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MULTI-ORDER FRACTIONAL OPERATOR IN A TIME-DIFFERENTIAL FORMAL WITH BALANCE FUNCTION

S. Harikrishnan, Rabha W. Ibrahim*, and K. Kanagarajan

Abstract. Balance function is one of the joint factors to determine fall in risk theory. It helps to moderate the progression and riskiness of falls for detecting balance and fall risk factors. Nevertheless, the objective measures for balance function require expensive equipment with the assessment of any expertise. We establish the existence and uniqueness of a multi-order fractional differential equations based on ψ- Hilfer operator on time scales with balance function. This class describes the dynamic of time scales derivative. Our tool is based on the Schauder fixed point theorem. Here, sufficient conditions for Ulam-stability are given.

1. Introduction

Consider the dynamic equation on time scales with ψ-Hilfer fractional derivative (HFD) of the form

$$(1)\qquad \begin{cases} {}^{\mathbb{T}}\Delta^{\alpha,\beta;\psi}\mathfrak{u}(t) = h(t)\mathfrak{H}(t,\mathfrak{u}(t)), \quad t=[0,b]:=J\subseteq\mathbb{T}, \\ {}^{\mathbb{T}}\mathfrak{I}^{1-\gamma;\psi}\mathfrak{u}(0)=\mathfrak{u}_0, \quad \gamma=\alpha+\beta-\alpha\beta, \end{cases}$$

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∗ Corresponding author.

where ${}^{\mathbb{T}}\Delta^{\alpha,\beta;\psi}$ is ψ-HFD defined on $\mathbb{T}$, $\alpha \in (0,1)$, $\beta \in [0,1]$ and $\mathfrak{I}^{1-\gamma;\psi}$ is ψ-fractional internal of order $1-\gamma(\gamma = \alpha+\beta-\alpha\beta)$, $\mathbb{T}$ is a time scale (nonempty subset of Banach space), h is a balance function in J and $\mathfrak{H} : J \times \mathbb{T} \to R$ is a right-dense function.

Time scales calculus permits us to training the dynamic equations, which include both difference and differential equations, both of which are significant in realizing applications; for further info about the theoretical and potential applications of time scales, refer [1, 4, 5]. The dynamical conduct of FDEs on time scales is presently experiencing active studies. Several authors considered the existence and uniqueness solutions for problems involving classical fractional derivative [2, 3].

Motivated by the above works, here we establish the existence theory and stability criteria of FDEs on times scale. The properties of ψ-HFD and the qualitative analysis is briefly studied in [8]. Further substantial attention paid to Ulam stability consequences for FDEs. For Ulam-Hyers stability theory of FDEs and its recent development, one can refer to [10–13, 17]. Further the solution of generalized Ulam-Hyers-Rassias(UHR) is obtained.

2. Preliminaries

Throughout this study, let $C(J)$ be continuous function with norm

$$\|\mathfrak{u}\|_C = \max\{|\mathfrak{u}(\tau)| : \tau \in J\}.$$

We denote the space $C_\gamma(J)$ as follows

$$C_\gamma(J) := \{\mathfrak{g}(\tau) : J \to R|\, (\psi(\tau)-\psi(0))^\gamma\, \mathfrak{g}(\tau) \in C(J)\}, 0 \le \gamma < 1$$

the weighted space $C_\gamma(J)$ of the functions $\mathfrak{g}$ on the interval J.Thus, $C_\gamma(J)$ is the Banach space provided the norm

$$\|\mathfrak{g}\|_{C_\gamma} = \|(\psi(\tau)-\psi(0))^\gamma\, \mathfrak{g}(\tau)\|_C.$$

In the sequel, we need the following preliminaries, which can be found in [9].

Definition 2.1. Let time scale be $\mathbb{T}$. The forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by $\sigma(\tau) := \inf\{s \in \mathbb{T} : s > \tau\}$, while the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined by $\rho(\tau) := \sup\{s \in \mathbb{T} : s < \tau\}$.

