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ONE-DIMENSIONAL JUMPING PROBLEM INVOLVING p-LAPLACIAN

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Abstract. We get one theorem which shows existence of solutions for one-dimensional jumping problem involving p-Laplacian and Dirichlet boundary condition. This theorem is that there exists at least one solution when nonlinearities crossing finite number of eigenvalues, exactly one solutions and no solution depending on the source term. We obtain these results by the eigenvalues and the corresponding normalized eigenfunctions of the $p-$Laplacian eigenvalue problem when $1 < p < \infty$, variational reduction method and Leray-Schauder degree theory when $2 \leq p < \infty$.

1. Introduction

Let $\Omega = (c, d) \subset R$, $c < d$, is an open interval. Let $p \in (1, \infty)$ and p' by $\frac{1}{p} + \frac{1}{p'} = 1$. Let $L^p(\Omega, R)$ be $p-$Lebesgue space with its dual

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space $L^{p'}(\Omega)$ and $W^{1,p}(\Omega, R)$ be the Lebesgue Sobolev space. When $1 < p < \infty$, it was proved in [7] the eigenvalue problem

$$-(|u'|^{p-2}u')' = \lambda|u|^{p-2}u \qquad \text{in } \ \Omega,$$

$$u = 0 \qquad \text{on } \ \partial\Omega.$$

has a nondecreasing sequence of nonnegative eigenvalues λ_j obtained by the Ljusternik-Schnirelman principle tending to ∞ as $j \to \infty$, where the first eigenvalue λ_1 is simple and only eigenfunctions associated with λ_1 do not change sign, the set of eigenvalues is closed, the first eigenvalue λ_1 is isolated. Thus there are two sequences of eigenfunctions $(\phi_j)_j$ and $(\mu_j)_j$ corresponding to the eigenvalues λ_j such that the first eigenfunction $\phi_1 > 0$ in the sequence $(\phi_j)_j$ and the first eigenfunction $\psi_1 < 0$ in the sequence $(\psi_j)_j$.

In this paper we consider multiplicity of solutions $u \in W^{1,p}(\Omega, R)$ for the following one-dimensional jumping problem involving p-Laplacian and Dirichlet boundary value condition;

$$-(|u'|^{p-2}u')' = b|u|^{p-2}u^+ - a|u|^{p-2}u^- + s\phi_1^{p-1} \qquad \text{in } \ \Omega, \tag{1.1}$$

$$u = 0 \qquad \text{on } \partial\Omega,$$

where $s \in R$, $u^+ = \max\{u, 0\}$ and $u^- = -\min\{u, 0\}$.

$p-$Laplacian boundary value problems with $p-$growth conditions arise in applications of nonlinear elasticity theory, electro rheological fluids, non-Newtonian fluid theory in a porous medium (cf. [5], [11]. Our problems are characterized as a jumping problem. Jumping problem was first suggested in the suspension bridge equation as a model of the nonlinear oscillations in differential equation

$$u_{tt} + K_1 u_{xxxx} + K_2 u^+ = W(x) + \epsilon f(x, t),$$

$$u(0, t) = u(L, t) = 0, \qquad u_{xx}(0, t) = u_{xx}(L, t) = 0.$$

This equation represents a bending beam supported by cables under a load f. The constant b represents the restoring force if the cables stretch. The nonlinearity u^+ models the fact that cables resist expansion but do not resist compression. Choi and Jung (cf. [1], [3], [4]) and McKenna and Walter (cf.[10]) investigate the existence and multiplicity of solutions for the single nonlinear suspension bridge equation with Dirichlet boundary condition. In [2], the authors investigate the multiplicity of solutions of a semilinear equation

$$Au + bu^+ - au^- = f(x) \qquad \text{in } \ \Omega,$$

$$u = 0 \qquad \text{on } \Omega,$$

where Ω is a bounded domain in R^n, $n \geq 1$, with smooth boundary $\partial\Omega$ and A is a a second order linear partial differential operator when the forcing term is a multiple $s\phi_1$, $s \in R$, of the positive eigenfunction and the nonlinearity crosses eigenvalues.

Let us set the operator $-L$ by

$$-Lu = -(|u'|^{p-2}u')'.$$

Then (1.1) is equivalent to the equation

$$u = (-L)^{-1}(b|u|^{p-2}u^+ - a|u|^{p-2}u^- + s\phi_1^{p-1}).$$

Our main theorem is as follows:

THEOREM 1.1. *Let $a < b$, $-\infty < a < \lambda_1, \cdots, \lambda_n < b < \lambda_{n+1}$ and $s \in R$. Then*
(i) if $1 < p < \infty$ and $s > 0$, then (1.1) has no solution
(ii) if $1 < p < \infty$ and $s = 0$, then (1.1) has a unique trivial solution $u = 0$.
(iii) if $2 \leq p < \infty$, there exists $s_1 < 0$ such that for any s with $s_1 < s < 0$, (1.1) has at least one nontrivial solutions.

For the proof of Theorem 1.1 we use the eigenvalues and the corresponding eigenfunctions of the eigenvalue problem, use variational reduction method and calculate the Leray-Schauder degree of $u-(-L)^{-1}(b\ |u|^{p-2}u^+ - a|u|^{p-2}u^- + s\phi_1^{p-1})$ in the neighborhood of positive solution, and in the whole solution bounded subspace. The outline of the proof of Theorem 1.1 is as follows: In Section 2, we introduce some preliminaries. In Section 3, we prove (i) and (ii) of Theorem 1.1 and some lemmas by using eigenvalues and the corresponding eigenfunctions of the eigenvalue problem, calculate direct computations and Leray-Schauder degree. In Section 4, we prove (iii) of Theorem 1.1 for the case p such that p-Laplacian eigenvalue problem has the first eigenfunction $\psi_1 < 0$. In Section 5, we prove (iii) of Theorem 1.1 for the case p such that p-Laplacian eigenvalue problem has the first eigenfunction $\phi_1 > 0$.

