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COMMON FIXED POINT RESULTS FOR NON-COMPATIBLE R-WEAKLY COMMUTING MAPPINGS IN PROBABILISTIC SEMIMETRIC SPACES USING CONTROL FUNCTIONS

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Abstract. In common fixed point problems in metric spaces several versions of weak commutativity have been considered. Mappings which are not compatible have also been discussed in common fixed point problems. Here we consider common fixed point problems of non-compatible and R-weakly commuting mappings in probabilistic semimetric spaces with the help of a control function. This work is in line with research in probabilistic fixed point theory using control functions. Further we support our results by examples.

1. Introduction

The problem of finding common fixed points of more than one mappings have been considered in several contemporary works. In this context the commuting condition on a pair of mappings has been relaxed mainly in two directions. In one direction the notion of compatibility of various types have been introduced and common fixed point results of several types of compatible mappings have been established. This line of research was initiated by Junngck [11]. It has also been noted that fixed point problems of non-compatible mappings are also important and have been considered in a number of recent works [20, 23].

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In another direction weaker versions of commutativity have been considered in a large number of works. One such concept is R-weakly commutativity, which was introduced by Pant [19]. This is an extension of weakly commuting mappings [18, 28]. Some of other references dealing with R-weakly commutating mappings are [20–22] and [23]. Recently commutativity conditions have also been used to find coupled coincidence point in [5–7].

In metric fixed point theory a new direction was opened by Khan et al in [13]. They introduced a new contraction mapping principle and proved a fixed point result with the help of a control function which they called altering distance function. Altering distance function has been used in a number of papers in metric fixed point theory. Some of these results are noted in [17, 24] and [25].

K. Menger [15] introduced the notion of probabilistic metric space in 1942 and since then the theory of probabilistic metric spaces has emerged as an active area of research. Menger utilized distribution functions instead of non-negative real numbers as the values of the metric. The notion of probabilistic metric space corresponds to the situation when we do not know exactly the distance between two points, we only know the probabilities of possible values of the distances. Probabilistic generalizations of the metric spaces appears to be interesting in the investigation of physical quantities and useful in modelling some physical phenomena.

First fixed point result in probabilistic metric spaces appeared in literature in the work of Sehgal and Bharucha-Reid [27]. After that result fixed point and common fixed point properties for mappings defined on probabilistic spaces has been studied by many authors. Hadzic and Pap in [12] has given a comprehensive survey of this line of research.

Recently in [2] Choudhury and Das have extended the idea of altering distance function to probabilistic metric spaces and have established a generalization of Sehgal's contraction, where a probabilistic contraction mapping principle has been established through the application of a control function. The introduction of control function in probabilistic spaces opens new possibilities of establishing new fixed point results. Some recent works in probabilistic spaces where this control function have been utilized are noted in [1, 3, 4, 8–10, 14, 16, 29] and [30]. Here we make another use of this control function to fixed point problems in probabilistic spaces. Precisely in this work we use the control function to establish common fixed point results in probabilistic semimetric spaces for mappings which are non-compatible and satisfy R-weakly commuting condition. Our results are supported by examples.

2. Definitions and Mathematical Preliminaries

In this section we give some definitions and results which are needed for our results.

Definition 2.1. t-norm [12, 26]
A t-norm is a function $\Delta : [0,1] \times [0,1] \to [0,1]$ which satisfies the following conditions

(i) $\Delta(1,a) = a$,
(ii) $\Delta(a,b) = \Delta(b,a)$,
(iii) $\Delta(c,d) \geq \Delta(a,b)$ whenever $c \geq a$ and $d \geq b$,
(iv) $\Delta(\Delta(a,b),c) = \Delta(a,\Delta(b,c))$.

Definition 2.2. [12,26] A mapping $F : R \to R^+$ is called a distribution function if it is non-decreasing and left continuous with $\inf_{t\in R} F(t) = 0$ and $\sup_{t\in R} F(t) = 1$, where R is the set of real numbers and R^+ denotes the set of non-negative real numbers.

