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TRIPLE CENTRALIZERS OF C^*-ALGEBRAS

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Abstract. In this paper, we extend the concept of double centralizer to triple centralizer and we show that, the triple centralizer is a C^*-algebra. Some algebraic properties are investigated.

1. Introduction

An involution on an algebra $\mathscr{A}$ is a conjugate-linear map $a \longrightarrow a^*$ on $\mathscr{A}$, such that $a^{**} = a$ and $(ab)^* = b^*a^*$ for all $a, b \in \mathscr{A}$. The pair $(\mathscr{A}, *)$ is called a *-algebra. A Banach *-algebra is a *-algebra $\mathscr{A}$ together with a complete submultiplicative norm such that $\|a^*\| = \|a\|$ $(a \in \mathscr{A})$. If, in addition, $\mathscr{A}$ has a unit such that $\|1\| = 1$, we call $\mathscr{A}$ a unital Banach *-algebra. A C^*-algebra is a Banach *-algebra such that $\|a^*a\| = \|a\|^2$ $(a \in \mathscr{A})$.

The notion of double centralizers was introduced by Hochschild [5] and by Johnson [4]. For a C^*-algebra $\mathscr{A}$ a linear mapping $T^{'} : \mathscr{A} \to \mathscr{A}$ is said to be left centralizer on $\mathscr{A}$ if $T^{'}(xy) = T^{'}(x)\,y$, for all $x, y \in \mathscr{A}$. Similarly, a linear mapping $T^{''} : \mathscr{A} \to \mathscr{A}$ such that $T^{''}(xy) = xT^{''}(y)$ for all $x, y \in \mathscr{A}$, is called right centralizer on $\mathscr{A}$. A double centralizer on $\mathscr{A}$ is a pair $\left(T^{'}, T^{''}\right)$, where $T^{'}$ is a left centralizer, $T^{''}$ is a right centralizer and $xT^{'}(y) = T^{''}(x)\,y$ for all $x, y \in \mathscr{A}$. For example, $\left(T_c^{'}, T_c^{''}\right)$ is a double centralizer, where $T_c^{'}(x) = cx$ and $T_c^{''}(x) =$

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xc. The set of all double centralizers equipped with the multiplication $\left(T_1^{'}, T_1^{''}\right) . \left(T_2^{'}, T_2^{''}\right) = \left(T_1^{'}T_2^{'}, T_2^{''}T_1^{''}\right)$ is an algebra.

We associate to each C^*-algebra $\mathscr{A}$ a certain unital C^*-algebra $\mathcal{M}\left(\mathscr{A}\right)$ which contains $\mathscr{A}$ as an ideal. This algebra is of great importance in more advanced aspects of the theory, especially in certain approaches to K-theory. In ([7], Chapter 2), the concept of double centralizer is studied. The importance of the study of double centralizers is that it is unital. In this work, we generalized this notion for triplet $\left(T^{'}, T^{''}, T^{'''}\right)$. Some interesting properties and characterizations are introduced and discussed.

For a comprehensive account on double centralizers and its various applications we refer the reader to [2, 3, 6, 7].

2. Main Results

2.1. The triple centralizer algebra of a C^*-algebra. In concluding this section, we state the following definition for their importance in the material of our paper.

DEFINITION 2.1. A triple centralizer for a C^*-algebra $\mathscr{A}$ is a triplet $\left(T^{'}, T^{''}, T^{'''}\right)$ of bounded linear maps on $\mathscr{A}$, such that for all $x, y, z \in \mathscr{A}$

(2.1)
$$T^{'}\left(x, y, z\right) = T^{'}\left(x\right) yz,\ T^{''}\left(x, y, z\right) = xT^{''}\left(y\right) z,\ T^{'''}\left(x, y, z\right) = xyT^{'''}\left(z\right)$$

and

(2.2)
$$xT^{'}\left(y\right) z = T^{''}\left(x\right) yz,\ xyT^{''}\left(z\right) = xT^{'''}\left(y\right) z,\ xyT^{'}\left(z\right) = T^{'''}\left(x\right) yz.$$