PROPOSITION 2.2. *Suppose $\mathbb{T}$ is a time scale and $[a,b] \subset \mathbb{T}$, $\mathfrak{g}$ is increasing continuous function on $[a,b]$. If the extension of $\mathfrak{g}$ is given in the following form:*

$$\mathscr{G}(s) = \begin{cases} \mathfrak{g}(s); & s \in \mathbb{T} \\ \mathfrak{g}(\tau); & s \in (\tau, \sigma(\tau)) \notin \mathbb{T}. \end{cases}$$

Then we have

$$\int_a^b \mathfrak{g}(t)\Delta t \leq \int_a^b \mathscr{G}(t)dt.$$

DEFINITION 2.3. Let $\mathbb{T}$ be a time scale, $J \in \mathbb{T}$. The left-sided R-L fractional integral of order $\alpha \in R^+$ of function $\mathfrak{g}(\tau)$ is defined by

$$\left({}^{\mathbb{T}}\mathfrak{J}^{\alpha}\mathfrak{g}\right)(\tau) = \int_0^\tau \psi^{'}(s)\frac{(\psi(\tau)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)}\mathfrak{g}(s)\Delta s.$$

DEFINITION 2.4. Suppose $\mathbb{T}$ is a time scale, $[0,b]$ is an interval of $\mathbb{T}$. The R-L fractional derivative of order $\alpha \in [n-1,n)$, $n \in \mathbb{Z}^+$ of function $\mathfrak{g}(\tau)$ is defined by

$$\left({}^{\mathbb{T}}\Delta^{\alpha}\mathfrak{g}\right)(\tau) = \left(\frac{1}{\psi'(\tau)}\frac{d}{d\tau}\right)^n \int_0^\tau \psi^{'}(s)\frac{(\psi(\tau)-\psi(s))^{n-\alpha-1}}{\Gamma(n-\alpha)}\mathfrak{g}(s)\Delta s.$$

DEFINITION 2.5. [7] The ψ-HFD of order α and type β of function $\mathfrak{g}(\tau)$ is defined by

$${}^{\mathbb{T}}\Delta^{\alpha,\beta;\psi}\mathfrak{g}(t) = \left({}^{\mathbb{T}}\mathfrak{J}^{\beta(1-\alpha);\psi}\ {}^{\mathbb{T}}\Delta\left({}^{\mathbb{T}}\mathfrak{J}^{(1-\beta)(1-\alpha);\psi}\mathfrak{g}\right)\right)(\tau),$$

where ${}^{\mathbb{T}}\Delta := \frac{d}{d\tau}$.

REMARK 2.6. 1. Here ${}^{\mathbb{T}}\Delta^{\alpha,\beta;\psi}$ is also written as

$${}^{\mathbb{T}}\Delta^{\alpha,\beta;\psi} = {}^{\mathbb{T}}\mathfrak{J}^{\beta(1-\alpha);\psi}\ {}^{\mathbb{T}}\Delta\,{}^{\mathbb{T}}\mathfrak{J}^{(1-\beta)(1-\alpha);\psi} = {}^{\mathbb{T}}\mathfrak{J}^{\beta(1-\alpha);\psi}\ {}^{\mathbb{T}}\Delta^{\gamma;\psi},\ \gamma = \alpha+\beta-\alpha\beta.$$

2. Let $\beta = 0$, it transfers into R-L derivative given by ${}^{\mathbb{T}}\Delta^{\alpha} := {}^{\mathbb{T}}\Delta^{\alpha,0}$.
3. Let $\beta = 0$, it turns to be Caputo fractional derivative given by ${}^{\mathbb{T}}_{c}\Delta^{\alpha} := {}^{\mathbb{T}}\mathfrak{J}^{1-\alpha}\ {}^{\mathbb{T}}\Delta$.

Next, we review some lemmas which will be used to establish our existence results.

