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ON CORSINI HYPERGROUPS AND THEIR PRODUCTIONAL HYPERGROUPS

M. Al Tahan and B. Davvaz*

Abstract. In this paper, we consider a special hypergroup defined by Corsini and we name it Corsini hypergroup. First, we investigate some of its properties and find a necessary and sufficient condition for the productional hypergroup of Corsini hypergroups to be a Corsini hypergroup. Next, we study its regular relations, fundamental group and complete parts. Finally, we characterize all Corsini hypergroups of orders two and three up to isomorphism.

1. Introduction

Hyperstructure theory was introduced in 1934, at the eighth Congress of Scandinavian Mathematicians, when F. Marty [16] defined hypergroups as natural generalization of the concept of group based on the notion of hyperoperation, analyzed their properties and applied them to groups, algebraic functions and rational fractions. Where in a group, the composition of two elements is an element, while in a hypergroup, the composition of two elements is a set. In the following decades and nowadays, a number of different hyperstructures are widely studied from the theoretical point of view and for their applications to many subjects of pure and applied mathematics like geometry, topology, cryptography and code theory, graphs and hypergraphs, probability theory, binary relations, theory of fuzzy and rough sets, automata theory, economy, etc.

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∗ Corresponding author.

(see [7, 8]). New applications to groups were also presented by other researchers: Eaton, Ore, Krasner, Utumi, Drbohlav, Harrison, Roth, Mockor, Davvaz, Freni, Sureau and Haddad [7]. Surveys of the hyperstructure theory can be found in the books of Corsini [5], Davvaz [8, 9], Corsini and Leoreanu [7] and Vougiouklis [19]. Cyclic semihypergroups have been studied by Desalvo and Freni [11], Vougiouklis [20], Leoreanu [15]. Cyclic semihypergroups are important not only in the sphere of finitely generated semihypergroups but also for interesting combinatorial implications. Mousavi et al. [18] introduced a strongly regular equivalence relation on a hypergroup such that in a particular case the quotient, the set of equivalence classes, is a cyclic group. Many researchers worked on cyclic hypergroups and regular relations (see [1, 2]).

Our paper discusses a special type of hypergroups that is defined by Corsini in [6] and it is organized as follows: After an introduction, Section 2 presents some basic definitions concerning hyperstructures that are used throughout this paper. Section 3 presents Corsini hypergroups and studies it properties. Section 4 defines hyperrings using Corsini hypergroups and finds a necessary and sufficient condition for the productional hypergroup to be a Cosini hypergroup. Section 5 proves that every Corsini hypergroup has trivial fundamental group and presents some interesting results regarding its regular relations and complete parts. Section 6 characterizes all Corsini hypergroups of orders two and three up to isomorphism.

2. Basic definitions

In this section, we present some definitions related to hyperstructures from [3, 8, 9, 11, 17, 19, 20] that are used throughout this paper.

Let H be a non-empty set. Then, a mapping $\circ : H \times H \to \mathcal{P}^*(H)$ is called a *hyperoperation* on H, where $\mathcal{P}^*(H)$ is the family of all non-empty subsets of H. The couple $(H, \circ)$ is called a *hypergroupoid*. In the above definition, if A and B are two non-empty subsets of H and $x \in H$, then we define:

$$A \circ B = \bigcup_{\substack{a \in A \\ b \in B}} a \circ b,\ x \circ A = \{x\} \circ A \text{ and } A \circ x = A \circ \{x\}.$$

An element $e \in H$ is called an *identity* of $(H, \circ)$ if $x \in x \circ e \cap e \circ x$, for all $x \in H$ and it is called a *scalar identity* of $(H, \circ)$ if $x \circ e = e \circ x = \{x\}$,

for all $x \in H$. If e is a scalar identity of $(H, \circ)$, then e is the unique identity of $(H, \circ)$. An element $x \in H$ is called *idempotent* if $x \circ x = x$. The hypergroupoid $(H, \circ)$ is said to be *commutative* if $x \circ y = y \circ x$, for all $x, y \in H$. A hypergroupoid $(H, \circ)$ is called a *semihypergroup* if it is associative, i.e., if for every $x, y, z \in H$, we have $x \circ (y \circ z) = (x \circ y) \circ z$ and is called a *quasihypergroup* if for every $x \in H$, $x \circ H = H = H \circ x$. This condition is called the reproduction axiom. The couple $(H, \circ)$ is called a *hypergroup* if it is a semihypergroup and a quasihypergroup. A subset K of a hypergroup $(H, \circ)$ is called a *subhypergroup* of H if $(K, \circ)$ is a hypergroup. A subhypergroup K of a hypergroup $(H, \circ)$ is *normal* if $a \circ K = K \circ a$ for all $a \in H$. A *canonical hypergroup* is a non-empty set H endowed with a hyperoperation $\circ : H \times H \to \mathcal{P}^*(H)$, satisfying the following properties: (1) for any $x, y, z \in H, x \circ (y \circ z) = (x \circ y) \circ z$, (2) for any $x, y \in H, x \circ y = y \circ x$, (3) there exists $\imath \in H$ such that $\imath \circ x = x \circ \imath = x$, for any $x \in H$,(4) for every $x \in H$, there exists a unique element x' and we call it the opposite of x), (5) $z \in x \circ y$ implies that $y \in x' \circ z$ and $x \in z \circ y'$, that is $(H, \circ)$ is *reversible*. A hypergroup $(H, \circ)$ is called *total hypergroup* if $a \circ b = H$ for all $a, b \in H$.
A subset I of H is called a *hyperideal* of H if $IH \subseteq H$. A hypergroup H is said to be *simple* if H has no proper hyperideal [9].