2. Preliminaries

Let $L^p(\Omega, R)$ be the Lebesgue space defined by

$$L^p(\Omega, R) = \{u|\ u : \Omega \to R \text{ is measurable}, \int_\Omega |u|^p dx < \infty\}$$

which is endowed with the norm

$$\|u\|_{L^p(\Omega)} = \inf\{\lambda > 0| \int_\Omega |\frac{u(x)}{\lambda}|^p \leq 1\},$$

and $W^{1,p}(\Omega, R)$ be the Lebesgue Sobolev space defined by

$$W^{1,p}(\Omega, R) = \{u \in L^p(\Omega, R)|\ u' \in L^p(\Omega, R)\}$$

which is endowed with the norm

$$\|u\|_{W^{1,p}(\Omega,R)} = [\int_\Omega |u'(x)|^p dx]^{\frac{1}{p}} + [\int_\Omega |u(x)|^p dx]^{\frac{1}{p}}.$$

Let $1 < p < \infty$ and $h \in L^r(\Omega))$, $r > 1$. Then the problem

$$-(|u'|^{p-2}u')' = h(x) \qquad \text{in } \ \Omega, \tag{2.1}$$

$$u = 0 \qquad \partial\Omega$$

has a unique solution $u \in C^1(\bar{\Omega})$ which is of the form

$$u(x) = \int_\Omega g_p^{-1}\big(c_f - \int_\Omega h(\tau)d\tau\big)dy, \tag{2.2}$$

where $g_p(t) = |t|^{p-2}t$ for $t \neq 0$, $g_p(0) = 0$ and its inverse g_p^{-1} is $g_p^{-1}(t) = t^{\frac{1}{p-1}}$ if $t > 0$ and $g_p^{-1}(t) = -|t|^{\frac{1}{p-1}}$ if $t < 0$ and c_f is the unique constant such that $u = 0$ on $\partial\Omega$. By [[8], Lemma 2.1 or [9], Lemma 4.2], the solution operator S satisfies that $S : L^P(\Omega) \to C^1(\bar{\Omega})$ is continuous and by [[12], Corollary 2.3], the embedding $S : L^p(\Omega) \to C(\bar{\Omega})$ is continuous and compact. By [6], we also have *Poincaré*-type inequality.

Lemma 2.1. *Let $1 < p < \infty$. Then the embedding*

$$W^{1,p}(\Omega, R) \hookrightarrow L^p(\Omega, R)$$

is continuous and compact and for every $u \in C_0^\infty(\Omega, R)$, we have

$$\|u\|_{L^p(\bar{\Omega},R)} \leq C\|u\|_{W^{1,p}(\bar{\Omega},R)}$$

for a positive constant C independent of u.

By Lemma 2.1, we obtain the following:

LEMMA 2.2. *Assume that* $1 < p < \infty$, $f(x,u) \in L^p(\Omega)$. *Then the solutions of the problem*

$$-(|u'|^{p-2}u')' = f(x,u) \qquad \text{in } \ L^p(\Omega),$$
$$u = 0 \qquad \partial\Omega$$

belong to $W^{1,p}(\Omega)$.

LEMMA 2.3. *Assume that* $2 \le p < \infty$, $v(u) \in L^p(\Omega)$ *and* $h(x) \in L^p(\Omega)$. *Then there exists a constant* $C > 0$ *such that the solutions* u *of the problem*

$$-(|u'|^{p-2}u')' = v(u) + h(x) \ \text{in } \ L^p(\Omega),$$
$$u = 0 \qquad \partial\Omega$$

satisfies $\|u\|_{W^{1,p}(\Omega)} < C$.

Proof. For given $v(u) \in L^p(\Omega)$ and $h(x) \in L^p(\Omega)$, the equation

$$-Lu = v(u) + h(x) \text{ in } \ L^p(\Omega)$$

is equivalent to the equation

$$u = (-L)^{-1}(v(u) + h(x)).$$

We observe that

$$\begin{aligned}\|(-L)^{-1}(v(u)+h(x))\|_{L^p(\Omega)} &\le \|\frac{1}{\lambda_1}g_p^{-1}(v(u)+h(x))\|_{L^p(\Omega)} \\ &\le \|\frac{1}{\lambda_1}(v(u)+h(x))^{\frac{1}{p-1}}\|_{L^p(\Omega)}.\end{aligned} \tag{2.3}$$

If $2 \le p < \infty$, then the right hand side of (2.3) is bounded. Thus we prove the lemma. □

3. Proof of Theorem 1.1

Proof of (i) of Theorem 1.1 (For the case $s > 0$)

We first consider the case p such that $1 < p < \infty$ and p-Laplacian eigenvalue problem has the first eigenfunction $\psi_1 < 0$. We assume that $a < b$, $-\infty < a < \lambda_1, \cdots, \lambda_n < b < \lambda_{n+1}$ and $s > 0$. Then (1.1) can be rewritten as

$$-(|u'|^{p-2}u')' - \lambda_1|u|^{p-2}u = (b-\lambda_1)|u|^{p-2}u^+ - (a-\lambda_1)|u|^{p-2}u^- + s\phi_1^{p-1}. \tag{3.1}$$

Taking inner product both side of (3.1) with ψ_1, we have

$$\begin{aligned}&\Big\langle -(|u'|^{p-2}u')' - \lambda_1|u|^{p-2}u, \psi_1\Big\rangle\\ &= \Big\langle (b-\lambda_1)|u|^{p-2}u^+ - (a-\lambda_1)|u|^{p-2}u^- + s\phi_1^{p-1}, \psi_1\Big\rangle. \qquad (3.2)\end{aligned}$$

The left hand side of (3.2) is equal to 0. On the other hand, the right hand side of (3.2) is negative because $b-\lambda_1 > 0$, $-(a-\lambda_1) > 0$, $s\phi_1^{p-1} > 0$ for $s > 0$, and $\psi_1 < 0$. Thus if $s > 0$, then there is no solution for (1.1). We next consider the case p such that $1 < p < \infty$ and p-Laplacian eigenvalue problem has the first eigenfunction $\phi_1 > 0$. Taking inner product both side of (3.1) with ϕ_1, we have

$$\begin{aligned}&\Big\langle -(|u'|^{p-2}u')' - \lambda_1|u|^{p-2}u, \phi_1\Big\rangle\\ &= \Big\langle (b-\lambda_1)|u|^{p-2}u^+ - (a-\lambda_1)|u|^{p-2}u^- + s\phi_1^{p-1}, \phi_1\Big\rangle. \qquad (3.3)\end{aligned}$$

The left hand side of (3.3) is equal to 0. On the other hand, the right hand side of (3.3) is positive because $b-\lambda_1 > 0$, $-(a-\lambda_1) > 0$, $s\phi_1^{p-1} > 0$ for $s > 0$, and $\phi_1 > 0$. Thus if $s > 0$, then there is no solution for (1.1).