LEMMA 2.7. *If $\alpha > 0$ and $\beta > 0$, there exist*

$$\left[{}^{\mathbb{T}}\mathfrak{J}^{\alpha}\left(\psi(s)-\psi(0)\right)^{\beta-1}\right](\tau) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}\left(\psi(\tau)-\psi(0)\right)^{\beta+\alpha-1}$$

Lemma 2.8. *Let* $\alpha \geq 0$, $\beta \geq 0$ *and* $\mathfrak{g} \in L^1(J)$. *Then*

$$^{\mathbb{T}}\mathfrak{J}^{\alpha}\ ^{\mathbb{T}}\mathfrak{J}^{\beta}\mathfrak{g}(\tau) \overset{a.e}{=}\ ^{\mathbb{T}}\mathfrak{J}^{\alpha+\beta}\mathfrak{g}(\tau).$$

Lemma 2.9. *If* $\mathfrak{g} \in C_\gamma(J)$ *and* $^{\mathbb{T}}\mathfrak{J}^{1-\alpha}\mathfrak{g} \in C^1_\gamma(J)$, *then*

$$^{\mathbb{T}}\mathfrak{J}^{\alpha}\ ^{\mathbb{T}}\Delta^{\alpha}\mathfrak{g}(\tau) = \mathfrak{g}(\tau) - \frac{\left(^{\mathbb{T}}\mathfrak{J}^{1-\alpha}\mathfrak{g}\right)(0)}{\Gamma(\alpha)}\left(\psi(\tau)-\psi(0)\right)^{\alpha-1}.$$

Lemma 2.10. *Suppose* $\alpha > 0$, $a(\tau)$ *is a nonnegative function locally integrable on* $0 \leq \tau < b$ *(some* $b \leq \infty$*), and let* $g(\tau)$ *be a non-negative, non-decreasing continuous function defined on* $0 \leq \tau < b$, *such that* $g(\tau) \leq K$ *for some constant K. Further let* $\mathfrak{u}(\tau)$ *be a non-negative locally integrable on* $0 \leq \tau < b$ *function with*

$$|\mathfrak{u}(\tau)| \leq a(\tau) + g(\tau)\int_0^\tau \psi'(s)\left(\psi(\tau)-\psi(s)\right)^{\alpha-1}\mathfrak{u}(s)\Delta s,$$

with some $\alpha > 0$. *Then*

$$|\mathfrak{u}(\tau)| \leq a(\tau) + \int_0^\tau \left[\sum_{n=1}^{\infty}\frac{(g(\tau)\Gamma(\alpha))^n}{\Gamma(n\alpha)}\psi'(s)\left(\psi(\tau)-\psi(s)\right)^{n\alpha-1}\right]\mathfrak{u}(s)\Delta s.$$

Theorem 2.11. *(Schauder FPT) Let* $\mathscr{E}$ *be a Banach space and* $\mathscr{Q}$ *be a nonempty bounded convex and closed subset of* $\mathscr{E}$ *and* $\mathscr{N}: \mathscr{Q} \to \mathscr{Q}$ *is compact, and continuous map. Then* $\mathscr{N}$ *has at least one fixed point in* $\mathscr{Q}$.

3. Existence results

Lemma 3.1. [9] *The functional integral* $\mathfrak{u}$ *is solution of* (1) *if and only if* $\mathfrak{u}$ *satisfies the following integral equation*

(2)

$$\begin{aligned}\mathfrak{u}(\tau) =& \frac{\mathfrak{u}_0}{\Gamma(\gamma)}\left(\psi(\tau)-\psi(0)\right)^{\gamma-1}\\ &+ \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)\left(\psi(\tau)-\psi(s)\right)^{\alpha-1}h(s)\mathfrak{H}(s,\mathfrak{u}(s))\Delta s, \quad t>0.\end{aligned}$$

For further investigation, we give the following assumptions:

(H1) The function $\mathfrak{H}: J \times R \to R$ is a continuous function.