Definition 2.3. Probabilistic semimetric space [12, 26]
Let M be a non empty set, F is a function defined on $M \times M$ to the set of distribution functions. (M,F) is said to be a probabilistic semimetric spaces if the following are satisfied:

(i) $F_{x,y}(0) = 0$ for all $x, y \in M$,
(ii) $F_{x,y}(s) = 1$ for all $s > 0$ and $x, y \in M$ if and only if $x = y$,
(iii) $F_{x,y}(s) = F_{y,x}(s)$ for all $x, y \in M, s > 0$.

Definition 2.4. Menger space [12, 26]
A Menger space is a triplet (M,F,Δ) where M is a non empty set, F is a function defined on $M \times M$ to the set of distribution functions and Δ is a t-norm, such that the following are satisfied:

(i) $F_{x,y}(0) = 0$ for all $x, y \in M$,
(ii) $F_{x,y}(s) = 1$ for all $s > 0$ and $x, y \in M$ if and only if $x = y$,
(iii) $F_{x,y}(s) = F_{y,x}(s)$ for all $x, y \in M, s > 0$ and
(iv) $F_{x,y}(u+v) \geq \Delta\left(F_{x,z}(u), F_{z,y}(v)\right)$ for all $u, v \geq 0$ and $x, y, z \in M$.

A sequence $\{x_n\} \subset M$ converges to some point $x \in M$ if for given $\epsilon > 0, \lambda > 0$ we can find a positive integer $N_{\epsilon,\lambda}$ such that for all $n > N_{\epsilon,\lambda}$,

$$F_{x_n,x}(\epsilon) > 1 - \lambda.$$

Definition 2.5. Altering distance function [13]
A function $h : [0, \infty) \to [0, \infty)$ is an altering distance function if

(i) h is monotone increasing and continuous and
(ii) $h(t) = 0$ if and only if t =0.

Khan et al. proved the following generalization of Banach contraction mapping principle.

Theorem 2.6. [13] Let (X, d) be a complete metric space, h be an altering distance function and let $f : X \to X$ be a self mapping which satisfies the following inequality

$$h(d(fx, fy)) \leq c\, h(d(x, y))$$

for all $x, y \in X$ and for some $0 < c < 1$. Then f has a unique fixed point.

In fact Khan et al. proved a more general theorem (Theorem-2 in [13]) of which the above result is a corollary.

Definition 2.7. Φ-function [2]
A function $\phi : [0, \infty) \to [0, \infty)$ is said to be a Φ-function if it satisfies the following conditions:

(i) $\phi(t) = 0$ if and only if $t = 0$,
(ii) $\phi(t)$ is strictly increasing and $\phi(t) \to \infty$ as $t \to \infty$,
(iii) ϕ is left continuous in $(0, \infty)$ and
(iv) ϕ is continuous at 0.

An altering distance function with the additional property that $h(t) \to \infty$ as $t \to \infty$ generates a Φ-function in the following way.

$$\phi(t) = \begin{cases} \sup\{s : h(s) < t\}, & \text{if} \quad t > 0, \\ 0, & \text{if} \quad t = 0. \end{cases}$$

It can be easily seen that ϕ is a Φ-function.

The following result has been established in [2].

Theorem 2.8. [2] Let (M, F, Δ) be a complete Menger space with $\Delta(a, b) = min\{a, b\}$ and $f : M \to M$ be a self mapping such that the following inequality is satisfied.

$$F_{fx,fy}(\phi(t)) \geq F_{x,y}(\phi(t/c))$$

where ϕ is a Φ-function, $0 < c < 1$, $t > 0$ and $x, y \in M$. Then f has a unique fixed point.

Definition 2.9. Two self-mappings f and g of a probabilistic semimetric space (M, F) are called compatible if $\lim_{n\to\infty} F_{fgx_n,gfx_n}(t) = 1$ for all $t > 0$ whenever $\{x_n\}$ is a sequence such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$ for some z in M.

Definition 2.10. Two self-mappings f and g of a probabilistic semimetric space (M, F) are called (ϕ, R)-weakly commuting if there exists some real number $R > 0$ such that
$F_{fgx,gfx}(\phi(t)) \geq F_{fx,gx}(\phi(\frac{t}{R}))$ for all $t > 0$ and for all $z \in M$.

Now f and g will be called pointwise (ϕ, R)-weakly commuting if given x in M, there exists $R > 0$ such that
$F_{fgx,gfx}(\phi(t)) \geq F_{fx,gx}(\phi(\frac{t}{R}))$ for all $t > 0$.