Proof of (ii) of Theorem 1.1 (For the case $s = 0$)

We first consider the case p such that $1 < p < \infty$ and p-Laplacian eigenvalue problem has the first eigenfunction $\psi_1 < 0$. If $s = 0$, then (3.2) is reduced to the equation

$$\Big\langle -(|u'|^{p-2}u')' - \lambda_1|u|^{p-2}u, \psi_1\Big\rangle = \Big\langle (b-\lambda_1)|u|^{p-2}u^+ - (a-\lambda_1)|u|^{p-2}u^-, \psi_1\Big\rangle,$$

i.e.,

$$\begin{aligned}&\int_\Omega [(-(|u'|^{p-2}u')' - \lambda_1|u|^{p-2}u)\psi_1]dx = 0\\ &= \int_\Omega [((b-\lambda_1)|u|^{p-2}u^+ - (a-\lambda_1)|u|^{p-2}u^-)\psi_1]dx. \qquad (3.4)\end{aligned}$$

Since $b-\lambda_1 > 0$, $-(a-\lambda_1) > 0$ and $\psi_1 < 0$, the only possibility to hold (3.4) is that $u = 0$.

We next consider the case p such that $1 < p < \infty$ and p-Laplacian eigenvalue problem has the first eigenfunction $\phi_1 > 0$. If $s = 0$, then (3.3) is reduced to the equation

$$\Big\langle -(|u'|^{p-2}u')' - \lambda_1|u|^{p-2}u, \phi_1\Big\rangle = \Big\langle (b-\lambda_1)|u|^{p-2}u^+ - (a-\lambda_1)|u|^{p-2}u^-, \phi_1\Big\rangle,$$

i.e.,

$$\int_\Omega [(-(|u'|^{p-2}u')' - \lambda_1 |u|^{p-2}u)\phi_1]dx = 0$$

$$= \int_\Omega [((b-\lambda_1)|u|^{p-2}u^+ - (a-\lambda_1)|u|^{p-2}u^-)\phi_1]dx. \tag{3.5}$$

Since $b-\lambda_1 > 0$ and $-(a-\lambda_1) > 0$, the only possibility to hold (3.5) is that $u = 0$.

Lemma 3.1. *(A priori bound) Assume that $2 \le p < \infty$, $-\infty < a < \lambda_1, \cdots, \lambda_n < b < \lambda_{n+1}$, $s \in R$. Then there exist $s_1 < 0$, $s_2 > 0$ and a constant $C > 0$ depending only on a, b and s such that for any any s with $s_1 \le s \le s_2$, any solution u of (1.1) satisfies $\|u\|_{W^{1,p}(\Omega)} < C$.*

Proof. Suppose that the lemma is false. Then there exists a sequence $(u_n)_n$, $(a_n)_n$, $(b_n)_n$ and $(t_n)_n$ such that $-\infty < a_n < \lambda_1, \cdots, \lambda_n < b_n < \lambda_{n+1}$, $s_1 \le t_n \le s_2$, $\|u_n\|_{W^{1,p}(\Omega)} = \rho_n \to \infty$ and

$$-(|u_n'|^{p-2}u_n')' = b_n|u_n|^{p-2}u_n^+ - a_n|u_n|^{p-2}u_n^- + t_n\phi_1^{p-1} \quad \text{in } \Omega \tag{3.6}$$

or equivalently

$$u_n = (-L)^{-1}(b_n|u_n|^{p-2}u_n^+ - a_n|u_n|^{p-2}u_n^- + t_n\phi_1^{p-1}) \quad \text{in } \Omega.$$

Let us set $w_n = \frac{u_n}{\|u_n\|_{W^{1,p}(\Omega)}}$. Then $(w_n)_n$ is bounded, so there exists a subsequence, up to a subsequence $(w_n)_n$ such that $(w_n)_n \to w$ weakly for some w in $W^{1,p}(\Omega)$. Dividing (3.6) by $\|u_n\|^{p-1}_{W^{1,p}(\Omega)}$, we have

$$\frac{-(|u_n'|^{p-2}u_n')'}{\|u_n\|^{p-1}_{W^{1,p}(\Omega)}} = b_n\frac{|u_n|^{p-2}u_n^+}{\|u_n\|^{p-1}_{W^{1,p}(\Omega)}} - a_n\frac{|u_n|^{p-2}u_n^-}{\|u_n\|^{p-1}_{W^{1,p}(\Omega)}} + \frac{t_n\phi_1^{p-1}}{\|u_n\|^{p-1}_{W^{1,p}(\Omega)}} \quad \text{in } \Omega, \tag{3.7}$$

i.e.,

$$w_n = (-L)^{-1}(b_n|w_n|^{p-2}w_n^+ - a_n|w_n|^{p-2}w_n^- + \frac{t_n\phi_1^{p-1}}{\|u_n\|^{p-1}_{W^{1,p}(\Omega)}}) \quad \text{in } \Omega.$$

Since, by Lemma 2.1, the embedding $W^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ is compact, and by Lemma 2.3, when $2 \le p < \infty$, $(-L)^{-1}$ is compact operator, $(w_n)_n \to w$ strongly in $W^{1,p}(\Omega)$. Moreover $(a_n)_n$ and $(b_n)_n$ satisfying $-\infty < a_n < \lambda_1, \cdots, \lambda_n < b_n < \lambda_{n+1}$ converge strongly to some a and b with $-\infty < a < \lambda_1, \cdots, \lambda_n < b < \lambda_{n+1}$. Moreover $(t_n)_n$ with $s_1 \le t_n \le$

s_2 also converge strongly to some s with $s_1 \leq s \leq s_2$. Limiting (3.7) as $n \to \infty$, we have

$$-(|w'|^{p-2}w')' = b|w|^{p-2}w^+ - a|w|^{p-2}w^-. \tag{3.8}$$

By (ii) of Theorem 1.1, (3.8) has only trivial solution, which is absurd because $\|w\|_{W^{1,p}(\Omega)} = 1$. Thus the lemma is proved. □

We shall consider the Leray-Schauder degree on a large ball.