(H2) There exists a positive constants $L > 0$ such that

$$|\mathfrak{H}(\tau,\mathfrak{u}) - \mathfrak{H}(\tau,\mathfrak{v})| \leq L\,|\mathfrak{u}-\mathfrak{v}|.$$

(H3) There exists an increasing function $\varphi \in C_{1-\gamma}(J)$ and there exists $\lambda_\varphi > 0$ such that for any $\tau \in J$,

$$^{\mathbb{T}}\mathfrak{J}^{\alpha}\varphi(\tau) \leq \lambda_\varphi \varphi(\tau).$$

THEOREM 3.2. *Assume that (H1)-(H3) are fulfilled. Then, equation* (1) *has at least one solution.*

Proof. Consider the operator $\mathscr{P} : C_{1-\gamma,\psi}(J) \to C_{1-\gamma,\psi}(J)$. The equivalent Volterra integral equation (2) which can be written in the operator form

(3)

$$(\mathscr{P}\mathfrak{u})(\tau) = \mathfrak{u}_0(\tau) + \frac{1}{\Gamma(\alpha)} \int_0^{\tau} \psi^{'}(s)\left(\psi(\tau) - \psi(s)\right)^{\alpha-1} h(s)\mathfrak{H}(s, \mathfrak{u}(s))\Delta s$$

with

$$\mathfrak{u}_0(\tau) = \frac{\mathfrak{u}_0}{\Gamma(\gamma)}\left(\psi(\tau) - \psi(0)\right)^{\gamma-1}. \tag{4}$$

Define $B_r = \left\{\mathfrak{u} \in C_{1-\gamma,\psi}(J) : \|\mathfrak{u}\|_{C_{1-\gamma,\psi}} \leq r\right\}$.
Set $\tilde{\mathfrak{H}}(s) = \mathfrak{H}(s, 0)$,

$$\sigma = \frac{|\mathfrak{u}_0|}{\Gamma(\gamma)} + \frac{bB(\gamma,\alpha)}{\Gamma(\alpha)}\left(\psi(b) - \psi(0)\right)^{\alpha} \left\|\tilde{\mathfrak{H}}\right\|_{C_{1-\gamma,\psi}}$$

and

$$\omega = \frac{bLB(\gamma,\alpha)}{\Gamma(\alpha)}\left(\psi(b) - \psi(0)\right)^{\alpha}.$$

To verify Theorem 2.11, we divide the proof into three steps.
Step 1: We check that $\mathscr{P}(B_r) \subset B_r$.

$$
\begin{aligned}
&\left|(\psi(\tau)-\psi(0))^{1-\gamma}(\mathscr{P}\mathfrak{u})(\tau)\right| \\
&\leq \frac{|\mathfrak{u}_0|}{\Gamma(\gamma)} + \frac{(\psi(\tau)-\psi(0))^{1-\gamma}}{\Gamma(\alpha)} \int_0^\tau \psi'(s)(\psi(\tau)-\psi(s))^{\alpha-1} b\,|\mathfrak{H}(s,\mathfrak{u}(s))|\,\Delta s \\
&\leq \frac{|\mathfrak{u}_0|}{\Gamma(\gamma)} + \frac{(\psi(\tau)-\psi(0))^{1-\gamma}}{\Gamma(\alpha)} \int_0^\tau \psi'(s)(\psi(\tau)-\psi(s))^{\alpha-1} b\,|\mathfrak{H}(s,\mathfrak{u}(s))-\mathfrak{H}(s,0)|\,\Delta s \\
&\quad + \frac{(\psi(\tau)-\psi(0))^{1-\gamma}}{\Gamma(\alpha)} \int_0^\tau \psi'(s)(\psi(\tau)-\psi(s))^{\alpha-1} b\,|\mathfrak{H}(s,0)|\,\Delta s \\
&\leq \frac{|\mathfrak{u}_0|}{\Gamma(\gamma)} + \frac{(\psi(\tau)-\psi(0))^{1-\gamma}}{\Gamma(\alpha)} \int_0^\tau \psi'(s)(\psi(\tau)-\psi(s))^{\alpha-1} bL\,|\mathfrak{u}|\,\Delta s \\
&\quad + \frac{(\psi(\tau)-\psi(0))^{1-\gamma}}{\Gamma(\alpha)} \int_0^\tau \psi'(s)(\psi(\tau)-\psi(s))^{\alpha-1} b\left|\tilde{\mathfrak{H}}\right|\Delta s \\
&\leq \frac{|\mathfrak{u}_0|}{\Gamma(\gamma)} + \frac{bB(\gamma,\alpha)}{\Gamma(\alpha)}(\psi(b)-\psi(0))^{\alpha}\|\tilde{\mathfrak{g}}\|_{C_{1-\gamma,\psi}} + \frac{bLB(\gamma,\alpha)}{\Gamma(\alpha)}(\psi(b)-\psi(0))^{\alpha}\|u\|_{C_{1-\gamma,\psi}}.
\end{aligned}
$$

