-supertriple system.*

Proof. Let $(A, *, \alpha)$ be a non-Hom-associative superalgebra. Define the supercommutator by $[x, y] := x * y - (-1)^{|x||y|}y * x$ and the ternary operation by $\{x, y, z\} := [[x, y], \alpha(z)] - as_\alpha(x, y, z) + (-1)^{|x||y|}as_\alpha(y, x, z)$ for all homogeneous elements $x, y, z \in A$ and where the Hom-associator $as_\alpha(x, y, z)$ is given by (2). Then we have $\{x, y, z\} = -(-1)^{|x||y|}\{y, x, z\}$ and $\circlearrowleft_{(x,y,z)} (-1)^{|x||z|}\{x, y, z\} = 0$. Thus $(A, \{,,\}, \alpha)$ is a Hom-supertriple system. □

Remark 3.5. For $\alpha = Id$ (the identity map), we recover the supertriple system with ternary operation $\{x, y, z\} = [[x, y], z] - as(x, y, z) + (-1)^{|x||y|}as(y, x, z)$ that is associated to each nonassociative superalgebra and where $[x, y]$ and $as(x, y, z)$ are supercommutator and associator respectively for all homogeneous elements x, y, z.

Definition 3.6. A Hom-Lie supertriple system *is a Hom-supertriple system* $(A, \{,,\}, \alpha)$ *satisfying the ternary Hom-super Nambu identity (4).* When $\alpha = Id$, a Hom-Lie supertriple system reduces to a Lie supertriple system.

One can note that Hom-Bol superalgebras introduced in [11] may be viewed as some generalization of Hom-supertriple systems.

In the following, we give the definition of the basic object of this paper and we consider construction methods for Hom-LY superalgebras. These methods allow to find examples of Hom-LY superalgebras starting from ordinary LY superalgebras(thus from Malcev superalgebras) or even from Hom-Lie superalgebras. Recall that from a Malcev superalgebra $(A, *)$ if consider on A the super ternary operation

(5) $\{x, y, z\} := x*(y*z) - (-1)^{|x||y|}y*(x*z) + (x*y)*z, \forall\ x, y, z \in A,$

then $(A, *, \{,,\})$ is a Lie-Yamaguti superalgebra [12]. First, as the main tool, we show that the category of (multiplicative) Hom-LY superalgebras is closed under even self-morphisms (see Theorem 3.9).

Definition 3.7. A **Hom-Lie Yamaguti superalgebra** (Hom-LY superalgebra for short) is a binary-ternary Hom-superalgebra $(L, *, \{,,\}, \alpha)$ such that

$(SHLY1)$ $\alpha(x*y) = \alpha(x)*\alpha(y),$
$(SHLY2)$ $\alpha(\{x, y, z\}) = \{\alpha(x,)\alpha(y), \alpha(z)\},$
$(SHLY3)$ $x*y = -(-1)^{|x||y|}y*x,$
$(SHLY4)$ $\{x, y, z\} = -(-1)^{|x||y|}\{y, x, z\},$
$(SHLY5)$ $\circlearrowleft_{(x,y,z)} (-1)^{|x||z|}[(x*y)*\alpha(z) + \{x, y, z\}] = 0,$
$(SHLY6)$ $\circlearrowleft_{(x,y,z)} (-1)^{|x||z|}[\{x*y, \alpha(z), \alpha(u)\}] = 0,$
$(SHLY7)$ $\{\alpha(x), \alpha(y), u*v\} = \{x, y, u\}*\alpha^2(v)$
$+\quad (-1)^{|u|(|x|+|y|)}\alpha^2(u)*\{x, y, v\},$

$$(SHLY8)\ \{\alpha^2(x), \alpha^2(y), \{u, v, w\}\} = \{\{x, y, u\}, \alpha^2(v), \alpha^2(w)\} + (-1)^{|u|(|x|+|y|)}\{\alpha^2(u), \{x, y, v\}, \alpha^2(w)\} + (-1)^{(|x|+|y|)(|u|+|v|)}\{\alpha^2(u), \alpha^2(v), \{x, y, w\}\},$$

for all $u, v, w, x, y, z \in \mathcal{H}(L)$ and where $\circlearrowleft_{(x,y,z)}$ denotes the sum over cyclic permutation of x, y, z.

Note that the conditions $(SHLY1)$ and $(SHLY2)$ mean the multiplicativity condition of $(L, *, \{, , \}, \alpha)$.

Remark 3.8. (1) If $\alpha = Id$, then the Hom-LY superalgebra $(L, *, \{, , \}, \alpha)$ reduces to a LY superalgebra $(L, *, \{, , \})$ (see $(SLY1) - (SLY6)$).

(2) If $x * y = 0$, for all $x, y \in \mathcal{H}(L)$, then $(L, *, \{, , \}, \alpha)$ becomes a Hom-Lie supertriple system $(L, \{, , \}, \alpha^2)$ and, subsequently, a ternary Hom-Nambu superalgebra (since, by Definition 3.6, any Hom-Lie supertriple system is automatically a ternary Hom-Nambu superalgebra).

(3) If $\{x, y, z\} = 0$ for all $x, y, z \in \mathcal{H}(L)$, then the Hom-LY superalgebra $(L, *, \{, , \}, \alpha)$ becomes a Hom-Lie superalgebra $(L, *, \alpha)$.