LEMMA 3.2. *Assume that $2 \leq p < \infty$, $-\infty < a < \lambda_1, \cdots, \lambda_n < b < \lambda_{n+1}$. Then there exist a constant $R > 0$ depending on a, b, s, and $s_1 < 0$ and $s_2 > 0$ such that for any s with $s_1 \leq s \leq s_2$, the Leray-Schauder degree*

$$d_{LS}(u - (-L)^{-1}(b|u|^{p-2}u^+ - a|u|^{p-2}u^- + s\phi_1^{p-1}), B_R(0), 0) = 0,$$

where $-Lu = -(|u'|^{p-2}u')'$.

Proof. Let us consider the homotopy

$$F(x, u) = u - (-L)^{-1}(b|u|^{p-2}u^+ - a|u|^{p-2}u^- + s\phi_1^{p-1}). \tag{3.9}$$

By (ii) of Theorem 1.1, for any $s > 0$, (1.1) has no solution. Thus there exist $s_2 > 0$ and a large $R > 0$ such that (3.7) has no zero in $B_R(0)$ for any $s \geq s_2$, and by the a priori bound in Lemma 3.1, there exists $s_1 < 0$ such that for any s with $s_1 \leq s \leq s_2$, all solutions of

$$u - (-L)^{-1}(b|u|^{p-2}u^+ - a|u|^{p-2}u^- + s\phi_1^{p-1}) = 0$$

satisfy $\|u\|_{W^{1,p}(\Omega)} \leq R$ and (3.9) has no zero on ∂B_R for any $s_1 \leq s \leq s_2$. Since

$$d_{LS}(u - (-L)^{-1}(b|u|^{p-2}u^+ - a|u|^{p-2}u^- + s_2\phi_1^{p-1}), B_R(0), 0) = 0,$$

by homotopy arguments, for any $s_1 \leq s \leq s_2$, we have

$$d_{LS}(u - (-L)^{-1}(b|u|^{p-2}u^+ - a|u|^{p-2}u^- + s\phi_1^{p-1}), B_R(0), 0)$$

$$= d_{LS}(u - (-L)^{-1}(b|u|^{p-2}u^+ - a|u|^{p-2}u^- + s\phi_1^{p-1} + \lambda(s_2 - s)\phi_1^{p-1}), B_R(0), 0)$$

$$= d_{LS}(u - (-L)^{-1}(b|u|^{p-2}u^+ - a|u|^{p-2}u^- + s_2\phi_1^{p-1}), B_R(0), 0) = 0$$

for any $0 \leq \lambda \leq 1$. Thus the lemma is proved. □

LEMMA 3.3. *Let K be a compact set in $L^p(\Omega)$. Let $\xi > 0$ a.e. Then there exists a modulus of continuity $\alpha : R \to R$ depending only on K and ξ such that*

$$\|\|(|\tau| - \frac{\xi}{\eta})^+\|_{L^p(\Omega)} \leq \alpha(\eta) \qquad \text{for all } \tau \in K.$$

It follows that

$$\|\|(\eta\tau + \xi)^-\|_{L^p(\Omega)} \leq \eta\alpha(\eta).$$

and

$$\|\|(\eta\tau - \xi)^+\|_{L^p(\Omega)} \leq \eta\alpha(\eta) \qquad \text{for all } \tau \in K.$$

Proof. For any $\tau \in K$, Let $\tau_n = (|\tau| - \frac{\xi}{\eta})^+$. Since $0 \leq \tau_n \leq |\tau|$ and $\tau_n(x) \to 0$ as $\eta \to 0$ a.e., it follows that $\|\tau_n\|_{L^p(\Omega)} \to 0$ for all $\tau \in K$. We claim that for given $\epsilon > 0$, there exists $\delta > 0$ such that if $\tau \in K$, then $\|\tau_n\|_{L^p(\Omega)} \leq 2\epsilon$ for all $\eta \in [0, \delta]$. Choose $\{\tau_i,\ i = 1, \cdots, N\}$ as an ϵ net for K. Choose δ so that $\|(\tau_i)_\delta\|_{L^p(\Omega)} < \epsilon$ for $i = 1, \cdots, N$. Then for any $\tau \in K$, there exists τ_k, α, $\|\alpha\|_{L^P(\Omega)} < \epsilon$ that $\tau = \tau_K + \alpha$. Since $(u + v)^+ \leq u^+ + v^+$, we have $\|\tau_\delta\|_{L^P(\Omega)} \leq (\tau_K)_\delta + |\alpha|$ and therefore $\|\tau_\eta\|_{L^P(\Omega)} \leq \|\tau_\delta\|_{L^P(\Omega)} + \|\alpha\|_{L^p(\Omega)} \leq 2\epsilon$ □

4. Proof of (iii) of Theorem 1.1 the case p such that $2 \leq p < \infty$ and p-Laplacian eigenvalue problem has the first eigenfunction $\psi_1 < 0$

We assume that $2 \leq p < \infty$, $-\infty < a < \lambda_1, \cdots, \lambda_n < b < \lambda_{n+1}$. To study equation (1.1), we shall reduce an infinite dimensional problem on $L^p(\Omega)$ to a finite dimensional one.