Hence,

$$
\|(\mathscr{P}\mathfrak{u})\| \leq \sigma + \omega r \leq r.
$$

Which yields that $\mathscr{P}(B_r) \subset B_r$.
Next, the completely continuous of operator $\mathscr{P}$ is proved.
Step 2: The operator $\mathscr{P}$ is continuous.

Let $\mathfrak{u}_n$ be a sequence such that $\mathfrak{u}_n \to \mathfrak{u}$ in $C_{1-\gamma,\psi}(J)$.

$$
\begin{aligned}
&\left|(\psi(\tau)-\psi(0))^{1-\gamma}((\mathscr{P}\mathfrak{u}_n)(t)-(\mathscr{P}\mathfrak{u})(\tau))\right| \\
&\leq \frac{(\psi(\tau)-\psi(0))^{1-\gamma}}{\Gamma(\alpha)} \int_0^\tau \psi'(s)(\psi(\tau)-\psi(s))^{\alpha-1} b\,|\mathfrak{H}(s,\mathfrak{u}_n(s))-\mathfrak{H}(s,\mathfrak{u}(s))|\,\Delta s \\
&\leq \frac{b\,(\psi(\tau)-\psi(0))^{1-\gamma}}{\Gamma(\alpha)} \int_0^\tau \psi'(s)(\psi(\tau)-\psi(s))^{\alpha-1} \sup_{s\in J}|\mathfrak{H}(s,\mathfrak{u}_n(s))-\mathfrak{H}(s,\mathfrak{u}(s))|\,\Delta s \\
&\leq \frac{b\,(\psi(\tau)-\psi(0))^{1-\gamma}}{\Gamma(\alpha)} \int_0^\tau \psi'(s)(\psi(\tau)-\psi(s))^{\alpha-1} |\mathfrak{H}(s,\mathfrak{u}_n(s))-\mathfrak{H}(s,\mathfrak{u}(s))|\,ds, \\
&\qquad \text{(by Proposition 2.2)} \\
&\leq \frac{bB(\gamma,\alpha)}{\Gamma(\alpha)}(\psi(b)-\psi(0))^{\alpha}\|\mathfrak{H}(\cdot,\mathfrak{u}_n(\cdot))-\mathfrak{H}(\cdot,\mathfrak{u}(\cdot))\|_{C_{1-\gamma,\psi}},
\end{aligned}
$$

Since $\mathfrak{H}$ is continuous, Lebesgue dominated convergence theorem implies

$$\|\mathscr{P}\mathfrak{u}_n - \mathscr{P}\mathfrak{u}\|_{C_{1-\gamma,\psi}} \to 0 \quad \text{as} \quad n \to \infty.$$

Step 3: $\mathscr{P}(B_r)$ is relatively compact.
Thus $\mathscr{P}(B_r)$ is uniformly bounded. Let $\tau_1, \tau_2 \in J$, $\tau_1 < \tau_2$, then