Theorem 3.9. *Let $A_\alpha := (A, *, \{, , \}, \alpha)$ be a Hom-LY superalgebra and let "β" be an even endomorphism of the superalgebra $(A, *, \{, , \})$ such that $\beta\alpha = \alpha\beta$. Let $\beta^0 = Id$ and, for any $n \geq 1$ $\beta^n := \beta \circ \beta^{n-1}$. Define on A the operations*
$x *_\beta y := \beta^n(x * y)$,
$\{x, y, z\}_\beta := \beta^{2n}(\{x, y, z\})$,
for all $x, y, z \in \mathcal{H}(A)$. Then, $A_{\beta^n} := (A, *_\beta, \{, , \}_\beta, \beta^n\alpha)$ is a Hom-LY superalgebra for each $n \geq 1$.

Proof. First, we observe that the condition $\beta\alpha = \alpha\beta$ implies $\beta^n\alpha = \alpha\beta^n, n \geq 1$. Next we have
$(\beta^n\alpha)(x *_\beta y) = (\beta^n\alpha)(\beta^n(x) * \beta^n(y)) = \beta^n((\alpha\beta^n)(x) * (\alpha\beta^n)(y))$
$= (\alpha\beta^n)(x) *_\beta (\alpha\beta^n)(y) = (\beta^n\alpha)(x) *_\beta (\beta^n\alpha)(y)$ and we get $(SHLY1)$ for A_{β^n}. Likewise, the condition $\beta\alpha = \alpha\beta$ implies $(SHLY2)$. The identities $(SHLY3)$ and $(SHLY4)$ for A_{β^n} follow from the super skew-symmetry of "$*$" and "$\{, , \}$" respectively. Consider now $(SHLY5)$ for A_{β^n}. Then

$$\begin{aligned}
&(x *_\beta y) *_\beta (\beta^n\alpha)(z) + (-1)^{|x|(|y|+|z|)}(y *_\beta z) *_\beta (\beta^n\alpha)(x) \\
&+(-1)^{|z|(|x|+|y|)}(z *_\beta x) *_\beta (\beta^n\alpha)(y) \\
&+\{x, y, z\}_\beta + (-1)^{|x|(|y|+|z|)}\{y, z, x\}_\beta + (-1)^{|z|(|x|+|y|)}\{z, x, y\}_\beta \\
&= \beta^n(\beta^n(x * y) * (\beta^n(\alpha(z))) + (-1)^{|x|(|y|+|z|)}\beta^n(\beta^n(y * z) * (\beta^n(\alpha(x))) \\
&+(-1)^{|z|(|x|+|y|)}\beta^n(\beta^n(z * x) * (\beta^n(\alpha(y))) + \beta^{2n}(\{x, y, z\}) \\
&+(-1)^{|x|(|y|+|z|)}\beta^{2n}(\{y, z, x\} + (-1)^{|z|(|x|+|y|)}\beta^{2n}(\{z, x, y\}) \\
&= \beta^{2n}((x * y) * \alpha(z)) + (-1)^{|x|(|y|+|z|)}\beta^{2n}((y * z) * \alpha(x))
\end{aligned}$$

$$
\begin{aligned}
&+(-1)^{|z|(|x|+|y|)}\beta^{2n}((z*x)*\alpha(y))+\beta^{2n}(\{x,y,z\})\\
&+(-1)^{|x|(|y|+|z|)}\beta^{2n}(\{y,z,x\})+(-1)^{|z|(|x|+|y|)}\beta^{2n}(\{z,x,y\})\\
&=\beta^{2n}[(x*y)*\alpha(z)+(-1)^{|x|(|y|+|z|)}(y*z)*\alpha(x)\\
&+(-1)^{|z|(|x|+|y|)}(z*x)*\alpha(y)+\{x,y,z\}+(-1)^{|x|(|y|+|z|)}\{y,z,x\}\\
&+(-1)^{|z|(|x|+|y|)}\{z,x,y\}]\\
=\;&\beta^{2n}(0) \text{ (by } (SHLY5) \text{ for } A_\alpha)\\
=\;&0.
\end{aligned}
$$

Thus we get $(SHLY5)$ for A_{β^n}. Next, let remark that $\{x *_\beta y, (\beta^n\alpha)(z), (\beta^n\alpha)(u)\}_\beta = \beta^{3n}(\{x*y, \alpha(z), \alpha(u)\})$ and therefore

$$
\begin{aligned}
&\circlearrowleft_{(x,y,z)} (-1)^{|x||z|}\{x *_\beta y, (\beta^n\alpha)(z), (\beta^n\alpha)(u)\}_\beta\\
&=\circlearrowleft_{(x,y,z)} (-1)^{|x||z|}\beta^{3n}(\{x*y,\alpha(z),\alpha(u)\})\\
&=\beta^{3n}(\circlearrowleft_{(x,y,z)} (-1)^{|x||z|}(\{x*y,\alpha(z),\alpha(u)\}))\\
&=\beta^{3n}(0) \text{ (by } (HLY6) \text{ for } A_\alpha)\\
&=0.
\end{aligned}
$$