Let $i \in \mathbb{N}$. A hypergroup $(H, \circ)$ is cyclic if there exist $h \in H$ such that

$$H = h \cup h^2 \cup \cdots \cup h^i \cup \cdots .$$

If $H = h \cup h^2 \cup \cdots \cup h^s$ then H is a cyclic hypergroup with finite period. Otherwise, H is called a cyclic hypergroup with infinite period. Here, $h^i = \underbrace{h \circ h \circ \ldots \circ h}_{i \text{ times}}$. It is a *single-power cyclic hypergroup* if there exist $h \in H$ and $s \in \mathbb{N}$ such that

$$H = h \cup h^2 \cup \cdots \cup h^i \cup \cdots \text{ and } h \cup h^2 \cup \cdots \cup h^{i-1} \subset h^i, \text{ for all } i \in \mathbb{N}.$$

Let $(H, \star)$ and $(H', \star')$ be two hypergroups. A function $f : (H, \star) \to (H', \star')$ is said to be a *homomorphism* if $f(x_1 \star x_2) \subseteq f(x_1) \star' f(x_2)$ for all $x_1, x_2 \in H$. And it is called a *good homomorphism* if $f(x_1 \star x_2) = f(x_1) \star' f(x_2)$ for all $x_1, x_2 \in H$.

Two hypergroups are said to be *isomorphic* is there exists a bijective good homomorphism between them. If the isomorphism is from the hypergroup to itself, then we call it *automorphism* and the set of all automorphisms of the hypergroup $(H, \circ)$ is denoted by $Aut(H, \circ)$.

3. Properties of Corsini hypergroups

Corsini in [6] defined a hypergroupoid on a hypergraph, called it *hypergraph hypergroupoid* and proved that it satisfies some conditions. Moreover, he found a necessary and sufficient condition for the *hypergraph hypergroupoid* to be a hypergroup. In this section, we present these hypergroups and call them Corsini hypergroups, prove some results regarding their properties including cyclicity and automorphism groups that are not found in Corsini's paper.

Definition 3.1. Let H be a non empty set. Then $(H, \circ)$ is called a *Corsini hypergroupoid* if for all $(x, y) \in H^2$, the following conditions are satisfied:

1. $x \circ y = x \circ x \cup y \circ y$,
2. $x \in x \circ x$,
3. $y \in x \circ x \Longleftrightarrow x \in y \circ y$.

Corsini proved the following theorem in [6].

Theorem 3.2. *A hypergroupoid $(H, \circ)$ satisfying Definition 3.1 is a hypergroup if and only if the the following condition is valid:*

$$\forall (a, c) \in H^2, c \circ c \circ c - c \circ c \subseteq a \circ a \circ a.$$

Proposition 3.3. *A Corsini hypergroupoid is commutative.*

Proof. Let $(H, \circ)$ be a Corsini hypergroup and $(x, y) \in H^2$. Then, Definition 3.1 asserts that $x \circ y = x \circ x \cup y \circ y = y \circ y \cup x \circ x = y \circ x$. □

Proposition 3.4. *A Corsini hypergroupoid is regular.*

Proof. Let $(H, \circ)$ be a Corsini hypergroupoid and $(x, y) \in H^2$. Then, Definition 3.1 asserts that $x \in x \circ y = x \circ x \cup y \circ y$. Thus, each element in H is an identity. Consequently, the set of all inverses $I(x)$ of $x \in H$ is equal to H. □

Definition 3.5. A non empty subset M of a hypergroup $(H, \star)$ is linear if $a \star b \subseteq M$ and $a/b \subseteq M$, for all a, $b \in M$. Here, $a/b = \{x \in H \mid a \in x \star b\}$.

Proposition 3.6. *A Corsini hypergroup has no proper linear subsets.*

Proof. Let M be a linear subset of the Corsini hypergroup $(H, \circ)$ and $a \in M$. Having M a linear subset of $(H, \circ)$ implies that $a/a \subseteq M$. We have that

$$a/a = \{x \in H : a \in x \circ a\}.$$

The latter and Definition 3.1 imply that $a/a = H \subseteq M$. $\square$

Proposition 3.7. *A Corsini hypergroup has no proper normal subhypergroups.*

Proof. Let N be a proper normal subhypergroup of the Corsini hypergroup $(H, \circ)$. Then there exists an element $x \in H$ that is not an element in N. By Definition 3.1 and having N a normal subhypergroup of H we conclude that $x \in x \circ N = N$. $\square$

Proposition 3.8. *A Corsini hypergroup is simple.*

Proof. Let I be a hyperideal of the Corsini hypergroup $(H, \circ)$ and let $x \in H$. Definition 3.1 and having I a hyperideal of H imply that $x \in IH \subseteq I$. $\square$

Proposition 3.9. *Let $(H, \circ)$ be a cyclic Corsini hypergroup and $x \in H$ a generator of H. If $y \in x \circ x$ then y is a generator of H.*

Proof. Having that $y \in x \circ x$ implies that $x \in y \circ y$ by Definition 3.1. It is easy to see that $x^k \subseteq y^{2k}$. Thus, $H = x \cup x^2 \cup \ldots \subseteq y \cup y^2 \cup \ldots$. $\square$

Corollary 3.10. *Let $(H, \circ)$ be a cyclic Corsini hypergroup of period two. Then every element in H is a generator.*

Proof. Since $(H, \circ)$ is a cyclic Corsini hypergroup of period two, it follows that there exist $x \in H$ such that $H = x \cup x^2$. Then, Definition 3.1 implies that $H = x^2$. The latter implies that $y \in x \circ x$ for all $y \in H$. Proposition 3.9 implies that y is a generator of H. $\square$

Proposition 3.11. *Every cyclic Corsini hypergroup is a single power cyclic hypergroup.*