$$\begin{aligned}
&\left|(\mathscr{P}\mathfrak{u})(\tau_2)\left(\psi(\tau_2)-\psi(0)\right)^{1-\gamma} - (\mathscr{P}\mathfrak{u})(\tau_1)\left(\psi(\tau_1)-\psi(0)\right)^{1-\gamma}\right| \\
&\leq \left|\frac{(\psi(\tau_2)-\psi(0))^{1-\gamma}}{\Gamma(\alpha)}\int_0^{\tau_2}\psi^{'}(s)\left(\psi(\tau_2)-\psi(s)\right)^{\alpha-1} b\,\mathfrak{H}(s,\mathfrak{u}(s))\Delta s\right. \\
&\quad \left.-\frac{(\psi(\tau_1)-\psi(0))^{1-\gamma}}{\Gamma(\alpha)}\int_0^{\tau_1}\psi^{'}(s)\left(\psi(\tau_1)-\psi(s)\right)^{\alpha-1}\, b\,\mathfrak{H}(s,\mathfrak{u}(s))\Delta s\right| \\
&\leq \frac{b}{\Gamma(\alpha)}\int_0^{\tau_1}\psi^{'}(s)\Big|\left(\psi(\tau_2)-\psi(0)\right)^{1-\gamma}\left(\psi(\tau_2)-\psi(s)\right)^{\alpha-1} \\
&\qquad\qquad - \left(\psi(\tau_1)-\psi(0)\right)^{1-\gamma}\left(\psi(\tau_1)-\psi(s)\right)^{\alpha-1}\Big|\,|\mathfrak{H}(s,\mathfrak{u}(s))|\,\Delta s \\
&\quad + \frac{b\left(\psi(t_2)-\psi(0)\right)^{1-\gamma}}{\Gamma(\alpha)}\int_{\tau_1}^{\tau_2}\psi^{'}(s)\left(\psi(\tau_2)-\psi(s)\right)^{\alpha-1}|\mathfrak{H}(s,\mathfrak{u}(s))|\,\Delta s \\
&\leq \frac{b}{\Gamma(\alpha)}\int_0^{\tau_1}\psi^{'}(s)\Big|\left(\psi(\tau_2)-\psi(0)\right)^{1-\gamma}\left(\psi(\tau_2)-\psi(s)\right)^{\alpha-1} \\
&\qquad\qquad - \left(\psi(\tau_2)-\psi(0)\right)^{1-\gamma}\left(\psi(\tau_1)-\psi(s)\right)^{\alpha-1}\Big|\,|\mathfrak{H}(s,\mathfrak{u}(s))|\,ds \\
&\quad + \frac{b\,\left(\psi(\tau_2)-\psi(0)\right)^{1-\gamma}}{\Gamma(\alpha)}\left(\psi(\tau_2)-\psi(\tau_1)\right)^{\alpha+\gamma-1}B(\gamma,\alpha)\,\|\mathfrak{H}\|_{C_{1-\gamma,\psi}}.
\end{aligned}$$

Thus, right-hand part tends to zero. Hence along with the Arzëla-Ascoli theorem and from Step 1-3, it is concluded that $\mathscr{P}$ is completely continuous. Thus the proposed problem has at least one solution. □

THEOREM 3.3. *Assume that (H1) and (H3) are fulfilled. If*

$$\left(\frac{bLB(\gamma,\alpha)}{\Gamma(\alpha)}\left(\psi(b)-\psi(0)\right)^{\alpha}\right) < 1 \tag{5}$$

then there exists a unique solution for Eq. (1).

Proof. Define the operator $\mathscr{P} : C_{1-\gamma,\psi}(J) \to C_{1-\gamma,\psi}(J)$.
(6)

$$(\mathscr{P}\mathfrak{u})(\tau) = \mathfrak{u}_0(\tau) + \frac{1}{\Gamma(\alpha)}\int_0^{t}\psi^{'}(s)\left(\psi(\tau)-\psi(s)\right)^{\alpha-1}h(s)\,\mathfrak{H}(s,\mathfrak{u}(s))\Delta s$$

with $\mathfrak{u}_0(\tau) = \frac{\mathfrak{u}_0}{\Gamma(\gamma)} (\psi(\tau) - \psi(0))^{\gamma-1}$.
Let $\mathfrak{u}_1, \mathfrak{u}_2 \in C_{1-\gamma,\psi}(J)$ and $\tau \in J$, then