So that we get $(SHLY6)$ for A_{β^n}. Further, using $(SHLY7)$ for A_α and the condition $\alpha\beta=\beta\alpha$, we compute

$$
\begin{aligned}
&\{(\beta^n\alpha)(x),(\beta^n\alpha)(y),u *_\beta v\}_\beta=\beta^{3n}(\{\alpha(x),\alpha(y),u*v\})\\
&=\beta^{3n}(\{x,y,u\}*\alpha^2(v)+(-1)^{|u|(|x|+|y|)}\alpha^2(u)*\{x,y,v\})\\
&=\beta^{n}(\beta^{2n}(\{x,y,u\})*(\beta^{2n}\alpha^2)(v))\\
&+(-1)^{|u|(|x|+|y|)}\beta^{n}((\beta^{2n}\alpha^2)(u)*\beta^{2n}(\{x,y,v\}))\\
&=\{x,y,u\}_\beta *_\beta (\beta^{2n}\alpha^2)(v)+(-1)^{|u|(|x|+|y|)}(\beta^{2n}\alpha^2)(u) *_\beta \{x,y,v\}_\beta\\
&=\{x,y,u\}_\beta *_\beta (\beta^{n}\alpha)^2(v)+(-1)^{|u|(|x|+|y|)}(\beta^{n}\alpha)^2(u) *_\beta \{x,y,v\}_\beta.
\end{aligned}
$$

Thus $(SHLY7)$ holds for A_{β^n}. Using repeatedly the condition $\alpha\beta=\beta\alpha$ and the super identity $(SHLY8)$ for A_α, the verification of $(SHLY8)$ for A_{β^n} is as follows.

$$
\begin{aligned}
&\{(\beta^n\alpha)^2(x),(\beta^n\alpha)^2(y),\{u,v,w\}_\beta\}_\beta\\
=\;&\beta^{2n}(\{(\beta^{2n}\alpha^2)(x),(\beta^{2n}\alpha^2)(y),\beta^{2n}(\{u,v,w\})\})\\
=\;&\beta^{4n}(\{\alpha^2(x),\alpha^2(y),\{u,v,w\}\})\\
=\;&\beta^{4n}(\{\{x,y,u\},\alpha^2(v),\alpha^2(w)\})
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad \beta^{4n}((-1)^{|u|(|x||y|)}\{\alpha^2(u), \{x,y,v\}, \alpha^2(w)\}) \\
&+ \quad \beta^{4n}((-1)^{(|x|+|y|)(|u|+|v|)}\{\alpha^2(u), \alpha^2(v), \{x,y,w\}\}) \\
&= \quad \beta^{2n}(\{\beta^{2n}(\{x,y,u\}), (\beta^{2n}\alpha^2)(v), (\beta^{2n}\alpha^2)(w)\}) \\
&+ \quad (-1)^{|u|(|x||y|)}\beta^{2n}(\{(\beta^{2n}\alpha^2)(u), \beta^{2n}(\{x,y,v\}), (\beta^{2n}\alpha^2)(w)\}) \\
&+ \quad (-1)^{(|x|+|y|)(|u|+|v|)}\beta^{2n}(\{(\beta^{2n}\alpha^2)(u), (\beta^{2n}\alpha^2)(v), \beta^{2n}(\{x,y,w\})\}) \\
&= \quad \{\{x,y,u\}_\beta, (\beta^n\alpha)^2(v), (\beta^n\alpha)^2(w)\}_\beta \\
&+ \quad (-1)^{|u|(|x||y|)}\{(\beta^n\alpha)^2(u), \{x,y,v\}_\beta, (\beta^n\alpha)^2(w)\}_\beta \\
&+ \quad (-1)^{(|x|+|y|)(|u|+|v|)}\{(\beta^n\alpha)^2(u), (\beta^n\alpha)^2(v), \{x,y,w\}_\beta\}_\beta
\end{aligned}
$$

Thus $(SHLY8)$ holds for A_{β^n}. Therefore, we get that A_{β^n} is a Hom-LY superalgebra. This finishes the proof. □

In [18], D. Yau established a general method of construction of Hom-algebras from their corresponding untwisted algebras. From Theorem 3.1 we have the following method of construction of Hom-LY superalgebras from LY superalgebras(this yields examples of Hom-LY superalgebras). This method is an extension to binary-ternary superalgebras of D. Yau's result ([18], Theorem 2.3). Such an extension to binary-ternary algebras is first mentioned in [10], Corollary 4.5.

Corollary 3.10. *Let $(A, *, [,,])$ be a LY superalgebra and β an even endomorphism of $(A, *, [,,])$. If define on A a binary operation "$\tilde{*}$" and a ternary operation $\{,,\}$ by*
$x\tilde{*}y := \beta(x * y)$,
$\{x,y,z\} := \beta^2([x,y,z])$, for all $x,y,z \in \mathcal{H}(A)$
then $(A, \tilde{*}, \{,,\}, \beta)$ is a Hom-LY superalgebra.

Proof. The proof follows if observe that Corollary 3.10 is Theorem 3.9 when $\alpha = Id$ and $n = 1$. □

Proposition 3.11. *Let $(L, [,], \alpha)$ be a (multiplicative) Hom-Lie superalgebra. Define on $(L, [,], \alpha)$ a ternary operation by*

$$\{x,y,z\}_\alpha := [[x,y], \alpha(z)]. \tag{6}$$

Then, $(L, [,], \{,,\}_\alpha, \alpha)$ is a Hom-Lie-Yamaguti superalgebra.