Proof. Let $(H, \circ)$ be a cyclic Corsini hypergroup. Then there exist $x \in H$ such that $H = x \cup x^2 \cup \ldots$. Definition 3.1 implies that $x \in x^2$. We need to show that $x^k \subseteq x^{k+1}$ for all $k \geq 1$. Our statement is true for the case $k = 1$. We assume now by Mathematical induction that $x^{k-1} \subseteq x^k$. Having that

$$x^{k+1} = x^k \circ x = \bigcup_{y \in x^k} y \circ x \text{ and that } x^{k-1} \subseteq x^k$$

by our assumption imply that

$$x^{k+1} \supseteq \bigcup_{y \in x^{k-1}} y \circ x = x^k.$$

Therefore, $(H, \circ)$ is a single power cyclic hypergroup. $\square$

PROPOSITION 3.12. *Let $(H, \circ)$ be a Corsini hypergroup and $(K, \star)$ be any hypergroup. If there exist an isomorphism function $f : H \longrightarrow K$ then $(K, \star)$ is a Corsini hypergroup.*

Proof. Since f is a bijective function, it follows that for all $(y_1, y_2) \in K^2$ there exist $(x_1, x_2) \in H^2$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$. Moreover, for all $A, B \subseteq H$ we have $f(A \cup B) = f(A) \cup f(B)$. We need to show that the conditions of Definition 3.1 are satisfied for $(K, \star)$:

1. $y_1 \star y_2 = f(x_1) \star f(x_2) = f(x_1 \circ x_2) = f(x_1 \circ x_1 \cup x_2 \circ x_2) = f(x_1 \circ x_1) \cup f(x_2 \circ x_2)$. Having f a good homomorphism implies that

$$f(x_1 \circ x_1) \cup f(x_2 \circ x_2) = f(x_1) \star f(x_1) \cup f(x_2) \star f(x_2) = y_1 \star y_1 \cup y_2 \star y_2.$$

2. $y_1 \star y_1 = f(x_1) \star f(x_1) = f(x_1 \circ x_1)$. And having $(H, \circ)$ a Corsini hypergroup imples that $x_1 \in x_1 \circ x_1$. Thus, $y_1 = f(x_1) \in f(x_1 \circ x_1)$.
3. $y_1 \in y_2 \star y_2 \Longleftrightarrow f(x_1) \in f(x_2) \star f(x_2) \Longleftrightarrow f(x_1) \in f(x_2 \circ x_2)$. The latter is equivalent to $x_1 \in x_2 \circ x_2$ as f is a bijective function. We get now:

$$\begin{aligned} y_1 \in y_2 \star y_2 &\Longleftrightarrow x_2 \in x_1 \circ x_1 \\ &\Longleftrightarrow f(x_2) \in f(x_1 \circ x_1) \\ &\Longleftrightarrow f(x_2) \in f(x_1) \star f(x_1). \end{aligned}$$

$\square$

PROPOSITION 3.13. *Let $(H, \circ)$ be a Corsini hypergroup and $f : H \longrightarrow H$ be a bijective function. Then $f \in Aut(H, \circ)$ if and only if $f(x \circ x) = f(x) \circ f(x)$ for all $x \in H$.*

Proof. If $f \in Aut(H, \circ)$ then $f(x \circ y) = f(x) \circ f(y)$ for all $x, y \in H$. Thus, $f(x \circ x) = f(x) \circ f(x)$ for all $x \in H$.

Let f be a bijective function satisfying the condition $f(x \circ x) = f(x) \circ f(x)$ for all $x \in H$ and let $y \in H$. Having f a bijective function implies that for any sets $A, B \subseteq H$, $f(A \cup B) = f(A) \cup f(B)$. Thus, we get:

$$\begin{aligned} f(x \circ y) &= f(x \circ x \cup y \circ y) \\ &= f(x \circ x) \cup f(y \circ y) \\ &= f(x) \circ f(x) \cup f(y) \circ f(y) \\ &= f(x) \circ f(y). \end{aligned}$$

Thus, $f \in Aut(H, \circ)$. □

EXAMPLE 3.14. Let $(H, \circ)$ be a Corsini hypergroup with at least one idempotent element. Set $H = \{a_1, \ldots, a_k\} \cup \{a_{k+1}, \ldots\}$ where $\{a_1, \ldots, a_k\}, \{a_{k+1}, \ldots\}$ are sets of idempotent and non idempotent elements in H respectively. Define $f : H \longrightarrow H$ as follows:

$$f(x) = \begin{cases} x, & \text{if } x \text{ is not an idempotent element in } H; \\ a_{i+1}, & \text{if } x = a_i \text{ and } 1 \leq i < k; \\ a_1, & \text{if } x = a_k. \end{cases}$$

It is clear that $f \in Aut(H, \circ)$ as it satisfies conditions of Proposition 3.13.

4. Hyperrings and productional hypergroups using Corsini hypergroups

In this section, we use the definition of Corsini hypergroups to define a special Corsini hypergroup and present hyperrings using Corsini hypergroups. Also, we find a necessary and sufficient condition for the productional hypergroup to be a Cosini hypergroup.

4.1. Hyperrings using Corsini hypergroups.

DEFINITION 4.1. Let H be any non empty set, $(x, y) \in H^2$ and define $\star$ as follows:

$$x \star y = \{x, y\}.$$

PROPOSITION 4.2. $(H, \star)$ *is a Corsini hypergroup.*

Proof. It is easy to see that $(H, \star)$ satisfies Definition 3.1 and Theorem 3.2. □

DEFINITION 4.3. [19] Let $(H, \circ)$ and $(H, \star)$ be two hypergroups. We say that $\circ \leq \star$ if there is $f \in Aut(H, \star)$ such that $a \circ b \subseteq f(a) \star f(b)$ for all $a, b \in H$.

