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APPLICATIONS OF LINKING INEQUALITIES TO AN ASYMMETRIC BEAM EQUATION

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Abstract. We prove that an asymmetric beam equation has at least two solutions, one of which is a positive solution. To prove the existence of the other solution, we use linking inequalities.

1. Introduction

We investigate the existence of multiple solutions of the nonlinear beam equation in an interval $(-\frac{\pi}{2}, \frac{\pi}{2})$,

$$u_{tt} + u_{xxxx} + bu^+ - |u^-|^{p-1} = f(x,t) \text{ in } (-\frac{\pi}{2}, \frac{\pi}{2}) \times R, \tag{1}$$

$$u(\pm\frac{\pi}{2}, t) = u_{xx}(\pm\frac{\pi}{2}, t) = 0, \tag{2}$$

$$u \textit{ is } \pi - \textit{periodic in } t \textit{ and even in } x \textit{ and } t, \tag{3}$$

where the nonlinearity $-(bu^+)$ crosses eigenvalues and $u^+ = max\{u, 0\}$, $u^- = max\{-u, 0\}$. Here we suppose that $p > 2$ and $f = s\phi_{00} + \alpha h(x,t)(s > 0)$, h is bounded. This equation represents a bending beam supported by cables under a load f. The nonlinearity u^+ models the fact that cables resist expansion but do not resist compression.

Let L be the differential operator, $Lu = u_{tt} + u_{xxxx}$. Then the eigenvalue problem for $u(x,t)$

$$Lu = \lambda u \text{ in } (-\frac{\pi}{2}, \frac{\pi}{2}) \times R$$

with (2) and (3), has infinitely many eigenvalues

$$\lambda_{mn} = (2n+1)^4 - 4m^2 \qquad (m, n = 0, 1, 2, \cdots)$$

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and corresponding eigenfunctions $\phi_{mn}(m, n \geq 0)$ given by

$$\phi_{mn} = \cos 2mt \cos(2n+1)x$$

We note that all eigenvalues in the interval $(-19, 45)$ are given by

$$\lambda_{20} = -15 < \lambda_{10} = -3 < \lambda_{00} = 1 < \lambda_{41} = 17$$

Let Ω be the square $[-\frac{\pi}{2}, \frac{\pi}{2}] \times [-\frac{\pi}{2}, \frac{\pi}{2}]$ and H the Hilbert space defined by

$$H = \{u \in L^2(\Omega) : u \text{ is even in } x \text{ and } t\}.$$

Then the set of eigenfunctions $\{\phi_{mn}\}$ is an orthonormal base in H. Hence equation (1) with (2) and (3) is equivalent to

$$Lu + bu^+ = f \text{ in } H. \tag{4}$$

In [6], the authors showed by degree theory that equation (4) with constant load $1 + \epsilon h$ (h is bounded) has at least two solutions. In [1], the authors showed by a variational reduction method that equation (4) with constant load $1 + \epsilon h$ (h is bounded) has at least three solutions when condition (3) is replaced by

$$u \textit{ is } \pi - \textit{periodic in } t \textit{ and even in } x. \tag{5}$$

In [5], the author showed by linking method and category theory that the following asymmetric beam equation has multiple nontrivial solutions

$$Lu + bu^+ = |u^+|^{p-1} - |u^-|^{q-1} \text{ in } H. \tag{6}$$

McKenna and Walter [7] proved that if $3 < b < 15$ then at least two π-periodic solutions exist, one of which is large in amplitude. The existence of at least three solutions was later proved by Choi, Jung and McKenna [2] using a variational reduction method. Humphreys [4] proved that there exists an $\varepsilon > 0$ such that when $15 < b < 15 + \varepsilon$ at least four periodic solutions exist. Choi and Jung [1] suppose that $3 < b < 15$ and f is generated by eigenfunctions. Since Micheletti and Saccon [8] applied the limit relative category to studying multiple nontrivial solutions for a floating beam.

The main result of this paper is the following.

THEOREM 1.1. *Let* $\Lambda_i^- < -b (b > 0)$ *and* $f = se_1^+ (s > 0)$. *Let* u_p *be the positive solution of (1). Then problem (1) has at least two solutions, one of which is a positive solution.*

In this paper, we use a variational approach and look for critical points of a suitable functional I on a Hilbert space H. In Section 2, we find a suitable functional I on a Hilbert space H and prove the suitable version of the Palais-Smale condition for the topological method. In Section 3, we study the geometry of the sub-levels of I and find two linking type inequalities, relative to two different decompositions of the space H.