Proof. Straightforward calculations by verification of the identities $(SHLY1 - SHLY8)$ of Definition 3.7. □

3.2. Examples of Hom-Lie-Yamaguti superalgebras. In the following, we give some examples of Hom-Lie-Yamaguti superalgebras which are constructed firstly from the Example 2.8 in [1] and using Proposition 3.11 and secondly from Example 3.2 given in [3] (see also in [2]) and using Theorem 3.9 via Corollary 3.10 . Then, we obtain some families of Hom-Lie-Yamaguti superalgebras of dimension 5 and dimension 4 respectively.

Example 3.12. Consider the family of Hom-Lie superalgebra $osp(1,2)_\lambda = (osp(1,2), [,]_{\alpha_\lambda}, \alpha_\lambda)$ given in the example 2.8 in [1]. The Hom-Lie superalgebra bracket $[,]_{\alpha_\lambda}$ on the basis elements is given, for $\lambda \neq 0$, by:

$$[H,X]_{\alpha_\lambda} = 2\lambda^2 X, \quad [H,Y]_{\alpha_\lambda} = \frac{-2}{\lambda^2}Y, \quad [X,Y]_{\alpha_\lambda} = H, [Y,G]_{\alpha_\lambda} = \frac{1}{\lambda}F,$$
$$[X,F]_{\alpha_\lambda} = \lambda G, \quad [H,F]_{\alpha_\lambda} = -\frac{1}{\lambda}F, \quad [H,G]_{\alpha_\lambda} = \lambda G, \quad [G,F]_{\alpha_\lambda} = H,$$
$$[G,G]_{\alpha_\lambda} = -2\lambda^2 X, \quad [F,F]_{\alpha_\lambda} = \frac{2}{\lambda^2}Y,$$

where $\alpha_\lambda : osp(1,2) \to osp(1,2)$ is a linear map defined by $\alpha_\lambda(X) = \lambda^2 X$, $\alpha_\lambda(Y) = \frac{1}{\lambda^2}Y$, $\alpha_\lambda(H) = H$, $\alpha_\lambda(F) = \frac{1}{\lambda}F$, $\alpha_\lambda(G) = \lambda G$, and $osp(1,2) = V_0 \oplus V_1$ is a Lie superalgebra where V_0 is generated by $H = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, $X = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ and V_1 is generated by $F = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$, $G = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}$ such that $[H,X] = 2X$, $[H,Y] = -2Y$, $[X,Y] = H$, $[Y,G] = F$, $[X,F] = G$,
$[H,F] = -F$, $[H,G] = G$, $[G,F] = H$, $[G,G] = -2X$, $[F,F] = 2Y$.

Now, we define on $osp(1,2)_\lambda = (osp(1,2), [,]_{\alpha_\lambda}, \alpha_\lambda)$ a ternary operation $\{,,\}_{\alpha_\lambda}$ as in (6), then $sLY(1,2)_\lambda = (osp(1,2), [,]_{\alpha_\lambda}, \{,,\}_{\alpha_\lambda}, \alpha_\lambda)$ is a family of Hom-Lie-Yamaguti superalgebras where the super Lie bracket $[,]_{\alpha_\lambda}$ is defined as above and the ternary operation $\{,,\}_{\alpha_\lambda}$ defined as in Proposition 3.11 (we give only the ones with non zero values in the left hand side and using the identity $(SHLY4)$ of Definition 3.7, one can deduce the others values which are non zero) is given by:

$$\begin{aligned}
2H &= \{H,X,Y\}_{\alpha_\lambda} = \{H,Y,X\}_{\alpha_\lambda} = -2\{H,F,G\}_{\alpha_\lambda} = 2\{H,G,F\}_{\alpha_\lambda} \\
&= 2\{X,F,F\}_{\alpha_\lambda} = 2\{Y,G,G\}_{\alpha_\lambda} = -\{F,F,X\}_{\alpha_\lambda} = -\{G,G,Y\}_{\alpha_\lambda} \\
-2\lambda^4 X &= \frac{1}{2}\{H,X,H\}_{\alpha_\lambda} = -\{X,Y,X\}_{\alpha_\lambda} = -\{F,G,X\}_{\alpha_\lambda} = \{H,G,G\}_{\alpha_\lambda} \\
&= \{X,F,G\}_{\alpha_\lambda} = -\frac{1}{2}\{G,G,H\}_{\alpha_\lambda} \\
\frac{4}{\lambda^4}Y &= -\{H,Y,H\}_{\alpha_\lambda} = -2\{X,Y,Y\}_{\alpha_\lambda} = -2\{F,G,Y\}_{\alpha_\lambda} = -2\{H,F,F\}_{\alpha_\lambda} \\
&= 2\{Y,G,F\}_{\alpha_\lambda} = \{F,F,H\}_{\alpha_\lambda} \\
-\frac{2}{\lambda^2}F &= -2\{H,F,H\}_{\alpha_\lambda} = \{H,Y,G\}_{\alpha_\lambda} = 2\{H,G,Y\}_{\alpha_\lambda} = 2\{X,Y,F\}_{\alpha_\lambda} = \\
& \quad -\{F,F,G\}_{\alpha_\lambda} = 2\{X,F,Y\}_{\alpha_\lambda} = -2\{Y,G,H\}_{\alpha_\lambda} = 2\{F,G,F\}_{\alpha_\lambda} \\
-\lambda^2 G &= \{H,G,H\}_{\alpha_\lambda} = -\frac{1}{2}\{H,X,F\}_{\alpha_\lambda} = -\{H,F,X\}_{\alpha_\lambda} = -\{X,Y,G\}_{\alpha_\lambda} \\
&= \{X,F,H\}_{\alpha_\lambda} = \{Y,G,X\}_{\alpha_\lambda} = -\{F,G,G\}_{\alpha_\lambda} = \frac{1}{2}\{G,G,F\}_{\alpha_\lambda}.
\end{aligned}$$