PROPOSITION 4.4. *Let $(H, \circ)$ be any Corsini hypergroup. Then $\star \leq \circ$.*

Proof. Let $(x, y) \in H^2$ and $i : (H, \circ) \longrightarrow (H, \circ)$ be the identity function. Definition 3.1 implies that $x \star y = \{x, y\} \subseteq x \circ y = i(x) \circ i(y)$. □

DEFINITION 4.5. Let R be a non empty set with two hyperoperations ($+$ and $\cdot$). We say that $(R, +, \cdot)$ is a hyperring if $(R, +)$ is a commutative hypergroup, $(R, \cdot)$ is a semihypergroup and the hyperoperation $\cdot$ is distributive with respect to $+$, i.e., $x \cdot (y + z) = x \cdot y + x \cdot z$ for all $x, y, z \in R$.

If the hyperoperation $\cdot$ is weak distributive with respect to $+$, i.e., $x \cdot (y + z) \subseteq x \cdot y + x \cdot z$ for all $x, y, z \in R$, we say that $(R, +, \cdot)$ is a weak hyperring.

PROPOSITION 4.6. *Let $(H, \circ)$ be a Corsini hypergroup. Then $(H, \circ, \star)$ is a commutative weak hyperring.*

Proof. Since $(H, \circ)$ and $(H, \star)$ are Corsini hypergroups, it follows, by Proposition 3.3, that they are commutative. Let $(x, y, z) \in H^3$. We have that:

$$\begin{aligned} x \star (y \circ z) &= \bigcup_{a \in y \circ z} x \star a = \bigcup_{a \in y \circ y \cup z \circ z} \{x, a\} \\ &= x \cup \bigcup_{a \in y \circ y \cup z \circ z} a = x \cup y \circ y \cup z \circ z \end{aligned}$$

and

$$(x \star y) \circ (x \star z) = \{x, y\} \circ \{x, z\} = x \circ x \cup x \circ z \cup y \circ z \cup x \circ y = x \circ x \cup y \circ y \cup z \circ z.$$

Since $x \in x \circ x$, it follows that $x \star (y \circ z) \subseteq (x \star y) \circ (x \star z)$. □

PROPOSITION 4.7. *If $(H, \circ)$ is any Corsini hypergroup, then $(H, \star, \circ)$ is a commutative hyperring.*

Proof. Since $(H, \circ)$ and $(H, \star)$ are Corsini hypergroups, it follows, by Proposition 3.3, that they are commutative. Let $(x, y, z) \in H^3$. We have that:

$$x \circ (y \star z) = x \circ \{y, z\} = x \circ y \cup x \circ z = x \circ x \cup y \circ y \cup z \circ z.$$

and

$$\begin{aligned} (x \circ y) \star (x \circ z) &= (x \circ x \cup y \circ y) \star (x \circ x \cup z \circ z) \\ &= \bigcup_{a \in x \circ x \cup y \circ y, b \in x \circ x \cup z \circ z} \{a, b\}. \end{aligned}$$

It is clear that $x \circ (y \star z) = (x \circ y) \star (x \circ z)$. □

4.2. Productional hypergroups using Corsini hypergroups. Let $(H_1, \circ_1)$ and $(H_2, \circ_2)$ be hypergroups. We define $(H_1 \times H_2, \circ_1 \times \circ_2)$ as follows: For any $(h_1, k_1), (h_2, k_2) \in H_1 \times H_2$, we have

$$(h_1, k_1) \circ_1 \times \circ_2 (h_2, k_2) = (h_1 \circ_1 h_2, k_1 \circ_2 k_2).$$

$(H_1 \times H_2, \circ_1 \times \circ_2)$ is called *productional hypergroup.*

The aim of this subsection is to investigate the following question:

For a given two Corsini hypergroups $(H_1, \circ_1)$ and $(H_2, \circ_2)$, is their productional hypergroup $(H_1 \times H_2, \circ_1 \times \circ_2)$ a Corsini hypergroup?

We first show that this question has a negative solution by presenting our next example.

EXAMPLE 4.8. Let $H = \{x, y\}$ and $(H, \star)$ be the Corsini hypergroup defined in Definition 4.1. Then the productional hypergroup $(H \times H, \star \times \star)$ is not a Corsini hypergroup.

This can be deduced from Condition 1 of Definition 3.1:
We have that:

$$(x, x) \star \times \star (y, y) = (x \star y, x \star y) = (H, H) = \{(x, x), (x, y), (y, x), (y, y)\},$$

whereas

$$(x, x)\star\times\star(x, x)\cup(y, y)\star\times\star(y, y) = (x\star x, x\star x)\cup(y\star y, y\star y) = \{(x, x), (y, y)\}.$$

LEMMA 4.9. *Let $(H_1, \circ_1)$ be a total hypergroup and $(H_2, \circ_2)$ be a Corsini hypergroup. Then the productional hypergroup $(H_1 \times H_2, \circ_1 \times \circ_2)$ is a Corsini hypergroup.*

Proof. Let $(h_1, k_1), (h_2, k_2) \in H_1 \times H_2$. It is easy to show that Condition 2 and Condition 3 of Definition 3.1 are satisfied. We need to show that both Condition 1. of Definition 3.1 and Theorem 3.2 are satisfied. We have that

$$(h_1, k_1) \circ_1 \times \circ_2 (h_2, k_2) = (h_1 \circ_1 h_2, k_1 \circ_2 k_2) = \bigcup_{a \in H, b \in k_1 \circ_2 k_2} (a, b)$$

and that

$$\begin{aligned} &(h_1, k_1) \circ_1 \times \circ_2 (h_1, k_1) \cup (h_2, k_2) \circ_1 \times \circ_2 (h_2, k_2) \\ &= \bigcup_{x \in H_1, y \in k_1 \circ_2 k_1} (x, y) \cup \bigcup_{x' \in H_1, y' \in k_2 \circ_2 k_2} (x', y'). \end{aligned}$$