2. The Palais Smale condition

To begin with, we consider the associated eigenvalue problem

$$\begin{aligned} Lu = \lambda u \qquad &\text{in } (-\tfrac{\pi}{2}, \tfrac{\pi}{2}) \times R \\ u(\pm\tfrac{\pi}{2}, t) &= u_{xx}(\pm\tfrac{\pi}{2}, t) = 0 \\ u(x,t) = u(-x,t) &= u(x,-t) = u(x, t+\pi). \end{aligned} \tag{7}$$

A simple computation shows that equation (3) has infinitely many eigenvalues λ_{mn} and the corresponding eigenfunctions ϕ_{mn} given by

$$\lambda_{mn} = (2n+1)^4 - 4m^2,$$

$$\phi_{mn}(x,t) = \cos 2mt \cos(2n+1)x \qquad (m, n = 0, 1, 2, \cdots).$$

Let Ω be the square $[-\frac{\pi}{2}, \frac{\pi}{2}] \times [-\frac{\pi}{2}, \frac{\pi}{2}]$ and H the Hilbert space defined by

$$H = \{u \in L^2(\Omega) | u \text{ is even in } x \text{ and } t\}. \tag{8}$$

Then the set $\{\phi_{mn} | m, n = 0, 1, 2, \cdots\}$ is an orthogonal base of H and H consists of the functions

$$u(x,t) = \sum_{m,n=0}^{\infty} a_{mn}\phi_{mn}(x,t) \tag{9}$$

with the norm given by

$$\|u\|^2 = \sum_{m,n=0}^{\infty} a_{mn}^2. \tag{10}$$

We denote by $(\Lambda_i^-)_{i\geq 1}$ the sequence of the negative eigenvalues of equation (3), by $(\Lambda_i^+)_{i\geq 1}$ the sequence of the positive ones, so that

$$\cdots < \Lambda_1^- = -3 < \Lambda_1^+ = 1 < \Lambda_2^+ = 17 < \cdots.$$

We consider an orthonormal system of eigenfunctions $\{e_i^-, e_i^+, i \geq 1\}$ associated with the eigenvalues $\{\Lambda_i^-, \Lambda_i^+, i \geq 1\}$.

The following theorem is the uniqueness result.

PROPOSITION 2.1. *Let $b < -\Lambda_1^-$ and $p > 2$. Then the equation*

$$Lu + bu^+ - |u^-|^{p-1} = 0 \text{ in } H \tag{11}$$

has only the trivial solution.

Proof. We rewrite the above equation as

$$\begin{aligned} Lu - \Lambda_1^+ u &= -\Lambda_1^+ u - bu^+ + |u^-|^{p-1} \\ &= -\Lambda_1^+ u^+ - bu^+ + |u^-|^{p-1} + \Lambda_1 u^- + |u^-|^{p-1}. \end{aligned}$$

Multiplying across by e_1^+ and integrating over Ω,

$$\begin{aligned} 0 &= < [L - \Lambda_1^+]u, e_1^+ > \\ &= \int_\Omega (-\Lambda_1^+ u^+ - bu^+ + |u^-|^{p-1} + \Lambda_1 u^- + |u^-|^{p-1}) e_1^+ dxdt \geq 0, \end{aligned}$$

since the condition $b < -\Lambda_1^-$ imply that $-\Lambda_1^+ u^+ - bu^+ + |u^-|^{p-1} + \Lambda_1 u^- + |u^-|^{p-1} \geq 0$ for all real valued functions u and $e_1^+(x) > 0$ for all $x \in \Omega$. Therefore the only possibility to hold (1) is that $u \equiv 0$. □

THEOREM 2.2. *Let $b < -\Lambda_1^-, s > 0$ and $||h|| = 1$. Then there exists $\alpha_0 > 0$ such that for $\alpha < \alpha_0$ the equation*

$$Lu + bu^+ + |u^-|^{p-1} = se_1^+ + \alpha h(x,t) \text{ in } H \tag{12}$$

has a positive solution.

Proof. Since $b < -\Lambda_1^- < -\Lambda_1^+$, $b + \Lambda_1^+ > 0$ Thus the equation

$$Lu + bu^+ = se_1^+ \text{ in } H$$

has a positive solution $u_p = \frac{s}{b+\Lambda_1^+} e_1^+$, which is a positive solution of the equation

$$Lu + bu^+ + |u^-|^{p-1} = se_1^+ \text{ in } H.$$

Therefore there exists $\alpha_0 > 0$ such that for $\alpha < \alpha_0$ equation (1) has a positive solution. □

We set

$$H^+ = \text{closure of span}\{\text{eigenfunctions with eigenvalue} \geq 0\},$$

$$H^- = \text{closure of span}\{\text{eigenfunctions with eigenvalue} \leq 0\}.$$

We define the linear projections $P^- : H \to H^-, P^+ : H \to H^+$.

We also introduce two linear operators $R : H \to H^+, S : H \to H^-$ by

$$S(u) = \sum_{i=1}^{\infty} \frac{a_i^- e_i^-}{\sqrt{-\Lambda_i^-}}, R(u) = \sum_{i=1}^{\infty} \frac{a_i^+ e_i^+}{\sqrt{\Lambda_i^+}}$$

if

$$u = \sum_{i=1}^{\infty} a_i^- e_i^- + \sum_{i=1}^{\infty} a_i^+ e_i^+.$$

It is clear that S and R are compact and self adjoint on H.