Definition 2.11. Ψ-function
A function $\psi : [0,1] \times [0,1] \to [0,1]$ is said to be a Ψ-function if

(i) ψ is monotone increasing and continuous,
(ii) $\psi(x,x) > x$ for all $0 < x < 1$,
(iii) $\psi(0,0) = 0$,
(iv) $\psi(x,x) = 1$ if and only if $x = 1$.

An example of Ψ-function.

1. $\psi(x,y) = \frac{p\sqrt{x}+q\sqrt{y}}{p+q}$, p and q are positive real numbers.

3. Main Results

Theorem 3.1. *Let (M, F) be a probabilistic semimetric space and $f, g : M \to M$ be non-compatible pointwise $(\phi, R)-$weakly commuting mappings such that*

[i] $fM \subset gM$ *(2.1)*

[ii] $F_{fx,fy}(\phi(t)) > min\{F_{gx,gy}(\phi(t)), \psi(F_{fx,gx}(\phi(\frac{t_1}{a_1})), F_{fy,gy}(\phi(\frac{t_2}{b_1}))), \psi(F_{fy,gx}(\phi(\frac{t_3}{a_2})), F_{fx,gy}(\phi(\frac{t_4}{b_2})))\}$ *(2.2)*

where $x, y \in M$ *with* $x \neq y$, $a_1, a_2, b_1, b_2 > 0$, *with* $0 < a_1 + b_1 < 1, 0 < a_2 + b_2 < 1$, $t, t_1, t_2, t_3, t_4 > 0$ *with* $t_1 + t_2 = t = t_3 + t_4$, ϕ *is a* Φ*-function and* ψ *is a* Ψ*-function. If the range of* f *or* g *is a complete subspace of* M *then* f *and* g *have a unique common fixed point.*

Proof. The mappings f and g are non-compatible maps, hence there exists a sequence $\{x_n\}$ in M such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$ for some $z \in M$ but $\lim_{n\to\infty} F_{fgx_n,gfx_n}(\phi(t)) \neq 1$ for some $t > 0$ or the limit does not exists.

Let us suppose that the range of g is a complete subspace of M. Then there exists some point $u \in M$ such that $z = gu$ where $z = \lim_{n\to\infty} gx_n$. If possible let $fu \neq gu$.
Also let us take $t_1 = \frac{a_1 t}{a_1+b_1}$, $t_2 = \frac{b_1 t}{a_1+b_1}$, $t_3 = \frac{a_2 t}{a_2+b_2}$, $t_4 = \frac{a_2 t}{a_2+b_2}$, (2.3)
and $c_1 = a_1 + b_1$, $c_2 = a_2 + b_2$ and $max\{c_1, c_2\} = c$. (2.4)
Now we have

$$\begin{aligned}
F_{fx_n,fu}(\phi(t)) &> min\{F_{gx_n,gu}(\phi(t)), \psi(F_{fx_n,gx_n}(\phi(\tfrac{t_1}{a_1})), F_{fu,gu}(\phi(\tfrac{t_2}{b_1}))),\\
&\qquad \psi(F_{fu,gx_n}(\phi(\tfrac{t_3}{a_2})), F_{fx_n,gu}(\phi(\tfrac{t_4}{b_2})))\}\\
&\geq min\{F_{gx_n,gu}(\phi(t)), \psi(F_{fx_n,gx_n}(\phi(\tfrac{t}{c})), F_{fu,gu}(\phi(\tfrac{t}{c}))),\\
&\qquad \psi(F_{fu,gx_n}(\phi(\tfrac{t}{c})), F_{fx_n,gu}(\phi(\tfrac{t}{c})))\}\\
&\geq min\{F_{gx_n,gu}(\phi(t)), \psi(F_{fx_n,gx_n}(\phi(t)), F_{fu,gu}(\phi(t))),\\
&\qquad \psi(F_{fu,gx_n}(\phi(t)), F_{fx_n,gu}(\phi(t)))\} \qquad (2.5)
\end{aligned}$$