Having $(H_2,\circ_2)$ a Corsini hypergroup implies that $k_1\circ_2 k_1\cup k_2\circ_2 k_2 = k_1\circ_2 k_2$. The latter implies that

$$\begin{array}{l}(h_1,k_1)\circ_1\times\circ_2(h_1,k_1)\cup(h_2,k_2)\circ_1\times\circ_2(h_2,k_2)\\ =\bigcup\limits_{x\in H_1,y\in k_1\circ_2 k_1\cup k_2\circ_2 k_2}(x,y).\end{array}$$

Thus, Condition 1 of Definition 3.1 is satisfied.

Let (x,y) belong to

$$\begin{array}{l}(h_1,k_1)\circ_1\times\circ_2(h_1,k_1)\circ_1\times\circ_2(h_1,k_1)-(h_1,k_1)\circ_1\times\circ_2(h_1,k_1)\\ =(H,k_1\circ_2 k_1\circ_2 k_1)-(H,k_1\circ_2 k_1).\end{array}$$

It is clear that $y\in k_1\circ_2 k_1\circ_2 k_1-k_1\circ_2 k_1$. And since $(H_2,\circ_2)$ is a Corsini hypergroup, it follows by Theorem 3.2 that $y\in k_1\circ_2 k_1\circ_2 k_1-k_1\circ_2 k_1\subseteq k_1^{'}\circ_2 k_1^{'}\circ_2 k_1^{'}$ for all $k_1^{'}\in H_2$. Therefore,

$$(x,y)\in(h_1^{'},k_1^{'})\circ_1\times\circ_2(h_1^{'},k_1^{'})\circ_1\times\circ_2(h_1^{'},k_1^{'})\text{ for all }(h_1^{'},k_1^{'})\in H_1\times H_2.$$

Therefore, $(H_1\times H_2,\circ_1\times\circ_2)$ is a Corsini hypergroup. □

Corollary 4.10. *Let $(H_1,\circ_1)$ be a Corsini hypergroup and $(H_2,\circ_2)$ be a total hypergroup. Then the productional hypergroup $(H_1\times H_2,\circ_1\times\circ_2)$ is a Corsini hypergroup.*

Proof. Since $(H_1,\circ_1)$ is a Corsini hypergroup and $(H_2,\circ_2)$ is a total hypergroup, it follows by Lemma 4.9 that $(H_2\times H_1,\circ_2\times\circ_1)$ is a Corsini hypergroup. Having that $(H_1\times H_2,\circ_1\times\circ_2)$ and $(H_2\times H_1,\circ_2\times\circ_1)$ isomorphic and using Proposition 3.12, we get that $(H_1\times H_2,\circ_1\times\circ_2)$ is a Corsini hypergroup. □

Lemma 4.11. *Let $(H_1,\circ_1)$ and $(H_2,\circ_2)$ be Corsini hypergroups that are not total hypergroups. Then the productional hypergroup $(H_1\times H_2,\circ_1\times\circ_2)$ is not a Corsini hypergroup.*

Proof. Since $(H_1,\circ_1)$ and $(H_2,\circ_2)$ are not total hypergroups, it follows that there exist $h_1,h_2\in H_1$ and $k_1,k_2\in H_2$ such that $h_1\circ_1 h_1\neq h_2\circ_1 h_2$ and $k_1\circ_2 k_1\neq k_2\circ_2 k_2$. If such elements do not exist then $(H_1,\circ_1)$ or $(H_2,\circ_2)$ are total hypergroups. Thus, there exists $a\in h_1\circ_1 h_1$ that is not in $h_2\circ_1 h_2$ and $b\in k_2\circ_2 k_2$ that is not in $k_1\circ_2 k_1$. We have that:

$$\begin{array}{rl}(a,b)&\in(h_1,k_1)\circ_1\times\circ_2(h_2,k_2)\\ &=(h_1\circ_1 h_2,k_1\circ_2 k_2)\\ &=(h_1\circ_1 h_1\cup h_2\circ_1 h_2,k_1\circ_2 k_1\cup k_2\circ_2 k_2).\end{array}$$

And that

$$\begin{aligned}&(h_1,k_1)\circ_1\times\circ_2(h_1,k_1)\cup(h_2,k_2)\circ_1\times\circ_2(h_2,k_2)\\&=(h_1\circ_1 h_1,k_1\circ_2 k_1)\cup(h_2\circ_1 h_2,k_2\circ_2 k_2).\end{aligned}$$

It is clear that (a,b) is not an element in $(h_1,k_1)\circ_1\times\circ_2(h_1,k_1)\cup(h_2,k_2)\circ_1\times\circ_2(h_2,k_2)$. Thus, Condition 1. of Definition 3.1 is not satisfied. □

Theorem 4.12. *Let $(H_1,\circ_1)$ and $(H_2,\circ_2)$ be Corsini hypergroups. Then the productional hypergroup $(H_1\times H_2,\circ_1\times\circ_2)$ is a Corsini hypergroup if and only if $(H_1,\circ_1)$ or $(H_2,\circ_2)$ (or both) is a total hypergroup.*

Proof. The proof follows from Lemma 4.9 and Lemma 4.11. □

Corollary 4.13. *There are infinite Corsini hypergroups, up to isomorphism, that are not total hypergroups.*

Proof. Proposition 4.9 asserts that starting from a Corsini hypergroup $(H_1,\circ_1)$ and a total hypergroup $(H_2,\circ_2)$, we can get infinite Corsini hypergroups that are of the form $(H_1\times H_2,\circ_1\times\circ_2)$.