Let V be the $n-$dimensional subspace of $L^p(\Omega)$ spanned by ψ_1, ψ_2, $\cdots$, ψ_n, and W be the orthogonal complement of V in $L^p(\Omega)$. Let P be an orthogonal projection from $L^p(\Omega)$ onto V. Then every element $u \in L^p(\Omega)$ is expressed by

$$u = v + w,$$

where $v = Pu$, $w = (I - P)u$. Hence equation (1.1) is equivalent to a pair of equations

$$\begin{gathered}(I - P)\big(- (|(v + w)'|^{p-2}(v + w)')'\big) \\ = (I - P)\big(b|v + w|^{p-2}(v + w)^+ - a|v + w|^{p-2}(v + w)^- + s\phi_1^{p-1}\big), \quad (4.1) \\ w|\partial\Omega = 0,\end{gathered}$$

$$
\begin{aligned}
&P\big(-(|(v+w)'|^{p-2}(v+w)')'\big)\big) \\
&= P\big(b|v+w|^{2m-2}(v+w)^+ - a|v+w|^{2m-2}(v+w)^- + s\phi_1^{p-1}\big), \qquad (4.2) \\
&\qquad v|\partial\Omega = 0.
\end{aligned}
$$

We can consider (4.1) and (4.2) as a system of two equations in two unknowns v, w.

LEMMA 4.1. *Let* $2 \le p < \infty$, $-\infty < a < \lambda_1, \cdots, \lambda_n < b < \lambda_{n+1}$. *For fixed* $v \in V$, *(4.1) has a unique solution* $w = \beta(v,s)$. *Furthermore,* $\beta(v,s)$ *is continuous on* $V \times R$.

Proof. We suppose that for fixed $v \in V$, (4.1) has two solutions w_1, w_2. Then we have

$$
\begin{aligned}
&(I-P)[\big(-(|(v+w_1)'|^{p-2}(v+w_1)')'\big) - \big(-(|(v+w_2)'|^{p-2}(v+w_2)')'\big)] \\
&= (I-P)[\big(b|v+w_1|^{p-2}(v+w_1)^+ - a|v+w_1|^{p-2}(v+w_1)^-\big) \\
&\qquad -\big(b|v+w_2|^{p-2}(v+w_2)^+ - a|v+w_2|^{p-2}(v+w_2)^-\big)]. \qquad (4.3)
\end{aligned}
$$

Taking the inner product of (4.3) with $w_1 - w_2$, we have

$$
\begin{aligned}
&\big\langle (I-P)[\big(-(|(v+w_1)'|^{p-2}(v+w_1)')'\big) \\
&\qquad -\big(-(|(v+w_2)'|^{p-2}(v+w_2)')'\big)], w_1 - w_2\big\rangle \\
&= \big\langle (I-P)[\big(b|v+w_1|^{p}(v+w_1)^+ - a|v+w_1|^{p}(v+w_1)^-\big) \\
&\qquad -\big(b|v+w_2|^{p-2}(v+w_2)^+ - a|v+w_2|^{p-2}(v+w_2)^-\big)], w_1 - w_2\big\rangle. \qquad (4.4)
\end{aligned}
$$

The left hand side of (4.4) is equal to

$$
\begin{aligned}
&\big\langle (I-P)[\big(-(|(v+w_1)'|^{p-2}(v+w_1)')'\big) \\
&\qquad\qquad -\big(-(|(v+w_2)'|^{p-2}(v+w_2)')'\big)], w_1 - w_2\big\rangle \\
&= (p-1)\int_\Omega [(I-P)[\big((|\nabla(v+w_2+\theta(w_1-w_2))|^{p-2} \\
&\qquad\qquad \nabla(v+w_2+\theta(w_1-w_2))(\nabla(w_1-w_2))^2\big)]dx \\
&\ge (p-1)\lambda_{n+1}\int_\Omega [(I-P)\big(|(v+w_2)+\theta(w_1-w_2)|^{p-2}(w_1-w_2)^2\big)]dx. \qquad (4.5)
\end{aligned}
$$

by mean value theorem. On the other hand, the right hand side of (4.5) is equal to

$$
\big\langle (I-P)[\big(b|v+w_1|^{p-2}(v+w_1)^+ - a|v+w_1|^{p-2}(v+w_1)^-\big)
$$

$$-\big(b|v+w_2|^{p-2}(v+w_2)^+ - a|v+w_2|^{p-2}(v+w_2)^-\big)], w_1 - w_2\big\rangle$$
$$\le (p-1)b\int_\Omega [[I-P]|v+w_2+\theta(w_1-w_2)|^{p-2}(w_1-w_2)^2]dx \quad (4.6)$$

for $0<\theta<1$. On the other hand, by (4.5) and (4.6), we have

$$(p-1)\lambda_{n+1}\int_\Omega [(I-P)\big(|(v+w_2)+\theta(w_1-w_2)|^{p-2}(w_1-w_2)^2\big)]dx$$
$$\le \quad (p-1)b\int_\Omega [[I-P]|v+w_2+\theta(w_1-w_2)|^{p-2}(w_1-w_2)^2]dx,$$

which is a contradiction because $b<\lambda_{n+1}$. Thus $w_1=w_2$. Thus for fixed $v\in V$, every solution of (4.1) is a unique solution $w=\beta(v,s)\in W$ which satisfies (4.1). It follows that, by the standard argument principle, that $\beta(v,s)$ is continuous in v. Standard bootstrap arguments show that $\beta(v,s)$ is a smooth solution of (4.1) □

LEMMA 4.2. *Let $2\le p<\infty$, $-\infty<a<\lambda_1,\cdots,\lambda_n<b<\lambda_{n+1}$. Let f denote the real values function defined on $W^{1,p}(\Omega,R)$ by*

$$f(u,s)=\int_\Omega [\frac{1}{p}|u'|^p - \frac{b}{p}|u|^{p-2}|u^+|^2 - \frac{a}{p}|u|^{p-2}|u^-|^2 - s\phi^{p-1}u]dx.$$

If $\tilde f : V\times R\to R$ is defined by

$$\tilde f(v,s)=f(v+\beta(v,s),s),$$

then $\tilde f$ had a continuous Frechét derivative $D\tilde f$ with respect to v and u is a solution of (1.1) if and only if $u=v+\beta(v,s)$, where $v=Pu$, and $D\tilde f(v,s)=0$.