EXAMPLE 2.1. Let $\mathscr{A}$ be a C^*-algebra. If $c \in \mathscr{A}$ and $T_c{'}, T_c{''}$ and $T_c{'''}$ are the linear maps on $\mathscr{A}$ defined by $T_c{'}\left(x\right) = cx$,$T_c{''}\left(x\right) = xc$ and $T_c{'''}\left(x\right) = c^{\frac{1}{2}}xc^{\frac{1}{2}}$, then $(T_c{'}, T_c{''}, T_c{'''})$ is a triple centralizer on $\mathscr{A}$.

REMARK 2.1. If $\mathscr{A}$ is a commutative C^*-algebra, then $T_c{'} = T_c{''} = T_c{'''}$.

The following lemma is a simple consequence of the classical paper related to Johnson (see [4]).

LEMMA 2.1. *Let $x, y, z, w, w' \in \mathscr{A}$.*
1. If $zx = zy$, then $x = y$.
2. If $zwx = zw'x$, then $w = w'$.

As an application of Lemma 2.1, we offer the following theorem:

THEOREM 2.1. *Let* $\left(T^{'}, T^{''}, T^{'''}\right) \in \mathcal{M}(\mathscr{A})$. *Then*

(a) $T^{'}$, $T^{''}$ *and* $T^{'''}$ *are continuous maps from* $\mathscr{A}$ *to* $\mathscr{A}$.
(b) $T^{'}(xyz) = T^{'}(x)\, yz$ *for all* x, y, z *in* $\mathscr{A}$.
(c) $T^{''}(xyz) = xT^{''}(y)\, z$ *for all* x, y, z *in* $\mathscr{A}$.
(d) $T^{'''}(xyz) = xyT^{'''}(z)$ *for all* x, y, z *in* $\mathscr{A}$.

Proof. We prove only the statements made about $T^{'}$. The statements made about $T^{''}$ and $T^{'''}$ are proved analogously. Let $x, y, z, w \in \mathscr{A}$. Let $(x_i)_{i\in I}$ be a net in $\mathscr{A}$, and let α and β be complex numbers. Then

$$\begin{aligned} zT'(\alpha x + y)\, w &= T''(z)(\alpha x + y)\, w \\ &= T''(z)(\alpha xw + yw) \\ &= \alpha T''(z)(xw) + T''(z)\, yw \\ &= \alpha z T'(x)\, w + zT'(y)\, w \\ &= z(\alpha T'(x) + T'(y))\, w \end{aligned}$$

and therefore

$$T'(\alpha x + y) = \alpha T'(x) + T'(y).$$

Now, suppose that

$$\lim_{i\to\infty} \|x_i - x\| = \lim_{i\to\infty} \left\|T^{'}(x_i) - y\right\| = 0,$$

then

$$\begin{aligned} \left\|zwT^{'}(x) - zwy\right\| &\le \left\|zwT^{'}(x) - zwT^{'}(x_i)\right\| + \left\|zwT^{'}(x_i) - zwy\right\| \\ &\le \left\|T^{'''}(z)\right\| \|wx - wx_i\| + \|zw\| \left\|T^{'}(x_i) - y\right\|. \end{aligned}$$

Since the last term of this inequality tends to zero, we have

$$zwT^{'}(x) = zwy,$$

for all $z \in \mathscr{A}$. Thus $y = T^{'}(x)$ and $T^{'}$ has a closed graph. By the closed graph theorem, $T^{'}$ is continuous.