Let $(H_1,\circ_1)$ be the Corsini hypergroup defined in Definition 4.1. It is clear that $(H_1\times H_2,\circ_1\times\circ_2)$ is not a total hypergroup. The latter and the existence of infinite total hypergroups result in the existence of infinite Corsini hypergroups that are not total hypergroups. □

Next, we generalize our work on the productional hypergroup of two Corsini hypergroups to k number of Corsini hypergroups.
Let $(H_i,\circ_i)$ be an H_v- group for $i\in\{1,\ldots,k\}$, $\circ=\circ_1\times\circ_2\times\ldots\times\circ_k$ and define $(H_1\times H_2\times\ldots\times H_k,\circ)$ as follows:

$$(h_1,\ldots,h_k)\circ(h_1^{'},\ldots,h_k^{'})=(h_1\circ_1 h_1^{'},\ldots,h_k\circ_k h_k^{'}).$$

We present our next theorem which generalizes our results of Theorem 4.12.

Theorem 4.14. *Let $(H_i,\circ_i)$ be Corsini hypergroups for $i\in\{1,\ldots,k\}$. Then $(H_1\times H_2\times\ldots\times H_k,\circ)$ is a Corsini hypergroup if and only if the number of total hypergroups $(H_i,\circ_i)$ is at least $k-1$.*

Proof. Suppose that the number of total hypergroups $(H_i,\circ_i)$ is at least $k-1$. Without loss of generality, we can consider $(H_i,\circ_i)$ as total hypergroups for $i\in\{1,\ldots,k-1\}$. It is clear that $(H,\star)=(H_1\times\ldots\times H_k,\circ_1\times\ldots\times\circ_{k-1})$ is a total hypergroup. Lemma 4.9 implies that $(H_1\times H_2\times\ldots\times H_k,\circ)$ is a Corsini hypergroup.

Suppose, for contradiction, that there exist more than one non total hypergroup. Without loss of generality, we can set the non total hypergroups as $(H_1, \circ_1)$ and $(H_2, \circ_2)$. Using the same argument as that in the proof of Lemma 4.11, we get our result. □

5. Fundamental group and regular relations of Corsini hypergroups

In this section, we study equivalence relations on Corsini hypergroups, find the fundamental group of these hypergroups and determine their complete parts.

Definition 5.1. Let $(H, \circ)$ be a semihypergroup and R be an equivalence relation on H. If A and B are non-empty subsets of H, then

1. $A\overline{R}B$ means that for every $a \in A$ there exists $b \in B$ such that aRb and for every $b' \in B$ there exists $a' \in A$ such that $a'Rb'$;
2. $A\overline{\overline{R}}B$ means that for every $a \in A$ and $b \in B$, we have aRb.

The equivalence relation R is called:

1. regular on the right (on the left) if for all $x \in H$, from aRb, it follows that $(a \circ x)\overline{R}(b \circ x)$ $((x \circ a)\overline{R}(x \circ b)$ respectively);
2. strongly regular on the right (on the left) if for all $x \in H$, from aRb, it follows that $(a \circ x)\overline{\overline{R}}(b \circ x)$ $((x \circ a)\overline{\overline{R}}(x \circ b)$ respectively);
3. regular (strongly regular) if it is regular (strongly regular) on the right and on the left.

Theorem 5.2. [5,8] *Let $(H, \circ)$ be a hypergroup and R an equivalence relation on H. Then R is strongly regular if and only if $(H/R, \otimes)$, the set of all equivalence classes, is a group.*

The main tools connecting the class of hyperstructures with the classical algebraic structures are the fundamental relations. The fundamental relation has an important role in the study of semihypergroups and especially of hypergroups.

Definition 5.3. For all $n > 1$, we define the relation β_n on a semihypergroup $(H, \circ)$ as follows:

$$x\beta_n y \text{ if there exist } a_1, \ldots, a_n \text{ in } H \text{ such that} \{x, y\} \subseteq \prod_{i=1}^{n} a_i$$

and we set $\beta = \bigcup_{n \geq 1} \beta_n$, where $\beta_1 = \{(x, x) \mid x \in H\}$ is the diagonal relation on H.

This relation was introduced by Koskas [14] and studied mainly by Corsini [5], Davvaz [8], Davvaz and Leoreanu-Fotea [10], Freni [12], Vougiouklis [19], and many others. Clearly, the relation β is reflexive and symmetric. Denote by β^* the transitive closure of β.

The $\beta^\star$ is called the *fundamental equivalence relation* on H. The $\beta^\star$ is the smallest strongly regular relation on H and if H is a hypergroup then $\beta = \beta^\star$ [12]. In this case, H/β^* is called the *fundamental group.*

PROPOSITION 5.4. *A Corsini hypergroup has a trivial fundamental group.*

Proof. Let $(H, \circ)$ be a Corsini hypergroup and $(x, y) \in H^2$. Definition 3.1 asserts that $\{x, y\} \subset x \circ y$. The latter implies that $x\beta_2 y$. We get now that $x\beta y$. Since $(H, \circ)$ is a hypergroup, it follows that $\beta = \beta^\star$. Consequently, $H/\beta^\star$ has only one equivalence class. $\square$

DEFINITION 5.5. Let $(H, \circ)$ be an H_v- group and A be a non empty subset of H. A is a complete part of H if for any natural number n and for all hyperproducts $P \in H_H(n)$, the following implication holds:

$$A \cap P \neq \emptyset \Longrightarrow P \subseteq A.$$

PROPOSITION 5.6. *A Corsini hypergroup has no proper complete parts.*

Proof. Let A be a complete part of the Corsini hypergroup $(H, \circ)$ and $a \in A$. Definition 3.1 asserts that for all $b \in H$, $a \in A \cap (a \circ b) \neq \emptyset$. Having A a complete part of H implies that $b \in a \circ b \subseteq A$. $\square$

