Korean J. Math. **26** (2018), No. 4, pp. 701–708
https://doi.org/10.11568/kjm.2018.26.4.701

GENERALIZED NORMALITY IN RING EXTENSIONS INVOLVING AMALGAMATED ALGEBRAS

Tae In Kwon† and Hwankoo Kim*

Abstract. In this paper, seminormality and t-closedness in ring extensions involving amalgamated algebras are studied. Let $R \subseteq T$ be a ring extension with ideals $I \subseteq J$, respectively such that J is contained in the conductor of R in T. Assume that T is integral over R. Then it is shown that $(R \bowtie I, T \bowtie J)$ is a seminormal (resp., t-closed) pair if and only if (R, T) is a seminormal (resp., t-closed) pair.

1. Introduction

All rings considered here are assumed to be commutative rings with identity. First, we recall some definitions and properties.

1. Let $R \subseteq T$ be an extension of commutative rings. Then we say that R is *seminormal* (resp., *t-closed*) *in* T if an element $t \in T$ is in R whenever $t^2, t^3 \in R$ (resp., whenever there exists r in R such that $t^2 - rt, t^3 - rt^2 \in R$). Also we say that (R, T) is a *normal* (resp., *seminormal, t-closed*) *pair* if, for each ring C between R and T (R and T included), C is integrally closed (resp., seminormal, t-closed) in T. Clearly every normal pair is a t-closed pair and every t-closed pair is a seminormal pair [9, Proposition 1.3].

Received September 8, 2018. Revised December 4, 2018. Accepted December 5, 2018.

2010 Mathematics Subject Classification: 13F45, 13B10, 13B22.

Key words and phrases: semi normal (pair); t-closed (pair); amalgamated algebra.

† This research was supported by Changwon National University in 2017–2018.

∗Corresponding author.

2. A commutative ring R is called *seminormal* (resp. *t-closed*) if, whenever $(a,c) \in R^2$ satisfies $a^3 = c^2$, there exists $b \in R$ such that $b^2 = a$ and $b^3 = c$ (resp., whenever $(a,r,c) \in R^3$ satisfies $a^3 + arc - c^2 = 0$, there exists $b \in R$ such that $b^2 - rb = a$ and $b^3 - rb^2 = c$).

The concepts of seminormality and t-closedness were introduced and studied in various contexts in [10] and [6] respectively, and further investigated in much literature, for example [4, 5, 7–9, 11].

On the other hand, in [2], the authors introduced the concept of amalgamated algebras along an ideal as follows. Let R and T be commutative rings with identity, let J be an ideal of T, and let $f : R \to T$ be a ring homomorphism. In this setting, they defined the following subring of $R \times T$:

$$R \bowtie^f J = \{(r, f(r) + j) \mid r \in R, j \in J\}$$

called the *amalgamation of R with T along J with respect to f*. Other classical constructions (such as the $A + XB[X]$ construction, the $D + M$ construction, and the Nagatas idealization) can be studied as particular cases of the amalgamation. Recently, in [5], the author defined the more general construction with algebras rather than ideals. Let J be an $f(R)$-subalgebra of T. Then $R \bowtie^f J = \{(r, f(r)+j) \mid r \in R, j \in J\}$ is referred to as a *general bowtie ring*, or a *general bowtie extension* of R. Then, among other things, he considered integrality of extensions of bowtie rings.

Our primary focus of this paper will be investigating seminormality and t-closedness in ring extensions of the form $R \bowtie^f I \subset T \bowtie^f J$, where $f : T \to T'$ is a ring homomorphism, I is an $f(R)$-subalgebra of T', and J is an $f(T)$-subalgebra of T' with $I \subseteq J$.

2. Main results

Let $R \subseteq T$ be a ring extension, $f : T \to T'$ a ring homomorphism, I an $f(R)$-subalgebra of T', and J an $f(T)$-subalgebra of T' with $I \subseteq J$. It is shown that $R \bowtie^f I$ is integrally closed in $T \bowtie^f J$ if and only if R is integrally closed in T, $f(R) + I$ is integrally closed in $f(R) + J$, and $J \cap (f(R) + I) = I$ [5, Theorem 6.1.10]. The following result is an analog of this result.

Proposition 2.1. *Let $R \subseteq T$ be a ring extension, $f : T \to T'$ a ring homomorphism, I an $f(R)$-subalgebra of T', and J an $f(T)$-subalgebra of T' with $I \subseteq J$. Then $R \bowtie^f I$ is seminormal (resp., t-closed) in $T \bowtie^f J$ if and only if R is seminormal (resp., t-closed) in T and $f(R)+I$ is seminormal (resp., t-closed) in $f(R)+J$.*

Proof. The "t-closed" case can be proved similarly to that of the "seminormal" case. Thus we only prove the "seminormal" case.