These Hom-Lie-Yamaguti superalgebras are not Lie-Yamaguti superalgebras for $\lambda \neq \pm 1$. Indeed, the left hand side of the identity $(SLY3)$ leads to $\circlearrowleft_{(H,Y,X)} (-1)^{|H||X|}([[H,Y]_{\alpha_\lambda}, X]_{\alpha_\lambda} + \{H,Y,X\}_{\alpha_\lambda}) = 2\left(\frac{1-\lambda^4}{\lambda^2}\right) H$ which does not vanish for $\lambda \neq \pm 1$.

Example 3.13. From the example $M^3(3,1)$ of a non-Lie Malcev superalgebra given in [3] (see also in [2]) with the multiplication $[,]$, if define a ternary operation by (5), we obtain a Lie-Yamaguti superalgebra $(sLY(3,1), [,], \{,,\})$ of dimension 4, defined with respect to a basis (e_1, e_2, e_3, e_4), where $(sLY(3,1))_0 = span(e_1, e_2, e_3)$ and $(sLY(3,1))_1 = span(e_4)$, by the following multiplication table:

$$\begin{aligned}
& [e_1,e_3] = -e_1 \quad (= -[e_3,e_1]), \ \ [e_2,e_3] = 2e_2 \quad (= -[e_3,e_2]), \ \ [e_3,e_4] \\
= \ & -e_4 \quad (= -[e_4,e_3]), [e_4,e_4] = e_1 + e_2, \\
& \{e_1,e_3,e_3\} = 2e_1 \quad (= -\{e_3,e_1,e_3\}), \ \ \{e_2,e_3,e_3\} = 8e_2 \\
& (= -\{e_3,e_2,e_3\}), \{e_3,e_4,e_3\} = -2e_4 \quad (= -\{e_4,e_3,e_3\}), \ \ \{e_3,e_4,e_4\} \\
& = e_1 - 2e_2 \quad (= -\{e_4,e_3,e_4\}), \ \ \{e_4,e_4,e_3\} = e_1 + 4e_2.
\end{aligned}$$

Consider the even superalgebra endomorphism $\alpha_1 : sLY(3,1) \to sLY(3,1)$ with respect to the same basis defined by $\alpha_1(e_1) = a^2 e_1, \quad \alpha_1(e_2) =$

a^2e_2, $\alpha_1(e_3) = be_1 + ce_2 + e_3$, $\alpha_1(e_4) = a^2e_4$ for all $a, b, c \in \mathbb{K}$. For each such even superalgebra endomorphism α_1, using Theorem 3.9 via Corollary 3.10, there is a Hom-Lie-Yamaguti superalgebra $sLY(3,1)_{\alpha_1} = (sLY(3,1), [,]_{\alpha_1}, \{,,\}_{\alpha_1})$ where

$$\begin{aligned}
&[e_1, e_3]_{\alpha_1} = -a^2e_1 \quad (= -[e_3, e_1]_{\alpha_1}), \ [e_2, e_3]_{\alpha_1} = 2a^2e_2 \quad (= -[e_3, e_2]_{\alpha_1}),\\
&[e_3, e_4]_{\alpha_1} = -a^2e_4 \quad (= -[e_4, e_3]_{\alpha_1}), \quad [e_4, e_4]_{\alpha_1} = a^2(e_1 + e_2),\\
&\{e_1, e_3, e_3\}_{\alpha_1} = 2a^4e_1 \quad (= -\{e_3, e_1, e_3\}_{\alpha_1}), \ \{e_2, e_3, e_3\}_{\alpha_1}\\
= \ &8a^4e_2 \quad (= -\{e_3, e_2, e_3\}_{\alpha_1}),\\
&\{e_3, e_4, e_3\}_{\alpha_1} = -2a^4e_4 \quad (= -\{e_4, e_3, e_3\}_{\alpha_1}), \ \{e_3, e_4, e_4\}_{\alpha_1}\\
= \ &a^4(e_1 - 2e_2) \quad (= -\{e_4, e_3, e_4\}_{\alpha_1})\\
&\{e_4, e_4, e_3\}_{\alpha_1} = a^4(e_1 + 2e_2).
\end{aligned}$$

While for example

$$\circlearrowleft_{(e_3,e_4,e_4)} (-1)^{|e_3||e_4|}([[e_3, e_4]_{\alpha_1}, e_4]_{\alpha_1} + \{e_3, e_4, e_4\}_{\alpha_1}) = -2a^4e_2.$$

If $a \neq 0$ then, the left hand side of the identity $(SLY3)$ does not vanish. Therefore these Hom-Lie-Yamaguti superalgebras are not Lie-Yamaguti superalgebras for $a \neq 0$.