Letting $n \to \infty$ we have from (2.5),

$$\begin{aligned}
F_{gu,fu}(\phi(t)) &\geq min\{F_{gu,gu}(\phi(t)), \psi(F_{gu,gu}(\phi(t)), F_{fu,gu}(\phi(t))),\\
&\qquad \psi(F_{fu,gu}(\phi(t)), F_{gu,gu}(\phi(t)))\}\\
&\geq min\{1,\ \psi(1, F_{fu,gu}(\phi(t))),\ \psi(F_{fu,gu}(t)), 1)\}. \qquad (2.6)
\end{aligned}$$

We claim $F_{fu,gu}(\phi(t)) = 1$ for all $t > 0$. If not then for some $t > 0$, $0 < F_{fu,gu}(\phi(t) < 1$ and by the property of ψ we have from (2.6)

$$\begin{aligned}
F_{gu,fu}(\phi(t)) &\geq min\{\psi(1, F_{fu,gu}(t))), \psi(F_{fu,gu}(\phi(t)), 1)\}\\
&\geq min\{\psi(F_{fu,gu}(\phi(t)), F_{fu,gu}(\phi(t))),\\
&\qquad \psi(F_{fu,gu}(\phi(t)), F_{fu,gu}(\phi(t)))\}
\end{aligned}$$

$= \psi(F_{fu,gu}(\phi(t)), F_{fu,gu}(\phi(t)))$
$> F_{fu,gu}(\phi(t))$, which is a contradiction.

Therefore for all $t > 0$ we have , $F_{gu,fu}(\phi(t)) = 1$. (2.7)
From the property of ϕ it follows that given $s > 0$ we can find $t > 0$ such that $s > \phi(t) > 0$.
Therefore for all $s > 0$ we have , $F_{gu,fu}(s) = 1$ that is, $gu = fu$.

Now f and g are (ϕ, R)-weakly commuting maps, therefore there exists $R_1 > 0$ such that for all $t > 0$

$$F_{fgu,gfu}(\phi(t)) \geq F_{fu,gu}(\phi(\tfrac{t}{R_1})) = F_{fu,fu}(\phi(\tfrac{t}{R_1})) = 1.$$

Therefore we can say as in above $fgu = gfu$ and hence

$$ffu = fgu = gfu = ggu.$$

Now we claim that $fu = ffu$. If $fu \neq ffu$ then we have,

$$\begin{aligned} F_{fu,ffu}(\phi(t)) &> min\{F_{gu,gfu}(\phi(t)), \psi(F_{fu,gu}(\phi(\tfrac{t_1}{a_1})), F_{ffu,gfu}(\phi(\tfrac{t_2}{b_1}))), \\ &\qquad \psi(F_{ffu,gu}(\phi(\tfrac{t_3}{a_2})), F_{fu,gfu}(\phi(\tfrac{t_4}{b_2})))\} \\ &\geq min\{F_{fu,ffu}(\phi(t)), \psi(1,1), \\ &\qquad \psi(F_{ffu,fu}(\phi(\tfrac{t_3}{a_2})), F_{fu,ffu}(\phi(\tfrac{t_4}{b_2}))) \end{aligned}$$

By taking $t_3 = \frac{a_2 t}{a_2+b_2}$, $t_4 = \frac{a_2 t}{a_2+b_2}$ and $c_2 = a_2 + b_2$ we have from above,

$$\begin{aligned} F_{fu,ffu}(\phi(t)) &> min\{F_{fu,ffu}(\phi(t)), 1, \psi(F_{ffu,fu}(\phi(\tfrac{t}{c_2})), F_{fu,ffu}(\phi(\tfrac{t}{c_2})))\} \\ &\geq min\{F_{fu,ffu}(\phi(t)), 1, F_{ffu,fu}(\phi(\tfrac{t}{c_2}))\} \\ &\geq min\{F_{fu,ffu}(\phi(t)), 1, F_{ffu,fu}(\phi(t))\} \\ &= min\{F_{fu,ffu}(\phi(t)), 1\} \end{aligned}$$

If $min\{F_{fu,ffu}(\phi(t)), 1\} = F_{fu,ffu}(\phi(t))$, then we arrived at contradiction. On the other hand if $min\{F_{fu,ffu}(\phi(t)), 1\} = 1$, then we have $F_{fu,ffu}(\phi(t)) > 1$, which is impossible.

Therefore $fu = ffu = gfu$. Hence fu is a common fixed point of f and g.