$$\begin{aligned}
&\left|(\psi(\tau) - \psi(0))^{1-\gamma} ((\mathscr{P}\mathfrak{u}_1)(\tau) - (\mathscr{P}\mathfrak{u}_2)(\tau))\right| \\
&\leq \frac{(\psi(\tau) - \psi(0))^{1-\gamma}}{\Gamma(\alpha)} \int_0^\tau \psi'(s) (\psi(\tau) - \psi(s))^{\alpha-1} b \left|\mathfrak{H}(s, \mathfrak{u}_1(s)) - \mathfrak{H}(s, \mathfrak{u}_2(s))\right| \Delta s \\
&\leq \frac{(\psi(\tau) - \psi(0))^{1-\gamma}}{\Gamma(\alpha)} \int_0^\tau \psi'(s) (\psi(\tau) - \psi(s))^{\alpha-1} b \left|\mathfrak{H}(s, \mathfrak{u}_1(s)) - \mathfrak{H}(s, \mathfrak{u}_2(s))\right| ds \\
&\leq \frac{bL (\psi(\tau) - \psi(0))^{1-\gamma}}{\Gamma(\alpha)} \int_0^\tau \psi'(s) (\psi(\tau) - \psi(s))^{\alpha-1} \left|\mathfrak{u}_1(s) - \mathfrak{u}_2(s)\right| ds \\
&\leq \frac{bLB(\gamma, \alpha)}{\Gamma(\alpha)} (\psi(b) - \psi(0))^{\alpha} \left\|\mathfrak{u}_1 - \mathfrak{u}_2\right\|_{C_{1-\gamma,\psi}}.
\end{aligned}$$

Then,

$$\left\|\mathscr{P}\mathfrak{u}_1 - \mathscr{P}\mathfrak{u}_2\right\|_{C_{1-\gamma,\psi}} \leq \frac{bLB(\gamma, \alpha)}{\Gamma(\alpha)} (\psi(b) - \psi(0))^{\alpha} \left\|\mathfrak{u}_1 - \mathfrak{u}_2\right\|_{C_{1-\gamma,\psi}}.$$

From (5), it follows that $\mathscr{P}$ has a unique fixed point which is solution of problem (1). □

4. Stability analysis

Next, we shall give the definitions and the criteria generalized UHR stability.

DEFINITION 4.1. Equation (1) is generalized UHR stable with respect to $\varphi \in C_{1-\gamma}(J)$ if there exists a real number $c_{\mathfrak{H},\varphi} > 0$ such that for each solution $\mathfrak{v} \in C_{1-\gamma}(J)$ of the inequality

$$\left|{}^{\mathbb{T}}\Delta^{\alpha,\beta}\mathfrak{v}(\tau) - h(\tau)\mathfrak{H}(\tau, \mathfrak{v}(\tau))\right| \leq \varphi(t), \tag{7}$$

there exists a solution $\mathfrak{u} \in C_{1-\gamma}^{\gamma}(J)$ of equation (1) with

$$|\mathfrak{v}(\tau) - \mathfrak{u}(\tau)| \leq c_{\mathfrak{H},\varphi}\varphi(\tau).$$

THEOREM 4.2. *Assume that (H1), (H3), (H4) and* (5) *are satisfied. Then, the problem* (1) *is generalized UHR stable.*

Proof. Let $\mathfrak{v} \in C_{1-\gamma}(J)$ be solution of the following inequality (7) and let $\mathfrak{u} \in C_{1-\gamma}(J)$ be the unique solution of the ψ-Hilfer type dynamics equation (1). By Lemma 3.1,

$$\mathfrak{u}(\tau) = \mathfrak{u}_0(\tau) + \frac{1}{\Gamma(\alpha)} \int_0^\tau \psi^{'}(s) \left(\psi(\tau) - \psi(s)\right)^{\alpha-1} h(s)\mathfrak{H}(s, \mathfrak{u}(s))\Delta s.$$

By integration of (7) we obtain

$$\left|\mathfrak{v}(\tau) - \mathfrak{v}_0(\tau) - \frac{1}{\Gamma(\alpha)} \int_0^\tau \psi^{'}(s) \left(\psi(\tau) - \psi(s)\right)^{\alpha-1} h(s)\mathfrak{H}(s, \mathfrak{v}(s))\Delta s\right| \leq \lambda_\varphi \varphi(\tau). \tag{8}$$