4. *nth* derived binary-ternary Hom-superalgebras

In this section, we extend the notion of *nth* derived (binary) Hom-superalgebras to the case of ternary and binary-ternary Hom-superalgebras. We also show that the category of Hom-Lie-Yamaguti superalgebras is closed under the process of taking *nth* derived (binary-ternary) Hom-superalgebras.

Definition 4.1. Let $A := (A, \{,,\}, \alpha)$ be a ternary Hom-superalgebra and $n \geq 0$ an integer. Define on A the *nth* derived ternary operation $\{,,\}^{(n)}$ by

$$\{x, y, z\}^{(n)} := \alpha^{2^{n+1}-2}(\{x, y, z\}), \forall\ x, y, z \in \mathcal{H}(A). \tag{7}$$

Then $A^{(n)} := (A, \{,,\}^{(n)}, \alpha^{2^n})$ will be called the *nth derived ternary Hom-superalgebra* of A. Now denote $\{,,\}^{(n)} = \alpha^{2^{n+1}-2} \circ \{,,\}$. Then we note that $A^{(0)} = (A, \{,,\}, \alpha)$, $A^{(1)} = (A, \{,,\}^{(1)} = \alpha^2 \circ \{,,\}, \alpha^2)$, and $A^{(n+1)} = (A^{(n)})^{(1)}$.

We also get the following:

DEFINITION 4.2. Let $A := (A, *, \{,,\}, \alpha)$ be a binary-ternary Hom-superalgebra and $n \geq 0$ an integer. Define on A the nth derived binary operation and the nth derived ternary operation by

$$x *^{(n)} y := \alpha^{2^n-1}(x * y) \tag{8}$$

$$\{x, y, z\}^{(n)} := \alpha^{2^{n+1}-2}(\{x, y, z\}), \forall\, x, y, z \in \mathcal{H}(A). \tag{9}$$

Then $A^{(n)} := (A, *^{(n)}, \{,,\}^{(n)}, \alpha^{2^n})$ will be called the *nth derived (binary-ternary) Hom-superalgebra* of A. Denote $*^{(n)} = \alpha^{2^n-1} \circ *$ and $\{,,\}^{(n)} = \alpha^{2^{n+1}-2} \circ \{,,\}$. Then we note that $A^{(0)} = (A, *, \{,,\}, \alpha)$, $A^{(1)} = (A, *^{(1)} = \alpha \circ *,\ \{,,\}^{(1)} = \alpha^2 \circ \{,,\}, \alpha^2)$, and $A^{(n+1)} = (A^{(n)})^{(1)}$.

Observe that from Definition 4.2, if set $\{x, y, z\} = 0, \forall\, x, y, z \in \mathcal{H}(A)$, then one recovers the nth derived (binary) Hom-superalgebra of Definition 2.3 and if $x * y = 0\ \forall\, x, y \in \mathcal{H}(A)$, then one has the nth derived (ternary) Hom-superalgebra (see Definition 4.1).

In the following result, we show that the category of Hom-Lie-Yamaguti superalgebras is closed under the process of taking nth derived Hom-superalgebras.

THEOREM 4.3. *Let $A_\alpha := (A, [,], \{,,\}, \alpha)$ be a Hom-Lie-Yamaguti superalgebra. Then, for each $n \geq 0$ the nth derived Hom-superalgebra*

$$A^{(n)} := (A, [,]^{(n)} = \alpha^{2^n-1} \circ [,], \{,,\}^{(n)} = \alpha^{2^{n+1}-2} \circ \{,,\}, \alpha^{2^n})$$

is a Hom-Lie-Yamaguti superalgebra.

Proof. The identities $(SHLY1)-(SHLY4)$ for $A^{(n)}$ are obvious. The checking of $SHLY5$ for $A^{(n)}$ is as follows.

$$\begin{aligned}
&\circlearrowleft_{(x,y,z)} ((-1)^{|x||z|}[[x, y]^{(n)}, (\alpha^{2^n})(z)]^{(n)} + (-1)^{|x||z|}\{x, y, z\}^{(n)})\\
=&\circlearrowleft_{(x,y,z)} ((-1)^{|x||z|}\alpha^{2^n-1}([\alpha^{2^n-1}([x, y]), (\alpha^{2^n-1}(\alpha(z)))])\\
&+ \circlearrowleft_{(x,y,z)} (\alpha^{2^{n+1}-2}((-1)^{|x||z|}\{x, y, z\}))\\
=&\ \alpha^{2^{n+1}-2}(\circlearrowleft_{(x,y,z)} ((-1)^{|x||z|}[[x, y], \alpha(z)]) + \circlearrowleft_{(x,y,z)} ((-1)^{|x||z|}\{y, z, x\}))\\
=&\ \alpha^{2^{n+1}-2}(\circlearrowleft_{(x,y,z)} ((-1)^{|x||z|}[[x, y], \alpha(z)] + (-1)^{|x||z|}\{x, y, z\}))\\
=&\ \alpha^{2^{n+1}-2}(0) \text{ (by } (SHLY5) \text{ for } A_\alpha)\\
=&\ 0
\end{aligned}$$

and thus we get $(SHLY5)$ for $A^{(n)}$. Next,

$$\begin{aligned}
& \{[x,y]^{(n)}, (\alpha^{2^n})(z), (\alpha^{2^n})(u)\}^{(n)} = \{\alpha^{2^n-1}([x,y]), (\alpha^{2^n})(z), (\alpha^{2^n})(u)\}^{(n)} \\
= \;& \alpha^{2^{n+1}-2}(\{\alpha^{2^n-1}([x,y]), (\alpha^{2^n})(z), (\alpha^{2^n})(u)\}) \\
= \;& \alpha^{2^{n+1}-2}\alpha^{2^n-1}(\{[x,y], \alpha(z), \alpha(u)\}) = \alpha^{3(2^n-1)}(\{[x,y], \alpha(z), \alpha(u)\}).
\end{aligned}$$