PROPOSITION 5.7. *Let $(H, \circ)$ be a Corsini hypergroup and R an equivalence relation on it. Then R is a strongly regular relation on H if and only if H/R is the trivial group.*

Proof. Theorem 5.2 asserts that if H/R is the trivial group and H is a hypergroup then R is strongly regular relation on H.
Let R be a strongly regular relation on H. For all $x \in H$, if aRb then $(a \circ x)\overline{\overline{R}}(b \circ x)$. Since R is an equivalence relation, it follows that aRa and thus $(a \circ x)\overline{\overline{R}}(a \circ x)$ for all $x \in H$. The latter and having $x \in a \circ x$, $a \in a \circ x$ imply that aRx. Thus, H/R contains only one equivalence class. $\square$

PROPOSITION 5.8. *Let $(H, \circ)$ be a Corsini hypergroup with the property that all its elements are idempotents. Then every equivalence relation on H is a regular relation.*

Proof. Let $(a,b,x) \in H^3$, R an equivalence relation on H and aRb. Definition 3.1 implies that $a \circ x = a \circ a \cup x \circ x$. Since a, b and x are idempotent elements in H, it follows that $a \circ x = \{a, x\}$ and $b \circ x = \{b, x\}$. Having aRb and xRx implies that $a \circ x\overline{R}b \circ x$. □

Remark 5.9. If $(H, \circ)$ is a Corsini hypergroup with the property that all its elements are idempotents then $(H, \circ) = (H, \star)$ defined in Section 4, Definition 4.1.

6. Corsini hypergroups of orders two and three

In this section, we characterize all Corsini hypergroups of orders two and three up to isomorphism.
Using some computations, we can show that we have nine hypergroups of order two up to isomorphism. The following Theorem proves that only two among them are Corsini hypergroups.

Theorem 6.1. *There are only two Corsini hypergroups of order two up to isomorphism in which one of them is cyclic.*

Proof. Let $H = \{a, b\}$. Definition 3.1 asserts that $H \subseteq a \circ a \cup b \circ b = a \circ b = b \circ a$. Moreover, we have that $a \circ a = a$ or $a \circ a = H$. If $a \circ a = H$ then by Definition 3.1 we get that $b \circ b = H$ and thus, $(H, \circ)$ is the total hypergroup.
If $a \circ a = a$ then Definition 3.1 asserts that $b \circ b = b$. Therefore, we get the two following hypergroupoids:

$\circ_1$	a	b
a	H	H
b	H	H

and

$\circ_2$	a	b
a	a	H
b	H	b

Theorem 3.2 asserts that $(H, \circ_1)$ and $(H, \circ_2)$ are Corsini hypergroups. Moreover, $(H, \circ_1)$ is a cyclic hypergroup whereas $(H, \circ_2)$ is not a cyclic hypergroup. □

Using Lemma 4.9, we can construct the Corsini hypergroup $(H \times H, \circ_1 \times \circ_2)$ (of order four) as follows:

$\circ_1 \times \circ_2$	(a,a)	(a,b)	(b,a)	(b,b)
(a,a)	$\{(a,a),(b,a)\}$	$H \times H$	$\{(a,a),(b,a)\}$	$H \times H$
(a,b)	$H \times H$	$\{(a,b),(b,b)\}$	$H \times H$	$\{(a,b),(b,b)\}$
(b,a)	$\{(a,a),(b,a)\}$	$H \times H$	$\{(a,a),(b,a)\}$	$H \times H$
(b,b)	$H \times H$	$\{(a,b),(b,b)\}$	$H \times H$	$\{(a,b),(b,b)\}$

.

Theorem 6.2. *There are only four Corsini hypergroups of order three up to isomorphism in which two of them are cyclic.*

Proof. Let $H = \{a, b, c\}$. Definition 3.1 asserts that $a \circ a = H$ or $a \circ a = \{a, b\}$ or $a \circ a = \{a, c\}$ or $a \circ a = a$.

- Case $a \circ a = H$. For all $x \in H$, we have that $a \circ x = a \circ a \cup x \circ x = H$. Since $b \in a \circ a$, it follows that there are two cases for $b \circ b$: $b \circ b = H$ or $b \circ b = \{a, b\}$. If $b \circ b = H$ then Definition 3.1 and having $c \in a \circ a$ imply that $c \circ c = H$. Thus $(H, \circ)$ is the total hypergroup given by the following table:

$\circ_1$	a	b	c
a	H	H	H
b	H	H	H
c	H	H	H

.

If $b \circ b = \{a, b\}$ then Definition 3.1 and having $c \in a \circ a$ imply that $c \circ c = \{a, c\}$. We get now that $b \circ c = c \circ b = b \circ b \cup c \circ c = H$. Thus $(H, \circ)$ is given by the following table:

$\circ_2$	a	b	c
a	H	H	H
b	H	$\{a,b\}$	H
c	H	H	$\{a,c\}$

.

Theorem 3.2 asserts that $(H, \circ_2)$ is a hypergroup.

- Case $a \circ a = \{a, b\}$. Definition 3.1 asserts that $b \circ b = H$ or $b \circ b = \{a, b\}$. If $b \circ b = H$ then $c \circ c = \{b, c\}$ and we get a hypergroup isomorphic to $(H, \circ_2)$. If $b \circ b = \{a, b\}$ then Definition 3.1 asserts that $c \circ c = c$. Simple computations imply that $(H, \circ)$ may be presented by the following table:

$\circ_3$	a	b	c
a	$\{a, b\}$	$\{a, b\}$	H
b	$\{a, b\}$	$\{a, b\}$	H
c	H	H	c

.