Let $x, y, z \in \mathscr{A}$, then

$$zT'(xy) = T''(z)\, xy = ((T''z)\, x)\, y = zT'(x)\, y$$

therefore

$$T'(xy) = T'(x)\, y$$

and the proof is completed. □

Remark 2.2. In the above theorem, we have shown that if $\left(T^{'}, T^{''}, T^{'''}\right) \in \mathcal{M}(\mathscr{A})$, then $T^{'}$, $T^{''}$ and $T^{'''}$ are continuous and linear. Therefore, $T^{'}$ may be given the usual norm

$$\left\|T^{'}\right\| = \sup_{\|x\|=1} \left\|T^{'}(x)\right\|.$$

Similarly the above definition holds for $T^{''}$ and $T^{'''}$.

Lemma 2.2. *It is easily checked that for all $x \in \mathscr{A}$*

$$\|x\| = \sup_{\|y\|=1} \|xy\|.$$

The next result will be used in the proof of the main theorem which follows.

Theorem 2.2. *If $\left(T^{'}, T^{''}, T^{'''}\right)$ is a triple centralizer on a C^{*}-algebra $\mathscr{A}$, then $\|T^{'}\| = \|T^{''}\| = \|T^{'''}\|$.*

Proof. Since

$$\|xT'(y)z\| = \|T''(x)yz\| \leq \|T''\|$$

hence $\|T'(y)\| \leq \|T''\|$, on taking the supremum over $y \in \mathscr{A}$ with $\|y\| = 1$, we deduce that

$$\left\|T^{'}\right\| \leq \left\|T^{''}\right\|.$$

Also

$$\|T''(x)yz\| = \|xT'(y)z\| \leq \|T'\|$$

and therefore $\|T''(x)\| \leq \|T'\|$, taking the supremum over $x \in \mathscr{A}$ with $\|x\| = 1$, we obtain

$$\left\|T^{''}\right\| \leq \left\|T^{'}\right\|.$$

Thus, $\left\|T^{'}\right\| = \left\|T^{''}\right\|$. Similar results may be stated for $\|T''\|$ and $\|T'''\|$. However the details are left to the interested reader. □

Definition 2.2. If $\left(T^{'}, T^{''}, T^{'''}\right)$ and $\left(S^{'}, S^{''}, S^{'''}\right)$ be in $\mathcal{M}(\mathscr{A})$, we define their product to be $\left(T^{'}, T^{''}, T^{'''}\right).\left(S^{'}, S^{''}, S^{'''}\right) = \left(T^{'}S^{'}, T^{''}S^{''}, S^{'''}T^{'''}\right)$. If $T^{'} : \mathscr{A} \to \mathscr{A}$, define $T^{'^{*}} : \mathscr{A} \to \mathscr{A}$ by setting $T^{'^{*}}(x) = \left(T^{'}(x^{*})\right)^{*}$. Then $T^{'^{*}}$ is linear and the map $T^{'} \to T^{'^{*}}$ is an isometric conjugate-linear map form $\mathcal{B}(\mathscr{A})$ to itself such that $T^{'^{**}} = T^{'}$ and $\left(T_1^{'}T_2^{'}\right)^{*} = {T_2^{'}}^{*}{T_1^{'}}^{*}$. If $\left(T^{'}, T^{''}, T^{'''}\right)$ is a triple centralizer on $\mathscr{A}$, so is $\left(T^{'}, T^{''}, T^{'''}\right)^{*} = \left(T^{'''^{*}}, T^{''^{*}}, T^{'^{*}}\right)$.

Remark 2.3. It is easy to see that $(T', T'', T''') \to (T', T'', T''')^*$ is an involution on $\mathcal{M}(\mathscr{A})$.