Definition 2.3. *Let $I_b : H \to R$ be defined by*

$$I_b(u) = \frac{1}{2}\|P^+u\|^2 - \frac{1}{2}\|P^-u\|^2 + \frac{b}{2}\|[Au]^+\|^2 - \int_\Omega G(Au)dxdt$$

where $A = R + S$ and $G(s) = \int_0^s g(x,t,\tau)d\tau$, $g(x,t,\tau) = se_1^+ - |\tau^-|^{p-1}$.

It is straightforward that

$$\nabla I_b(u) = P^+u - P^-u + bA(Au)^+ - Ag(Au).$$

Following the idea of Hofer (see [3]) one can show that

Proposition 2.4. *$I_b \in C^{1,1}(H, R)$. Moreover $\nabla I_b(u) = 0$ if and only if $w = (R + S)(u)$ is a weak solution of (P), that is,*

$$\int_\Omega (w(v_{tt} + v_{xxxx}) + b[w]^+v)dxdt = \int_\Omega g(w)vdxdt$$

for all smooth $v \in H$.

In this section, we suppose $b > 0$. Under this assumption, we have a concern with multiplicity of solutions of equation (1). Here we suppose that f is defined by equation (2).

In the following, we consider the following sequence of subspaces of $L^2(R^N)$:

$$H_n = (\oplus_{i=1}^n H_{\Lambda_i^-}) \oplus (\oplus_{i=1}^n H_{\Lambda_i^+})$$

where H_Λ is the eigenspace associated to Λ.

Lemma 2.5. *The functional I_b satisfies $(P.S.)_\gamma^*$ condition, with respect to (H_n), for all γ.*

For the proof we refer [2], [5].

3. Linking theory and main result

Fixed Λ_i^- and $\Lambda_i^- < -b < \Lambda_{i-1}^-$. We prove the Theorem via a linking argument.

First of all, we introduce a suitable splitting of the space H. Let

$$Z_1 = \oplus_{j=i+1}^{\infty} H_{\Lambda_j^-}, Z_2 = H_{\Lambda_i^-}, Z_3 = \oplus_{j=1}^{i-1} H_{\Lambda_j^-} \oplus H^+$$

LEMMA 3.1. *There exists R such that* $\sup_{v \in Z_1 \oplus Z_2, \|v\|=R} I_b(v) \leq 0$.

Proof. If $v \in Z_1 \oplus Z_2$ then

$$I_b(v) = -\frac{1}{2}\|v\|^2 + \frac{b}{2}\|[Sv]^+\|^2 - \int_\Omega G(Sv)dxdt.$$

Since

$$\frac{b}{2}\|[Sv]^+\|^2 - \int_\Omega G(Sv)dx = \int_\Omega \frac{b}{2}([Sv]^+)^2 - \frac{1}{p}([Sv]^-)^p dxdt,$$

there exists R such that $-\frac{1}{4}\|v\|^2 + \frac{b}{2}\|[Sv]^+\|^2 - \int_\Omega G(Sv)dx \leq 0$ for all $\|v\| = R$. Hence

$$I_b(v) \leq -\frac{1}{4}\|v\|^2 \leq 0$$

□

LEMMA 3.2. *There exists ρ such that* $\inf_{u \in Z_2 \oplus Z_3, \|u\|=\rho} I_b(u) > 0$.

For the proof we refer [5].

DEFINITION 3.3. *Let H be an Hilbert space, $Y \subset H$, $\rho > 0$ and $e \in H \setminus Y$, $e \neq 0$. Set:*

$$\begin{aligned}
B_\rho(Y) &= \{x \in Y \mid \|x\| \leq \rho\}, \\
S_\rho(Y) &= \{x \in Y \mid \|x\| = \rho\}, \\
\triangle_\rho(e, Y) &= \{\sigma e + v \mid \sigma \geq 0, v \in Y, \|\sigma e + v\| \leq \rho\}, \\
\Sigma_\rho(e, Y) &= \{\sigma e + v \mid \sigma \geq 0, v \in Y, \|\sigma e + v\| = \rho\} \cup \{v \mid v \in Y, \|v\| \leq \rho\}.
\end{aligned}$$

THEOREM 3.4. *Let $\Lambda_i^- < -b (b > 0)$ and $f = se_1^+ (s > 0)$. Let u_p be the positive solution of (1). Then problem (1) has at least two solutions, one of which is a positive solution.*

Proof. Let $e \in Z_2$. By Lemma 3.1 and Lemma 3.2, for a suitable large R and a suitable small ρ, we have the linking inequality

$$\sup I_b(\Sigma_R(e, Z_1)) < \inf I_b(S_\rho(Z_2 \oplus Z_3)). \tag{13}$$

Moreover $(P.S.)^*_\gamma$ holds. By standard linking arguments, it follows that there exists a critical point u for I_b with $\alpha \le I_b(u) \le \beta$, where $\alpha = \inf I_b(S_\rho(Z_2 \oplus Z_3))$ and $\beta = \sup I_b(\Delta_R(e, Z_1))$. Since $\alpha > 0$ and $I_b(u_p) = 0$, $u \ne u_p$. Therefore then problem (1) has at least two solutions, one of which is a positive solution. □

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