For uniqueness let fu and fv be the two common fixed points of f and g. If $fu \neq fv$ then we have,

$$F_{fu,fv}(\phi(t)) > min\{F_{gu,gv}(\phi(t)), \psi(F_{fu,gu}(\phi(\tfrac{t_1}{a_1})), F_{fv,gv}(\phi(\tfrac{t_2}{b_1}))),$$

$$\begin{aligned}&\psi(F_{fv,gu}(\phi(\tfrac{t_3}{a_2})),F_{fu,gv}(\phi(\tfrac{t_4}{b_2})))\}\\ = min\{&F_{gu,gv}(\phi(t)),\psi(1,1),\\ &\psi(F_{fv,fu}(\phi(\tfrac{t_3}{a_2})),F_{fu,fv}(\phi(\tfrac{t_4}{b_2})))\end{aligned}$$

Now taking t_1, t_2, t_3, t_4 as in (2.3) and c_1, c_2, c as in (2.4) we have ,

$$\begin{aligned}F_{fu,fv}(\phi(t)) &> min\{F_{fu,fv}(\phi(t)),\psi(1,1),\psi(F_{fv,fu}(\phi(\tfrac{t}{c})),F_{fu,fv}(\phi(\tfrac{t}{c})))\}\\ &\geq min\{F_{fu,fv}(\phi(t)),1,F_{fv,fu}(\phi(\tfrac{t}{c}))\}\\ &\geq min\{F_{fu,fv}(\phi(t)),1,F_{fv,fu}(\phi(t))\}\\ &= min\{F_{fu,fv}(\phi(t)),1\}\end{aligned}$$

If $min\{F_{fu,fv}(\phi(t)),1\} = F_{fu,fv}(\phi(t))$ then we have $F_{fu,fv}(\phi(t)) > F_{fu,fv}(\phi(t))$, which is a contradiction. Again if $min\{F_{fu,fv}(\phi(t)),1\} = 1$ then we have $F_{fu,fv}(\phi(t)) > 1$ which is impossible.

Therefore $fu = fv$. This completes the proof of the theorem. □

Taking $a_1 = b_1 = a_2 = b_2 = a$ in above theorem we get the following corollary.

Corollary 3.2. *Let (M, F) be a probabilistic semimetric space and $f, g : M \to M$ be non-compatible pointwise $(\phi, R)-$weakly commuting mappings such that*

[i] $fM \subset gM$ (2.8)

[ii] $$F_{fx,fy}(\phi(t)) > min\{F_{gx,gy}(\phi(t)),\psi(F_{fx,gx}(\phi(\tfrac{t_1}{a})),F_{fy,gy}(\phi(\tfrac{t_2}{a}))),\\ \psi(F_{fy,gx}(\phi(\tfrac{t_3}{a})),F_{fx,gy}(\phi(\tfrac{t_4}{a})))\} \quad (2.9)$$

where $x, y \in M$ with $x \neq y$, $0 < a < \frac{1}{2}$, $t, t_1, t_2, t_3, t_4 > 0$ with $t_1 + t_2 = t$ and $t_3 + t_4 = t$, ϕ is a Φ-function and ψ is a Ψ-function. If the range of f or g is a complete subspace of M then f and g have a unique common fixed point.

Theorem 3.3. *Let (M, F) be a probabilistic semimetric space, (A, S) and (B, T) be pairwise $(\phi, R)-$weakly commuting self-mappings on (M, F, Δ) such that*

[i] $AM \subset TM$ and $BM \subset SM$ (2.10)

$$[ii]\ F_{Ax,By}(\phi(t)) > min\{F_{Sx,Ty}(\phi(t)), \psi(F_{Ax,Sx}(\phi(\tfrac{t_1}{a_1})), F_{By,Ty}(\phi(\tfrac{t_2}{b_1}))), \psi(F_{Ax,Ty}(\phi(\tfrac{t_3}{a_2})), F_{By,Sx}(\phi(\tfrac{t_4}{b_2})))\} \qquad (2.11)$$

where $x, y \in M$ with $x \neq y$, $a_1, a_2, b_1, b_2 > 0$, with $0 < c_1 = a_1 + b_1 < 1$, $0 < c_2 = a_2 + b_2 < 1$, $t, t_1, t_2, t_3, t_4 > 0$ with $t_1 + t_2 = t = t_3 + t_4$, ϕ is a Φ-function and ψ is a Ψ-function. Let (A, S) or (B, T) be a non-compatible pair of mappings. If the range of one of the mappings is a complete subspace of M then A, B, S and T have a unique common fixed point.