Therefore

$$\begin{aligned}
& \circlearrowleft_{(x,y,z)} ((-1)^{|x||z|}\{[x,y]^{(n)}, (\alpha^{2^n})(z), (\alpha^{2^n})(u)\}^{(n)}) \\
= \;& \circlearrowleft_{(x,y,z)} (\alpha^{3(2^n-1)}((-1)^{|x||z|}\{[x,y], \alpha(z), \alpha(u)\})) \\
= \;& \alpha^{3(2^n-1)}(\circlearrowleft_{(x,y,z)} ((-1)^{|x||z|}\{[x,y], \alpha(z), \alpha(u)\})) \\
= \;& \alpha^{3(2^n-1)}(0) \text{ (by } (SHLY6) \text{ for } A_\alpha) \\
= \;& 0
\end{aligned}$$

So that we get $(SHLY6)$ for $A^{(n)}$. Further, using $(SHLY7)$ for A_α and the condition of multiplicativity and the linearity of α, we compute

$$\begin{aligned}
& \{(\alpha^{2^n})(x), (\alpha^{2^n})(y), [u,v]^{(n)}\}^{(n)} \\
= \;& \alpha^{2^{n+1}-2}(\{(\alpha^{2^n})(x), (\alpha^{2^n})(y), \alpha^{2^n-1}([u,v])\}) \\
= \;& \alpha^{2^{n+1}-2}\alpha^{2^n-1}(\{\alpha(x), \alpha(y), [u,v]\}) \\
= \;& \alpha^{2^{n+1}-2}\alpha^{2^n-1}([\{x,y,u\}, \alpha^2(v)] + (-1)^{|u|(|x|+|y|)}[\alpha^2(u), \{x,y,v\}]) \\
= \;& \alpha^{2^n-1}([\{x,y,u\}^{(n)}, (\alpha^{2^n})^2(v)] + (-1)^{|u|(|x|+|y|)}[(\alpha^{2^n})^2(u), \{x,y,v\}^{(n)}]) \\
= \;& [\{x,y,u\}^{(n)}, (\alpha^{2^n})^2(v)]^{(n)} + (-1)^{|u|(|x|+|y|)}[(\alpha^{2^n})^2(u), \{x,y,v\}^{(n)}]^{(n)}.
\end{aligned}$$

Thus $(SHLY7)$ holds for $A^{(n)}$. Using repeatedly the condition of the multiplicativity and the identity $(SHLY8)$ for A_α, the verification for $(SHLY8)$ for $A^{(n)}$ is as follows.

$$\begin{aligned}
& \{(\alpha^{2^n})^2(x), (\alpha^{2^n})^2(y), \{u,v,w\}^{(n)}\}^{(n)} \\
= \;& \alpha^{2^{n+1}-2}(\{(\alpha^{2^n})^2)(x), (\alpha^{2^n})^2)(y), \alpha^{2^{n+1}-2}(\{u,v,w\})\}) \\
= \;& \alpha^{2^{n+1}-2}(\{(\alpha^{2^{n+1}})(x), (\alpha^{2^{n+1}})(y), \alpha^{2^{n+1}-2}(\{u,v,w\})\}) \\
= \;& (\alpha^{2^{n+1}-2})^2(\{\alpha^2(x), \alpha^2(y), \{u,v,w\}\}) \\
= \;& (\alpha^{2^{n+1}-2})^2(\{\{x,y,u\}, \alpha^2(v), \alpha^2(w)\}) \\
+ \;& (\alpha^{2^{n+1}-2})^2((-1)^{|u|(|x||y|)}\{\alpha^2(u), \{x,y,v\}, \alpha^2(w)\}) \\
+ \;& (\alpha^{2^{n+1}-2})^2((-1)^{(|x|+|y|)(|u|+|v|)}\{\alpha^2(u), \alpha^2(v), \{x,y,w\}\})
\end{aligned}$$