Theorem 2.3. *If $\mathscr{A}$ is a C^*-algebra, then $\mathcal{M}(\mathscr{A})$ is a C^*-algebra under the multiplication, involution and norm as defined above.*

Proof. Easy computation and simplification yield

$$T_1^{'}T_2^{'}(xyz) = T_1^{'}\Big(T_2^{'}(x)\,yz\Big) = T_1^{'}\Big(T_2^{'}(x)\Big)\,yz,$$

$$T_1^{''}T_2^{''}(xyz) = T_1^{''}\Big(\alpha T_2^{''}(y)\,z\Big) = \alpha T_1^{''}\Big(T_2^{''}(y)\Big)\,z,$$

$$T_2^{'''}T_1^{'''}(xyz) = T_2^{'''}\Big(xyT_1^{'''}(z)\Big) = xyT_2^{'''}\Big(T_1^{'''}(z)\Big),$$

for each $x, y, z \in \mathscr{A}$. Also

$$xT_1^{'}T_2^{'}(y)\,z = xT_1^{'}\Big(T_2^{'}(y)\Big)\,z{=}T_1^{''}\Big(T_2^{''}(x)\Big)\,yz,$$

$$xT_2^{'''}T_1^{'''}(y)\,z = xT_2^{'''}\Big(T_1^{'''}(y)\Big)\,z{=}xyT_2^{''}\Big(T_1^{''}(z)\Big),$$

since $\left(T^{'}, T^{''}, T^{'''}\right)^* = \Big(T^{'''*}, T^{''*}, T^{'*}\Big)$, then

$$\begin{aligned} T^{'*}(xyz) &= \Big(T^{'}(xyz)^*\Big)^* = \left(L\left(z^*y^*x^*\right)\right)^* = \Big(T^{'}\left(z^*\right)y^*x^*\Big)^* \\ &= \alpha\beta\Big(T^{'}\left(\gamma^*\right)\Big)^* = xyT^{'*}(z), \end{aligned}$$

$$\begin{aligned} T^{''*}(xyz) &= \Big(T^{''}(xyz)^*\Big)^* = \Big(T^{''}\left(z^*y^*x^*\right)\Big)^* = \Big(z^*T^{''}\left(y^*\right)x^*\Big)^* \\ &= x\Big(T^{''}\left(y^*\right)\Big)^* z = xT^{''*}(y)\,z, \end{aligned}$$

$$\begin{aligned} T^{'''*}(xyz) &= \Big(T^{'''}(xyz)^*\Big)^* = \Big(T^{'''}\left(z^*y^*x^*\right)\Big)^* = \Big(z^*y^*T^{'''}\left(x^*\right)\Big)^* \\ &= \Big(T^{'''}\left(x^*\right)\Big)^* yz = T^{'''*}(x)\,yz. \end{aligned}$$

To prove that $\mathcal{M}(\mathscr{A})$ is a $*$-algebra we must show that

$$\left\|\Big(T^{'}, T^{''}, T^{'''}\Big)\right\| = \left\|\Big(T^{'}, T^{''}, T^{'''}\Big)^*\right\|.$$

It is easy to see that

$$\begin{aligned}\left\|(T',T'',T''')^*\right\| &= \left\|\left(T'''^*,T''^*,T'^*\right)\right\| = \left\|T'^*\right\| \\ &= \sup\left\{\left\|T'^*\left(xy\right)\right\|:\ \ \|x\|=\|y\|=1\right\} \\ &= \sup\left\{\left\|(T'(xy)^*)^*\right\|:\ \ \|x\|=\|y\|=1\right\} \\ &= \sup\left\{\left\|\left(T'^*\left(y^*x^*\right)\right)^*\right\|:\ \ \|x^*\|=\|y^*\|=1\right\}\end{aligned}$$

therefore, we can state that

$$\left\|\left(T',T'',T'''\right)\right\| = \left\|\left(T',T'',T'''\right)^*\right\|.$$

Now, we show that $\mathcal{M}\left(\mathscr{A}\right)$ is a C^*-algebra.