Proof. The function f has a continuous *Frechét* derivative with respect to its first variable given by

$$Df(u,s)(z)=\int_\Omega [|u'|^{p-2}u'\cdot w' - b|u|^{p-2}u^+w + a|u|^{p-2}u^-w - s\phi_1^{p-1}w]dx$$

and by standard regularity arguments, solution of (1.1) coincide with solutions of $Df(u,s)=0$. □

LEMMA 4.3. *Let $2\le p<\infty$ such that p-Laplacian eigenvalue problem has the first eigenfunction $\psi_1<0$, $-\infty<a<\lambda_1,\cdots,\lambda_n<b<\lambda_{n+1}$ and let $s<0$. Then there exist $s_1<0$ such that for any any s with $s_1\le s<0$, $\tilde f(0,s)$ has a strict local minimum at $v=0$.*

Proof. Since $u = 0$ is a solution of (1.1) when $s = 0$, $\beta(0,0) = 0$. We claim that if $s < 0$, then $\beta(0, s) < 0$. In fact, since $s\phi^{p-1} \in V^{\perp}$, by (4.1), we have

$$-(|\beta(0,s)'|^{p-2}\beta(0,s)')' = b|\beta(0,s)|^{p-2}\beta(0,s)^{+} - a|\beta(0,s)|^{p-2}\beta(o,s)^{-} + s\phi_1^{p-1}. \tag{4.7}$$

Taking the inner product of (4.7) with ψ_1, we have

$$\begin{aligned}&\big\langle -(|\beta(0,s)'|^{p-2}\beta(0,s)')' - b|\beta(0,s)|^{p-2}\beta(0,s)^{+} + a|\beta(0,s)|^{p-2}\beta(o,s)^{-}, \psi_1\big\rangle \\ &\quad = \big\langle s\phi_1^{p-1}, \psi_1\big\rangle. \end{aligned}\tag{4.8}$$

We suppose that $\beta(0, s) \geq 0$. Then the left hand side of (4.8) is equal to

$$(\lambda_1 - b)|\beta(0,s)|^{p-2}\psi_1^2 < 0,$$

on the other hand, the right hand side of (4.8) is $s\phi_1^{p-1}\psi_1 > 0$ because $s < 0$ and $\psi_1 < 0$, which is a contradiction. Thus $\beta(0, s) < 0$. Since $I + \beta$ is continuous, there exists a small neighborhood O_1 such that if $v \in O_1$, then $v + \beta(v, s) < 0$. We claim that $\beta(v, s) = \beta(0, s)$. In fact, if $s < 0$, $v \in V$, $v \in O_1$ and $w = \beta(v, s)$, then we have

$$\begin{aligned}&(I-P)\big(-(|(v+w)'|^{p-2}(v+w)')' - b|v+w|^{p-2}(v+w)^{+} \\ &\qquad + a|v+w|^{p-2}(v+w)^{-} - s\phi_1^{p-1}\big) \\ =&(I-P)\big(-(|(v+w)'|^{p-2}(v+w)')' - a|v+w|^{p-2}(v+w) - s\phi_1^{p-1}\big) \\ =&-(|w'|^{p-2}w')' - a|w|^{p-2}w - s\phi_1^{p-1} = 0\end{aligned}$$

on Ω and $w = 0$ on $\partial\Omega$. By Lemma 4.1, $\beta(v, s) = \beta(0, s)$. Therefore, if $s < 0$, $v \in V$ and $v \in O_1$, then $v + \beta(v, s) < 0$ and we have

$$\begin{aligned}&\tilde{f}(v,s) \\ =&f(v+\beta(v,s)) \\ =&\int_\Omega [\frac{1}{p}|(v+\beta(v,s))'|^p - \frac{a}{p}|(v+\beta(v,s))|^{p-2}|u^-|^2 - s\phi^{p-1}((v+\beta(v,s)))]dx \\ =&\int_\Omega [\frac{1}{p}|v'|^p - \frac{a}{p}|v|^{p-2}|v^-|^2]dx + C,\end{aligned}$$

where

$$C = \int_\Omega [\frac{1}{p}|\beta(v,s)'|^p - \frac{a}{p}|\beta(v,s)|^{p-2}|\beta(v,s)^-|^2] - s\phi_1^{p-1}\beta(v,s)]dx$$

$$= \int_\Omega [\frac{1}{p}|\beta(0,s)'|^p - \frac{a}{p}|\beta(0,s)|^{p-2}|\beta(0,s)^-|^2] - s\phi_1^{p-1}(\beta(0,s))]dx$$
$$= \tilde{f}(0,s).$$

If $v \in V$, then $v = c_1\psi_1 + \cdots + c_n\psi_n$. Thus we have

$$\int_\Omega \frac{1}{p}|v'|^p dx \geq \lambda_1 \int_\Omega |v|^p dx$$

and

$$\int_\Omega \frac{1}{p}|v'|^p dx \leq \lambda_{n+1} \int_\Omega |v|^p dx.$$

It follows that if $s < 0$, $v \in V$ and $v \in O_1$, then we have

$$\tilde{f}(v,s) - \tilde{f}(0,s) = \int_\Omega [\frac{1}{p}|v'|^p - \frac{a}{p}|v|^{p-2}|v^-|^2]dx \geq (\lambda_1 - a)\int_\Omega |v|^p dx > 0.$$

Thus for $s < 0$, $\tilde{f}(0,s)$ has a strict local minimum at $v = 0$. □

Proof of (iii) of Theorem 1.1 for the case p such that $2 \leq p < \infty$ *and p-Laplacian eigenvalue problem has the first eigenfunction* $\psi_1 < 0$