  Theorem 3.2 asserts that $(H, \circ_3)$ is a hypergroup.
- Case $a \circ a = \{a, c\}$. Using Dedfinition 3.1, we get that $c \circ c = H$ and $b \circ b = \{b, c\}$ or $c \circ c = \{a, c\}$ and $b \circ b = b$. Thus we get hypergroups isomorphic to $(H, \circ_2)$ and $(H, \circ_3)$ respectively.
- Case $a \circ a = a$. Since b and c are not elements of $a \circ a$ then Definition 3.1 implies that either $b \circ b = c \circ c = \{b, c\}$ or $b \circ b = b$ and $c \circ c = c$. If $b \circ b = c \circ c = \{b, c\}$ then $(H, \circ)$ is isomorphic to $(H, \circ_3)$. If $b \circ b = b$ and $c \circ c = c$ then simple computations imply that $(H, \circ)$ may be presented by the following table:

$\circ_4$	a	b	c
a	a	$\{a, b\}$	$\{a, c\}$
b	$\{a, b\}$	b	$\{b, c\}$
c	$\{a, c\}$	$\{b, c\}$	c

.

  Theorem 3.2 asserts that $(H, \circ_4)$ is a hypergroup.

It is clear that $(H, \circ_1)$ and $(H, \circ_2)$ are cyclic hypergroups whereas $(H, \circ_3)$ and $(H, \circ_4)$ are not cyclic hypergroups. $\square$

7. Conclusion

After introducing the notion of hypergroups by Marty, many research papers studiedthis important concept from different perspectives. This paper studied a special type of hypergroups; Corsini hypergroups in which we investigated some interesting properties about these hypergroups, studied their complete parts and fundamental groups. Several interesting results were obtained such as characterizing all Corsini hypergroups of orders two and three up to isomorphism.

For future research, it will be interesting to characterize infinite Corsini hypergroups up to isomorphism.

References

[1] M. Al- Tahan and B. Davvaz, *On a special single-power cyclic hypergroup and its automorphisms*, Discrete Mathematics, Algorithms and Applications **7** (4) (2016), 12 pages.

[2] M. Al- Tahan and B. Davvaz, *On some properties of single power cyclic hypergroups and regular relations*, J. Algebra Appl. **16** (11) (2017), 14 pages.

[3] R. Ameri, M. Amiri-Bideshki, A.B. Saeid and S. Hoskova-Mayerova, *Prime filters of hyperlattices*, An. Stiint. Univ. "Ovidius" Constanta Ser. Mat. **24**(2) (2016), 15–26.

[4] R. Ameri, A. Kordi and S. Hoskova-Mayerova, *Multiplicative hyperring of fractions and coprime hyperideals*, An. Stiint. Univ. "Ovidius" Constanta Ser. Mat. **25**(1) (2017), 5–23.

[5] P. Corsini, *Prolegomena of Hypergroup Theory*, Second edition, Aviani Editore, Italy, 1993.

[6] P. Corsini, *Hypergraphs and hypergroups*, Algebra Universalis **35** (4) (1996), 548–555.

[7] P. Corsini and V. Leoreanu, *Applications of Hyperstructures Theory*, Advances in Mathematics, Kluwer Academic Publisher, 2003.

[8] B. Davvaz, *Polygroup Theory and Related Systems*, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2013. viii+200 pp.

[9] B. Davvaz, *Semihypergroup Theory*, Elsevier, 2016.

[10] B. Davvaz and V. Leoreanu-Fotea, *Hyperring Theory and Applications*, International Academic Press, USA, 2007.

[11] M. De Salvo and D. Freni, *Cyclic semihypergroups and hypergroups*, (Italian) Atti Sem. Mat. Fis. Univ. Modena **30** (1) (1981), 44–59.

[12] D. Freni, *A note on the core of a hypergroup and the transitive closure β^* of β*, Riv. Mat. Pura Appl., **8** (1991), 153–156.

[13] S. Hoskova-Mayerova and A. Maturo, *Algebraic hyperstructures and social relations*, Ital. J. Pure Appl. Math. **39** (2018), 701–709.

[14] M. Koskas, *Groupoides, demi-hypergroupes et hypergroupes*, J. Math. Pure Appl. **49** (1970), 155–192.

[15] V. Leoreanu, *About the simplifiable cyclic semihypergroups*, Ital. J. Pure Appl. Math. **7** (2000), 69–76.

[16] F. Marty, *Sur une generalization de la notion de group*, In 8*th* Congress Math. Scandenaves, (1934), 45–49.

[17] J. Mittas, *Hypergroups canoniques*, Math. Balkanica **2** (1972), 165–179.

[18] S.Sh. Mousavi, V. Leoreanu-Fotea and M. Jafarpour, *Cyclic groups obtained as quotient hypergroups*, An. Stiint. Univ. Al. I. Cuza Iasi. Mat. (N.S.) **61** (1) (2015), 109–122.

[19] T. Vougiouklis, *Hyperstructures and Their Representations*, Aviani editor. Hadronic Press, Palm Harbor, USA, 1994.

[20] T. Vougiouklis, *Cyclicity in a special class of hypergroups*, Acta Univ. Carolin. Math. Phys. **22** (1) (1981), 3–6.

M. Al Tahan
Department of Mathematics
Lebanese International University
Bekaa, Lebanon
E-mail: madeline.tahan@liu.edu.lb

B. Davvaz
Department of Mathematics
Yazd University
Yazd, Iran
E-mail: davvaz@yazd.ac.ir