($\Leftarrow$) Assume that R is seminormal in T and $f(R)+I$ is seminormal in $f(R)+J$. Let $(t, f(t)+j) \in T \bowtie^f J$ $(t \in T, j \in J)$ such that $(t, f(t)+j)^2, (t, f(t)+j)^3 \in R \bowtie^f I$. Then $(t^2, (f(t)+j)^2), (t^3, (f(t)+j)^3) \in R \bowtie^f I$. Thus $t^2, t^3 \in R$, and so $t \in R$ since R is seminormal in T. Also we have $f(t)+j \in f(R)+J$. Since $(f(t)+j)^2, (f(t)+j)^3 \in f(R)+I$, we have $f(t)+j \in f(R)+I$ since $f(R)+I$ is seminormal in $f(R)+J$.

($\Rightarrow$) Assume that $R \bowtie^f I$ is seminormal in $T \bowtie^f J$. Let $t \in T$ such that $t^2, t^3 \in R$. Then $(t, f(t)) \in T \bowtie^f J$ such that $(t, f(t))^2, (t, f(t))^3 \in R \bowtie^f I$. Thus by hypothesis, $(t, f(t)) \in R \bowtie^f I$, and so $t \in R$. Therefore, R is seminormal in T. Now let $f(r)+j \in f(R)+J$ $(r \in R, j \in J)$ such that $(f(r)+j)^2, (f(r)+j)^3 \in f(R)+I$. Then $(r, f(r)+j)^2, (r, f(r)+j)^2 \in R \bowtie^f I$. By hypothesis, $(r, f(r)+j) \in R \bowtie^f I$, and so $f(r)+j \in f(R)+I$. Therefore, $f(R)+I$ is seminormal in $f(R)+J$. □

Note that the extension $R \bowtie^f I \subset T \bowtie^f J$ generalizes ring extensions of the form $R \bowtie I \subset T \bowtie J$ (which belong to the special case where $T = T'$ and f is simply the identity map). Thus we have the following result.

Corollary 2.2. *Let $R \subseteq T$ be a ring extension with ideals $I \subseteq J$, respectively. Then $R \bowtie I$ is seminormal (resp., t-closed) in $T \bowtie J$ if and only if R is seminormal (resp., t-closed) in T.*

Proof. This follows from Proposition 2.1 and the fact that "R is seminormal (resp., t-closed) in T" implies that "$R+I(=R)$ is a seminormal (resp., t-closed) in $R+J(\subseteq T)$". □

We say that R is a *decent ring* (also called *complemented*) if the total ring of quotients of R, denoted by $\mathrm{Tot}(R)$, is a von Neumann regular ring (also called an absolutely flat ring). Recall from a comment after [10, Corollary 3.4] that for a decent ring R, by saying R is seminormal (resp., t-closed) we mean that R is seminormal (resp., t-closed) in $\mathrm{Tot}(R)$. Now we have an immediate application of Corollary 2.2.

Corollary 2.3. *Let R be a decent ring and I be an ideal of R. Then $R \bowtie I$ is seminormal (resp., t-closed) if and only if R is seminormal (resp., t-closed).*

Proof. It is enough to show that $R \bowtie I$ is a decent ring. This follows from two facts that (1) $\mathrm{Tot}(R \bowtie I)$ is canonically isomorphic to $\mathrm{Tot}(R) \bowtie I\mathrm{Tot}(R)$ [5, Theorem 5.3.3] and (2) for a proper ideal J of a commutative ring A, $A \bowtie J$ is a von Neumann regular ring if and only if A is a von Neumann regular ring [1, Theorem 2.1]. $\square$

It was shown that if T is von Neumann regular and is integral over a subring R, then (R, T) is a seminomal pair [11, Lemma 1.4]. In what follows we will consider seminormal (resp., t-closed) pairs in extensions of bowtie rings.

Proposition 2.4. *Let $R \subseteq T$ be a ring extension with ideals $I \subseteq J$, respectively. Assume that T is integral over R and T is von Neumann regular. Then $(R \bowtie I, T \bowtie J)$ is a seminormal pair.*

Proof. Recall that $R \bowtie I \subset T \bowtie J$ is integral if and only if $R \subset T$ is integral [5, Corollary 6.1.2]. Now the assertion follows from [1, Theorem 2.1] and [11, Lemma 1.4]. $\square$

Lemma 2.5. *Let $R \subseteq T$ be a ring extension, $f : T \to T'$ a ring homomorphism, I an $f(R)$-subalgebra of T', and J an $f(T)$-subalgebra of T' with $I \subseteq J$. If $(R \bowtie^f I, T \bowtie^f J)$ is a seminormal (resp., t-closed) pair, then (R, T) is a seminormal (resp., t-closed) pair.*

Proof. The "t-closed" case can be proved similarly to that of the "seminormal" case. Thus we only prove the "seminormal" case.