By Lemma 4.2, $\tilde{f}$ is C^1. By (ii) of Theorem 1.1, we can obtain the result that $\tilde{f}$ satisfies $(P.S.)$ condition. By Lemma 4.3, for $s < 0$, $\tilde{f}(0,s)$ has a strict local minimum at $v = 0$. Thus (1.1) has at least one weak solution which is of the form $u = 0 + \beta(0,s)$. □

5. Proof of Theorem 1,1 for the case p such that $2 \leq p < \infty$ and p-Laplacian eigenvalue problem has the first eigenfunction $\phi_1 > 0$

Lemma 5.1. *Assume that* $2 \leq p < \infty$, $-\infty < a < \lambda_1, \cdots, \lambda_n < b < \lambda_{n+1}$. *Then there exist a small constant* ϵ *and* $s_1 < 0$ *such that for any* s *with* $s_1 \leq s < 0$, *the Leray-Schauder degree*

$$d_{LS}(u - (-L)^{-1}(b|u|^{p-2}u^+ - a|u|^{p-2}u^- + s\phi_1^{p-1}), B_{\epsilon|s|}(u_0), 0) = (-1)^n,$$

where $u_0 = (\frac{s}{\lambda_1 - b})^{\frac{1}{p-1}}\phi_1 > 0$ *is a positive solution of (1.1).*

Proof. Let us set $M = (-L - bg_p)^{-1}$. Then (1.1) can be rewritten as

$$(-L - bg_p)(u) = b|u|^{p-2}u^+ - a|u|^{p-2}u^- - b|u|^{p-2}u + s\phi_1^{p-1}$$

or equivalently

$$u = M(b|u|^{p-2}u^+ - a|u|^{p-2}u^- - b|u|^{p-2}u + s\phi_1^{p-1}) = Tu. \tag{5.1}$$

The operator M is compact on $L^p(\Omega)$, and the set $K = M(\bar{B})$, where $\bar{B}$ is the closed unit ball in $L^p(\Omega)$. Then K is a compact set. Let us set $\gamma = \min\{b - \lambda_n, \lambda_{n+1} - b\}$. We can observe that if $2 \le p < \infty$, then $\|M(u)\|_{L^p(\Omega)} \le \|\frac{1}{\gamma}g_p^{-1}(u)\|_{L^p(\Omega)}$. Let α be the modulus continuity of Lemma 3.3 corresponding to K and $\xi = M\phi_1^{p-1} = \frac{1}{\lambda_1 - b}\phi_1$ and choose $\epsilon > 0$ so that

$$\alpha(\epsilon^{\frac{1}{p-1}}((b-a)^{\frac{1}{p-1}} + \gamma^{\frac{1}{p-1}}) \le \frac{\gamma}{4(b-a)^{\frac{1}{p-1}}((b-a)^{\frac{1}{p-1}} + \gamma^{\frac{1}{p-1}})}.$$

We have

$$\|b|u|^{p-2}u^+ - a|u|^{p-2}u^- - b|u|^{p-2}u\|_{L^p(\Omega)} \le (b-a)\||u|^{p-2}u^-\|_{L^p(\Omega)}.$$

It follows from that

$$\|M\big(b|u|^{p-2}u^+ - a|u|^{p-2}u^- - b|u|^{p-2}u\big)\|_{L^p(\Omega)} \le \frac{(b-a)^{\frac{1}{p-1}}}{\gamma^{\frac{1}{p-1}}}\|u^-\|_{L^p(\Omega)}. \tag{5.2}$$

For $u \in (\frac{|s|}{\lambda_1 - b})^{\frac{1}{p-1}}\phi_1 + (|s|\epsilon v)^{\frac{1}{p-1}}$ with $v \in \bar{B}$,

$$\begin{aligned}\|u^-\|_{L^p(\Omega)} &= \|((\frac{s}{\lambda_1 - b})^{\frac{1}{p-1}}\phi_1 + (|s|\epsilon)^{\frac{1}{p-1}}v^{\frac{1}{p-1}})^-\|_{L^p(\Omega)} \\ &\le \|((|s|\epsilon v)^{\frac{1}{p-1}})^-\|_{L^p(\Omega)} \le (|s|\epsilon)^{\frac{1}{p-1}}\end{aligned}$$

since $(\frac{s}{\lambda_1 - b})^{\frac{1}{p-1}}\phi_1 > 0$. Then $T(u) = M(b|u|^{p-2}u^+ - a|u|^{p-2}u^- - b|u|^{p-2}u + s\phi^{p-1})$ can be rewritten as

$$T(u) = (\frac{s}{\lambda_1 - b})^{\frac{1}{p-1}}\phi_1 + (|s|\epsilon)^{\frac{1}{p-1}}((b-a)^{\frac{1}{p-1}} + \gamma)w^{\frac{1}{p-1}}, \qquad w \in \bar{B}.$$

If u is a solution of (5.1), then $u = Tu$ and by Lemma 3.3,

$$\|u^-\|_{L^p(\Omega)} = \|((\frac{s}{\lambda_1 - b})^{\frac{1}{p-1}}\phi_1 + ((|s|\epsilon)^{\frac{1}{p-1}}((b-a)^{\frac{1}{p-1}} + \gamma)w^{\frac{1}{p-1}})_-\|_{L^p(\Omega)}$$

$$\le ((|s|\epsilon)^{\frac{1}{p-1}}((b-a)^{\frac{1}{p-1}} + \gamma)\alpha(\epsilon^{\frac{1}{p-1}}((b-a)^{\frac{1}{p-1}} + \gamma) < \frac{\gamma(|s|\epsilon)^{\frac{1}{p-1}}}{4(b-a)^{\frac{1}{p-1}}}. \tag{5.3}$$

Combining (5.2) with (5.3), we have

$$\|M(b|u|^{p-2}u^+ - a|u|^{p-2}u^- - b|u|^{p-2}u)\|_{L^p(\Omega)}$$
$$\leq \frac{(b-a)^{\frac{1}{p-1}}}{\gamma}\|u^-\|_{L^p(\Omega)} \leq \frac{1}{4}(|s|\epsilon)^{\frac{1}{p-1}} \leq \frac{1}{4}|s|\epsilon.$$