(2.3)
$$\begin{aligned}\left\|\left(T',T'',T'''\right)^*\left(T',T'',T'''\right)\right\| &= \left\|\left(T'''^*,T''^*,T'^*\right)\left(T',T'',T'''\right)\right\| \\ &= \left\|\left(T'''^*T',T''^*T'',T'''T'^*\right)\right\| \\ &= \left\|T'''^*T'\right\| \\ &\le \left\|T'''^*\right\|\left\|T'\right\| \\ &= \left\|T'\right\|^2 \\ &= \left\|\left(T',T'',T'''\right)\right\|^2\end{aligned}$$

therefore, we have

$$\left\|\left(T',T'',T'''\right)^*\left(T',T'',T'''\right)\right\| \le \left\|\left(T',T'',T'''\right)\right\|^2. \tag{2.4}$$

On the other hand, for $x,y,z\in\mathscr{A}$ we have

$$\begin{aligned}\left\|T''\left(xyz\right)\right\|^2 &= \left\|\left(T''\left(xyz\right)\right)^*T''\left(xyz\right)\right\| \\ &= \left\|T''^*\left(z^*y^*x^*\right)T''\left(xyz\right)\right\| \\ &= \left\|T''^*\left(z^*y^*x^*\right)xT''\left(y\right)z\right\| \\ &= \left\|T''T''^*\left(z^*y^*x^*\right)xyz\right\| \\ &\le \left\|T''T''^*\right\|\left\|(xyz)\right\|^2\end{aligned}$$

since $\left\| T^{''} T^{''*} \right\| = \left\| T^{'''*} T^{'} \right\|$ and by (2.3)

$$\left\| \left(T^{'}, T^{''}, T^{'''} \right) \right\|^2 \leq \left\| \left(T^{'}, T^{''}, T^{'''} \right)^* \left(T^{'}, T^{''}, T^{'''} \right) \right\|. \tag{2.5}$$

Now from (2.4) and (2.5) it follows that

$$\left\| \left(T^{'}, T^{''}, T^{'''} \right)^* \left(T^{'}, T^{''}, T^{'''} \right) \right\| = \left\| \left(T^{'}, T^{''}, T^{'''} \right) \right\|^2.$$

This completes the proof. □

2.2. Application of triple centralizer on semigroups. In this section, we investigate some algebraic properties of the triple centralizers. We begin by two basic definitions.

Definition 2.3. A left (rep. center and right) centralizer on semigroup G is a map $T : G \to G$ such that $T(\alpha\beta\gamma) = T(\alpha)\beta\gamma$ (resp. $T(\alpha\beta\gamma) = \alpha T(\beta)\gamma$ for center and $T(\alpha\beta\gamma) = \alpha\beta T(\gamma)$ for right) for all $\alpha, \beta, \gamma \in G$. A triple centralizer is an ordered triplet $\left\{T^{'}, T^{''}, T^{'''}\right\}$ of maps $G \to G$ such that $\alpha T^{'}(\beta)\gamma = T^{''}(\alpha)\beta\gamma$, $\alpha\beta T^{''}(\gamma) = \alpha T^{'''}(\beta)\gamma$ and $\alpha\beta T^{'}(\gamma) = T^{'''}(\alpha)\beta\gamma$. If T is a left (resp. center and right) centralizer on G then we shall write Txy (resp. xTy and xyT) for $T(xy)$. We denote the set of all left (resp. center and right) centralizers on G by $\Gamma_L(G)$ (resp. $\Gamma_C(G)$ and $\Gamma_R(G)$). It is clear that $\Gamma_L(G)$ (resp. $\Gamma_C(G)$ and $\Gamma_R(G)$) is a semigroup.
Each element ab of G generates a left (resp. center and right) centralizer on G defined by $L_{ab} : x \to abx$ (resp. $C_{ab} : x \to axb$ and $R_{ab} : x \to xab$) for all x in G.

Definition 2.4. Let a, b, c, d be elements of G. ab and cd is called left isoproductive if $ab \neq cd$ and $abx = cdx$ for all $x \in G$ (that is if $L_{ab} = L_{cd}$). If G has no pairs of left isoproductive elements we say that G is left faithful. If G is left and center and right faithful we shall say that it is faithful.