On the other hand, we have

$$\begin{aligned}
&|\mathfrak{v}(\tau) - \mathfrak{u}(\tau)| \\
&\leq \left|\mathfrak{v}(\tau) - \mathfrak{v}_0(\tau) - \frac{1}{\Gamma(\alpha)} \int_0^\tau \psi^{'}(s) \left(\psi(\tau) - \psi(s)\right)^{\alpha-1} h(s)\mathfrak{H}(s, \mathfrak{v}(s))\Delta s\right| \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_0^\tau \psi^{'}(s) \left(\psi(\tau) - \psi(s)\right)^{\alpha-1} h(s) \left|\mathfrak{H}(s, \mathfrak{v}(s)) - \mathfrak{g}(s, \mathfrak{u}(s))\right| \Delta s \\
&\leq \left|\mathfrak{v}(\tau) - \mathfrak{v}_0(\tau) - \frac{1}{\Gamma(\alpha)} \int_0^\tau \psi^{'}(s) \left(\psi(\tau) - \psi(s)\right)^{\alpha-1} h(s)\mathfrak{H}(s, \mathfrak{v}(s))\Delta s\right| \\
&\quad + \frac{bL}{\Gamma(\alpha)} \int_0^t \psi^{'}(s) \left(\psi(\tau) - \psi(s)\right)^{\alpha-1} |\mathfrak{v}(s) - \mathfrak{u}(s)|\, ds \\
&\leq \lambda_\varphi \varphi(\tau) + \frac{bL}{\Gamma(\alpha)} \int_0^t \psi^{'}(s) \left(\psi(\tau) - \psi(s)\right)^{\alpha-1} |\mathfrak{v}(s) - \mathfrak{u}(s)|\, ds.
\end{aligned}$$

By applying Lemma 2.10, we obtain

$$|\mathfrak{v}(\tau) - \mathfrak{u}(\tau)| \leq \left[(1 + \nu_1 bL\lambda_\varphi)\lambda_\varphi\right] \varphi(\tau),$$

where $\nu_1 = \nu_1(\alpha)$ is a constant,then for any $\tau \in J$:

$$|\mathfrak{v}(\tau) - \mathfrak{u}(\tau)| \leq c_{\mathfrak{g}} \epsilon \varphi(\tau),$$

Thus, the proof is completed. □

EXAMPLE 4.3. Consider the following equation

$$\begin{cases} {}^{\mathbb{T}}\Delta^{\alpha,\beta;\psi}\mathfrak{u}(t) = 0.5tu, \quad t = [0,1] := J \subseteq \mathbb{T}, \\ {}^{\mathbb{T}}\mathfrak{J}^{1-\gamma;\psi}\mathfrak{u}(0) = \mathfrak{u}_0, \quad \gamma = \alpha + \beta - \alpha\beta, \end{cases} \tag{9}$$

With the following parameters $\alpha = 0.5, \gamma = 0.5, L = 0.5$ and $\psi(1) = 1, \psi(0) = 0$ we have

$$\left(\frac{bLB(\gamma,\alpha)}{\Gamma(\alpha)} \left(\psi(b) - \psi(0)\right)^{\alpha}\right) = \frac{0.5 * 3.141}{1.77} = 0.887 < 1. \tag{10}$$

Hence, in view of Theorem 3.3, Eq.(9) has a unique stable solution.

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S. Harikrishnan
Department of Mathematics
Sri Ramakrishna Mission Vidyalaya
College of Arts and Science
Coimbatore-641020, India
E-mail: hkkhari1@gmail.com

Rabha W. Ibrahim
Senior IEEE member
94086547
E-mail: rabhaibrahim@yahoo.com

K. Kanagarajan
Department of Mathematics
Sri Ramakrishna Mission Vidyalaya
College of Arts and Science
Coimbatore-641020, India
E-mail: kanagarajank@gmail.com