Thus we have shown that any solution $u \in (\frac{s}{\lambda-b})^{\frac{1}{p-1}}\phi_1 + |s|\epsilon\bar{B}$ of (5.1) belong to $(\frac{s}{\lambda-b})^{\frac{1}{p-1}}\phi_1 + \frac{1}{4}|s|\epsilon\bar{B}$. This estimate holds if we replace $b|u|^{p-2}u^+ - a|u|^{p-2}u^- - b|u|^{p-2}u$ by $\lambda(b|u|^{p-2}u^+ - a|u|^{p-2}u^- - b|u|^{p-2}u)$ with $0 \leq \lambda \leq 1$. Thus the equation

$$u = (-L)^{-1}(s\phi_1^{p-1} + b|u|^{p-2}u + \lambda(b|u|^{p-2}u^+ - a|u|^{p-2}u^- - b|u|^{p-2}u))$$

has no solution on the boundary of the ball $B_{\epsilon|s|}((\frac{s}{\lambda_1-b})^{\frac{1}{p-1}}\phi_1)$ for $0 \leq \lambda \leq 1$. By the homotopy invariance degree,

$$d_{LS}(u - (-L)^{-1}(s\phi_1^{p-1} + b|u|^{p-2}u + \lambda(b|u|^{p-2}u^+ - a|u|^{p-2}u^- - b|u|^{p-2}u),$$
$$B_{\epsilon|s|}(\frac{s}{\lambda_1-b})^{\frac{1}{p-1}}\phi_1, 0)$$

is defined for $0 \leq \lambda \leq 1$ and is independent of λ. For $\lambda = 0$,

$$d_{LS}(u - (-L)^{-1}(s\phi_1^{p-1} + b|u|^{p-2}u, B_{\epsilon|s|}((\frac{s}{\lambda_1-b})^{\frac{1}{p-1}}\phi_1), 0) = (-1)^n.$$

since $u = s\frac{\phi_1}{\lambda_1-b}$ is the unique solution of the equation and since there are n eigenvalues $\lambda_1, \cdots, \lambda_n$ of $-\Delta_p$ to the left of b and thus the operator $I - b(-L)^{-1}$ has n negative eigenvalues, while all the rest are positive. When $\lambda = 1$, we have

$$d_{LS}((u - (-L)^{-1}(s\phi_1^{p-1} + b|u|^{p-2}u^+ + 1(b|u|^{p-2}u^+ - a|u|^{p-2}u^- - b|u|^{p-2}u),$$
$$B_{\epsilon|s|}((\frac{s}{\lambda_1-b})^{\frac{1}{p-1}}\phi_1), 0)$$
$$= d_{LS}(s\phi_1^{p-1} + b|u|^{p-2}u^+ - a|u|^{p-2}u^-, B_{\epsilon|s|}((\frac{s}{\lambda_1-b})^{\frac{1}{p-1}}\phi_1), 0).$$

Thus by the homotopy invariance of degree, we have

$$d_{LS}(s\phi_1^{p-1} + b|u|^{p-2}u^+ - a|u|^{p-2}u^-, B_{\epsilon|s|}((\frac{s}{\lambda_1-b})^{\frac{1}{p-1}}\phi_1), 0)$$
$$= d_{LS}(u - (-L)^{-1}(s\phi_1^{p-1} + b|u|^{p-2}u, B_{\epsilon|s|}((\frac{s}{\lambda_1-b})^{\frac{1}{p-1}}\phi_1), 0) = (-1)^n.$$

Thus the lemma is proved. □

Proof of (iii) of Theorem 1.1 for the case p such that $2 \le p < \infty$ *and p-Laplacian eigenvalue problem has the first eigenfunction* $\phi_1 > 0$

By Lemma 5.1, there is a solution $(\frac{s}{\lambda_1 - b})^{\frac{1}{p-1}}\phi_1 > 0$ in $B_{|s|\epsilon}((\frac{s}{\lambda_1-b})^{\frac{1}{p-1}}\phi_1)$. We may assume that $\epsilon < (\frac{s}{\lambda_1-a})^{\frac{1}{p-1}}$. Then there is a large ball B_R centred at origin and containing $B_{|s|\epsilon}((\frac{s}{\lambda_1-b})^{\frac{1}{p-1}}\phi_1)$. Since

$$d_{LS}(u-(-L)^{-1}(b|u|^{p-2}u^+ - a|u|^{p-2}u^- + s\phi_1^{p-1}), B_R(0), 0) = 0$$

and

$$d_{LS}(u-(-L)^{-1}(b|u|^{p-2}u^+ - a|u|^{p-2}u^- + s\phi_1^{p-1}), B_{|s|\epsilon}((\frac{s}{\lambda_1-b})^{\frac{1}{p-1}}\phi_1), 0) = (-1)^n,$$

we have

$$d_{LS}(u-(-L)^{-1}(b|u|^{p-2}u^+ - a|u|^{p-2}u^- + s\phi_1^{p-1}), B_R(0)\backslash B_{|s|\epsilon}((\frac{s}{\lambda_1-b})^{\frac{1}{p-1}}\phi_1)), 0)$$

$$= (-1)^n \neq 0.$$

Thus there exists the second solution in $B_R(0)\backslash\big(B_{|s|\epsilon}((\frac{s}{\lambda_1-b})^{\frac{1}{p-1}}\phi_1)\big)$. Thus there exist at least two solutions for problem (1.1). □

Proof of (iii) of Theorem 1.1

By Chapter 4 and Chapter 5, if $2 \le p < \infty$, there exists $s_1 < 0$ such that for any s with $s_1 < s < 0$, (1.1) has at least one nontrivial solutions. Thus (iii) of Theorem 1.1. is proved. □

Competing Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Authors's contributions

Tacksun Jung introduced the main ideas of multiplicity study for this problem. Q-Heung Choi participate in applying the method for solving this problem and drafted the manuscript. All authors contributed equally to read and approved the final manuscript.

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