Assume that $(R \bowtie^f I, T \bowtie^f J)$ is a seminormal pair. If (R, T) is not a seminormal pair, then there exists an intermediate ring S (possibly R itself) and a $t \in T \setminus R$ which satisfies $t^2, t^3 \in S$. Note that $S \bowtie^f J = \{(s, f(s) + j) \mid s \in S, j \in J\}$ is a ring lying between $R \bowtie^f I$ and $T \bowtie^f J$. Further, the element $(t, f(t)) \in T \bowtie^f J \setminus S \bowtie^f J$ satisfies $(t, f(t))^2, (t, f(t))^3 \in S \bowtie^f J$, contradicting that $S \bowtie^f J$ is seminormal in $T \bowtie^f J$ by hypothesis. $\square$

Let $R \subseteq T$ be an extension of commutative rings with (the same) identity. Consider the following conditions:

(a) T is integral over R.
(b) $\mathrm{Spec}(T) \longrightarrow \mathrm{Spec}(R)$ is a bijection.

(c) The residue field extensions are isomorphisms, i.e., for each $Q \in \operatorname{Spec}(T)$ the extension $R_P/PR_P \hookrightarrow T_Q/QR_Q$ is an isomorphism, where $P = Q \cap R$.

We recall some special extensions satisfying two or three conditions above including the condition (a).

1. R. G. Swan called the extension $R \subseteq T$ *subintegral* if (a), (b) and (c) are satisfied [10].
2. G. Picavet and M. Picavet-L'Hermitte called the extension $R \subseteq T$ *infra-integral* if (a) and (c) are satisfied [7].

LEMMA 2.6. *Let $R \subseteq T$ be an integral extension of commutative rings. Then (R, T) is a seminormal (resp., t-closed) pair if and only if (R_P, T_P) is a seminormal (resp., t-closed) pair for all maximal ideals P of R.*

Proof. Recall that for every multiplicatively closed subset S of a ring A, any intermediate ring for an extension $A_S \subseteq B_S$ of rings has the form C_S, for a suitable ring C between A and B (cf., [9, Proposition 1.5]). Thus the necessity follows from the fact that being seminormal (resp., t-closed) is stable under localization, while the sufficiency follows from the fact that localization preserves subintegrality (resp., infra-integrality) [10, Corollary 2.10] (resp., [7, Proposition 1.16]). $\square$

In [4, Proposition 4.3], it is shown that for a decent ring R and an ideal J of an extension ring T of R with $I := J \cap R$, if (R, T) is a normal pair, then $(R \bowtie I, T \bowtie J)$ is a normal pair.

PROPOSITION 2.7. *Let $R \subseteq T$ be a ring extension with ideals $I \subseteq J$, respectively such that $J \subseteq (R :_R T)$. Assume that T is integral over R. Then $(R \bowtie I, T \bowtie J)$ is a seminormal (resp., t-closed) pair if and only if (R, T) is a seminormal (resp., t-closed) pair.*

Proof. The "t-closed" case can be proved similarly to that of the "seminormal" case. Thus we only prove the "seminormal" case.

Assume that (R, T) is a seminormal pair. By Lemma 2.6, it suffices to show that $R \bowtie I$ is locally seminormal in $T \bowtie J$. Let $Q \in \operatorname{Spec}(R \bowtie I)$ and set $P := Q \cap R$.

Case 1: $I \nsubseteq P$. By [4, Proposition 4.2(b)], we have $(R \bowtie I)_Q \cong R_P$ and $(T \bowtie J)_{(R \bowtie I)\setminus Q} \cong T_{R \setminus P}$. Thus $((R \bowtie I)_Q, (T \bowtie J)_{(R \bowtie I)\setminus Q})$ can be identified with the seminormal pair $(R_P, T_{R\setminus P})$.

Case 2: $I \subseteq P$. By [4, Proposition 4.2(a)], we have $(R \bowtie I)_Q \cong R_P \bowtie I_P$ and $(T \bowtie J)_{(R \bowtie I)\setminus Q} \cong T_{R\setminus P} \bowtie J_{R\setminus P}$. Since J is contained in the conductor ideal of R in T, $J_{R\setminus P}$ is an ideal of R_P. Since $(R_P, T_{R\setminus P})$ is a seminormal pair, we can apply [7, Theorem 3.15] to show that $(R_P \bowtie I_P, T_{R\setminus P} \bowtie J_{R\setminus P})$ is a seminormal pair. This completes the proof that $R \bowtie I$ is locally seminormal in $T \bowtie J$, as required.

The converse follows from Lemma 2.5. □

If we refine the assumption $I \subseteq J$ by requiring that $J = I$, we are able to strengthen the result by removing the restriction that $J \subseteq (R :_R T)$ and $R \subseteq T$ be integral. In fact, we can give this result, which is an analog of [5, Theorem 6.2.4.], in the context of general bowtie rings.