Remark 2.4. We say that G has a left cancellation law if for $a, b, c, d \in G$, $xab = xcd$ then $ab = cd$. The rational extension of a commutative semigroup with cancellation law is the smallest group in which it can be embedded.

We can state the following result as well.

Theorem 2.4. *The left regular representation is a homomorphism of G into $\Gamma_L(G)$. It is an isomorphism if and only if G is left faithful, and is onto if and only if G has a left identity element. If G is commutative then $\Gamma_L(G) = \Gamma_C(G) = \Gamma_R(G)$, and if G is faithful and commutative then $\Gamma_L(G)$ is commutative. If G is commutative and has a cancellation law then $\Gamma_L(G)$ is a sub-semigroup of the rational extension of G and, in particular, has a cancellation law.*

Proof. Let G be faithful and commutative. Let $S, T \in \Gamma_L(G)$ and let $x, y, z, w \in G$. Then $S(Txy)zw = Szx.Txy = Txy.Szx = T(xy.Szw) = T(Sxyzw) = T(Sxy)zw$. Since this holds for all zw in G, $STxy = TSxy$ for all $x, y \in G$, that is $ST = TS$. If G has a cancellation law then G is certainly faithful, and if $\Re$ is the rational extension of G then each element T of $\Gamma_L(G)$ generates an element T' of $\Gamma_L(\Re)$ by $T'(x, y, z) = (Tx, y, z)$ for all $x, y, z \in G$. This gives an isomorphism of $\Gamma_L(G)$ into $\Gamma_L(\Re)$, and since $\Re$ has an identity, $\Gamma_L(\Re)$ is isomorphic with $\Re$. □

Remark 2.5. The set $\Gamma(G)$ of triple centralizers may be formed into an associative semigroup by defining multiplication by

$$\left\{S', S'', S'''\right\}\left\{T', T'', T'''\right\} = \left\{S' o T', S'' o T'', T''' o S'''\right\}$$

where o denotes composition of functions.

By $\alpha \to D_\alpha = \{L_\alpha, C_\alpha, R_\alpha\}$, each α in G generates an element D_α of $\Gamma(G)$. This map is called the triple representation of G. We now show the relationship between Γ_L and Γ.

Theorem 2.5. *Let G be right faithful and let $\left\{T', T'', T'''\right\}$ be a triple centralizer on G. Then T' is a left centralizer. Also if $\{S_1, T_1, T_2\}$, $\{S_2, T_1, T_2\}$ are triple centralizer then $S_1 = S_2$.*

Proof. For the first part we have $zwT'(x)y = zT''(w)xy = T'''(z)wxy = zT''(wx)y = zwT'(xy)$ for all x, y, z, w in G. For the second,

$$xyS_1(z) = xT_1(y)z = T_2(x)yz = xyS_2(z)$$

for all x, y, z in G. □

Now, the following result is evident:

Corollary 2.1. *If G is faithful and commutative then*

$$\Gamma_L(G) = \Gamma_C(G) = \Gamma_R(G) = \Gamma(G).$$

Finally and from a different view point we may state:

THEOREM 2.6. *The triple representation of G is a homomorphism of G into $\Gamma(G)$. If G is right faithful and has a right cancellation law, then $\Gamma(G)$ has a right cancellation law.*

Proof. Suppose that G is right faithful and has a right cancellation law, and that $S_1, S_2, T_1, T_2 \in G$ with $S_1T_1T_2 = S_2T_1T_2$. Then for all x, y, z in G,

$$\begin{aligned} S_1^{''}(x).yT^{'}(z) &= x\left[S_1^{'}\left(yT^{'}(z)\right)\right] = x\left[S_2^{'}\left(T^{''}(y)\,z\right)\right] \\ &= x\left[S_2^{'}\left(yT^{'}(z)\right)\right] = S_2^{''}(x)\,yT^{'}(z)\,. \end{aligned}$$

So that ${S_1}^{''} = {S_2}^{''}$. As G is right faithful then by Theorem 2.5 $S_1 = S_2$. □

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