$$\begin{aligned}
&= (\alpha^{2^{n+1}-2})(\{(\alpha^{2^{n+1}-2})(\{x,y,u\}),(\alpha^{2^{n+1}-2})(\alpha^2(v)),(\alpha^{2^{n+1}-2})(\alpha^2(w))\}) \\
&+ (\alpha^{2^{n+1}-2})((-1)^{|u|(|x||y|)}\{(\alpha^{2^{n+1}-2})(\alpha^2(u)),(\alpha^{2^{n+1}-2})(\{x,y,v\}), \\
&\quad (\alpha^{2^{n+1}-2})(\alpha^2(w))\}) + (\alpha^{2^{n+1}-2})((-1)^{(|x|+|y|)(|u|+|v|)}\{(\alpha^{2^{n+1}-2})(\alpha^2(u)), \\
&\quad (\alpha^{2^{n+1}-2})(\alpha^2(v)),(\alpha^{2^{n+1}-2})(\{x,y,w\})\}) \\
&= \{\{x,y,u\}^{(n)},(\alpha^{2^n})^2(v),(\alpha^{2^n})^2(w)\}^{(n)} \\
&+ (-1)^{|u|(|x||y|)}\{(\alpha^{2^n})^2(u),\{x,y,v\}^{(n)},(\alpha^{2^n})^2(w)\}^{(n)} \\
&+ (-1)^{(|x|+|y|)(|u|+|v|)}\{(\alpha^{2^n})^2(u),(\alpha^{2^n})^2(v),\{x,y,w\}^{(n)}\}^{(n)}
\end{aligned}$$

Thus $(SHLY8)$ holds for $A^{(n)}$. Therefore, we get that $A^{(n)}$ is a Hom-LY superalgebra. This finishes the proof. □

REMARK 4.4. (1) If $[x,y]^{(n)}=0$, for all $x,y\in\mathcal{H}(L)$, then the *nth* derived Hom-Lie-Yamaguti superalgebra becomes the *nth* derived Hom-Lie supertriple system.

(2) If $\{x,y,z\}^{(n)}=0$, for all $x,y,z\in\mathcal{H}(L)$, then the *nth* derived Hom-LY superalgebra becomes the *nth* derived Hom-Lie superalgebra.

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References

[1] K. Abdaoui, F. Ammar and A. Makhlouf, *Hom-Lie Superalgebras and Hom-Lie admissible Superalgebras*, Journal of algebra. **324** (7) (2010), 1513–1528.

[2] , K. Abdaoui, F. Ammar and A. Makhlouf, *Hom-alternative, Hom-Malcev and Hom-Jordan Superalgebras*, arXiv: 1304.1579v1 [math.RA] Apr 2013.

[3] H. Albuquerque and A. Elduque, *Classification of Mal'tsev Superalgebras of small dimensions,* Algebra and Logic **35** (6) (1996), 512–554.

[4] H. Ataguema, A. Makhlouf and S.D. Silvestrov, *Generalization of n-ary Nambu algebras and beyond,* J. Math. Phys. **50** (2009), 083501.

[5] S. Attan and A. Nourou Issa, *Hom-Bol algebras*, Quasigroups and related Systems **21** (2013), 131–146.

[6] Y. Fregier and A. Gohr, *Unital algebras of Hom-associative type and surjective or injective twistings,* J. of Gen. Lie Theo. and Appl. **3** (4) (2009), 285–295.

[7] D. Gaparayi and A. N. Issa, *A twisted generalization of Lie-Yamaguti algebras*, Int. J. Algebra. **6** (7) (2012), 339–352.

[8] D. Gaparayi and A. N. Issa, *Hom-Akivis superalgebras*, Journal of Algebra and Computational Applications. **6** (1) (2017), 36–51.

[9] J. T. Hartwig, D. Larsson and S. D. Silvestrov, *Deformations of Lie algebras using $\sigma-$derivations,* J. Algebras **292** (2006), 314–361.

[10] A. N. Issa, *Hom-Akivis algebras*, Comment. Math. Univ. Carolin. **52** (4) (2011), 485–500.
[11] A. N. Issa, *Supercommutator algebras of right (Hom)-alternative superalgebras*, arXiv:1710.02706v1[math.RA]
[12] A. Koulibaly and M.F. Ouedraogo, *Supersystèmes Triples de Lie associés aux superalgèbres de Malcev,* Africa Matematika **14** (3) (2002).
[13] D. Larsson and S. D. Silvestrov, *Quasi-Hom-Lie algebras, central extensions and 2-cycle-like identies,* J. Algebra. **288** (2005), 321–344.
[14] D. Larsson and S. D. Silvestrov, *Quasi-Lie algebras,* Comptemp.Math. **391** (2005).
[15] A. Makhlouf, *Hom-Alternative algebras and Hom-Jordan algebras,* Int. Elect. J. Alg. **8** (2010), 177–190.
[16] A. Makhlouf and Silvestrov S.D., *Hom-algebra structures,* J. Gen. Lie Theory Appl. **2** (2008), 51–64.
[17] S. Okubo, *Jordan-Lie Super Algebra and Jordan-Lie Triple System,* Journal of algebra **198** (1997), 388–411.
[18] D. Yau, *Hom-algebras and homology,* J. Lie Theory. **19** (2009), 409–421.
[19] D. Yau, *Hom-Novikov algebras,* J. Phys. **A 44** (2011), 085202.
[20] D. Yau, *Hom-Maltsev, Hom-alternative and Hom-Jordan algebras,* Int. Elect. J. Alg. **11** (2012), 177–217.
[21] D. Yau , *On n-ary Hom-Nambu and Hom-Nambu-Lie algebras,* J. Geom. Phys. **62** (2012), 506–522.
[22] T. Zhang and J. Li, *Representations and cohomologies of Hom-Lie-Yamaguti algebras with applications,* Colloquium Mathematicum **148** (2) (2017), 131–155.

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