Proposition 2.8. *Let $R \subseteq T$ be a ring extension, $f : T \to T'$ a ring homomorphism, and J an $f(T)$-subalgebra of T'. Then $(R \bowtie^f J, T \bowtie^f J)$ is a seminormal (resp., t-closed) pair if and only if (R, T) is a seminormal (resp., t-closed) pair.*

Proof. Assume that (R, T) is a seminormal (resp., t-closed) pair. By [5, Lemma 3.2.13], every ring between $R \bowtie^f J$ and $T \bowtie^f J$ is of the form $S \bowtie^f J$ for some ring $R \subseteq S \subseteq T$. Assume that (R, T) is a seminormal (resp., t-closed) pair. Then for each intermediate ring S, S is seminormal (resp., t-closed) in T. By Proposition 2.1, $S \bowtie^f J$ is seminormal (resp., t-closed) in $T \bowtie^f J$.

The converse follows immediately from Lemma 2.5. □

Now we end this paper by generating new related examples in the context of extensions of bowtie rings. As usual we denote by $\mathbb{C}$ the field of complex numbers.

Example 2.9. Let $T := \mathbb{C}[X, Y]$, where X, Y are indeterminates over $\mathbb{C}$.

1. Let $R := \mathbb{C}[X^2, Y, XY]$. Then it was shown that R is t-closed in T [7, Example 3.13] and R is a t-closed ring [8, Example 3.1]. Also note that (R, T) is not a t-closed pair since $\mathbb{C}[X^2, X^3, Y, XY]$ is not t-closed in T [7, Example 3.13]. Clearly T is integral over R. Now take $I := (X^2, XY)R$ and $J := (X)T$. Then by Corollary 2.2, $R \bowtie I$ is seminormal (resp., t-closed) in $T \bowtie J$.
2. Let M, N be two distinct maximal ideals of T and let $R := \mathbb{C} + (M \cap N)$. Then it was shown that T is integral over R and (R, T) is a seminormal pair but not a t-closed pair [11, Example 3.1].

Now take $J := M \cap N = MN$, the conductor of R in T, and take a nonzero proper subideal I of J. Then by Proposition 2.7, $(R \bowtie I, T \bowtie J)$ is a seminormal pair but not a t-closed pair.

Acknowledgements

We would like to thank the referee for her/his valuable comments.

References

[1] M. Chhiti and N. Mahdou, *Some homological properties of amalgamated duplication of a ring along an ideal*, Bull. Iranian Math. Soc. **38** (2012), 507–515.
[2] M. D'Anna, C.A. Finocchiaro, and M. Fontana, *Amalgamated algebras along an ideal*, in: M. Fontana, S. Kabbaj, B. Olberding, I. Swanson, editors. Commutative Algebra and Its Applications. Berlin, Walter de Gruyter, 2009, pp. 241–252.
[3] M. D'Anna, C.A. Finocchiaro, and M. Fontana, *Properties of chains of prime ideals in an amalgamated algebra along an ideal*, J. Pure Appl. Algebra **214** (2010), 1633–1641.
[4] D. Dobbs and J. Shapiro, *Normal pairs with zero-divisors*, J. Algebra Appl. **10** (2011), 335–356.
[5] T.S. Long, *Ring Extensions Involving Amalgamated Duplications*, Ph.D. Thesis, George Mason university, 2014.
[6] N. Onoda, T. Sugatani, and K. Yoshida, *Local quasinormality and closedness type criteria*, Houston J. Math. **11** (1985), 247–256.
[7] G. Picavet and M. Picavet-L'Herniitte, *Morphismes t-clos*, Commun. Algebra, **21** (1993), 179–219.
[8] G. Picavet and M. Picavet-L'Herniitte, *Anneaux t-clos*, Commun. Algebra, **23** (1995), 2643–2677.
[9] M. Picavet-L'Hermitte, *t-closed pairs*, in: P.-J. Cahen, M. Fontana, E. Houston, S.-E. Kabbaj, editors. Commutative Ring Theory. Lect. Notes Pure Appl. Math. 185, New York, Dekker, 1996, pp. 401-415.
[10] R.G. Swan, *On seminormality*, J. Algebra **67** (1980), 210–229.
[11] S. Visweswaran, *Some t-closed pairs*, Commun. Algebra **29** (10) (2001), 4425–4435 .

Tae In Kwon
Department of Mathematics
Changwon National University
Changwon 51140, Republic of Korea
E-mail: taekwon@changwon.ac.kr

Hwankoo Kim
Division of Computer & Information Engineering
Hoseo University
Asan 31499, Republic of Korea
E-mail: hkkim@hoseo.edu