Proof. Let B and T be non-compatible pair of mappings. Then there exists a sequence $\{x_n\}$ in M such that $\lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Tx_n = z$ for some $z \in M$ but $\lim_{n\to\infty} F_{BTx_n,TBx_n}\phi(t) \neq 1$ for some $t > 0$ or the limit does not exists. Since $BM \subset SM$, for each x_n there exists y_n in M such that $Bx_n = Sy_n$. Thus $Bx_n \to z$, $Tx_n \to z$ and $Sy_n \to z$ as $n \to \infty$.

We claim that $Ay_n \to z$ as $n \to \infty$,

that is, $\lim_{n\to\infty} F_{Ay_n,z}\phi(t) \to 1$ for all $t > 0$,

that is, there exists a positive integer N_1 such that $\lim_{n\to\infty} F_{Ay_n,z}\phi(t) > 1 - \lambda$ for all $t > 0$ and for all $n \geq N_1$.

Otherwise, we have for some $t' > 0$ a subsequence $\{Ay_m\}$ of $\{Ay_n\}$ and a positive number r, $0 < r \leq 1 - \lambda$ such that $\lim_{m\to\infty} F_{Ay_m,z}\phi(t') = r$.

$$\begin{aligned}
&F_{Ay_m,Bx_m}(\phi(t'))\\
&> min\{F_{Sy_m,Tx_m}(\phi(t')), \psi(F_{Ay_m,Sy_m}(\phi(\tfrac{t'_1}{a_1})), F_{Bx_m,Tx_m}(\phi(\tfrac{t'_2}{b_1}))),\\
&\qquad \psi(F_{Ay_m,Tx_m}(\phi(\tfrac{t'_3}{a_2})), F_{Bx_m,Sy_m}(\phi(\tfrac{t'_4}{b_2})))\}. \qquad (2.12)
\end{aligned}$$

Taking $t'_1 = \frac{a_1 t'}{a_1+b_1}$, $t'_2 = \frac{b_1 t'}{a_1+b_1}$, $t'_3 = \frac{a_2 t'}{a_2+b_2}$, $t'_4 = \frac{a_2 t'}{a_2+b_2}$ and c_1, c_2, c as in (2.4) we have from (2.12),

$$\begin{aligned}
&F_{Ay_m,Bx_m}(\phi(t'))\\
&> min\{F_{Sy_m,Tx_m}(\phi(t')), \psi(F_{Ay_m,Sy_m}(\phi(\tfrac{t'}{c})), F_{Bx_m,Tx_m}(\phi(\tfrac{t'}{c}))),\\
&\qquad \psi(F_{Ay_m,Tx_m}(\phi(\tfrac{t'}{c})), F_{Bx_m,Sy_m}(\phi(\tfrac{t'}{c})))\}.\\
&\geq min\{F_{Sy_m,Tx_m}(\phi(t')), \psi(F_{Ay_m,Sy_m}(\phi(t')), F_{Bx_m,Tx_m}(\phi(t'))),\\
&\qquad \psi(F_{Ay_m,Tx_m}(\phi(t')), F_{Bx_m,Sy_m}(\phi(t')))\}.
\end{aligned}$$

Letting $n \to \infty$ we get,

$$\begin{aligned}
r &\geq min\{F_{z,z}(\phi(t')),\ \psi(r, F_{z,z)}(\phi(t'))),\ \psi(r, F_{z,z}(\phi(t')))\}.\\
&\geq min\{1,\ \psi(r,1), \psi(r,1)\}
\end{aligned}$$

$$\begin{aligned} &= \psi(r,1) \\ &\geq \psi(r,r) \\ &> r, \text{ which is a contradiction.} \end{aligned}$$

Therefore, $Ay_n \to z$ as $n \to \infty$. (2.13)

Suppose that SM is a complete subspace of M. Then, since $Sy_n \to z$, there exists a point u in M such that $z = Su$. If $Au \neq Su$ then we have,

$$\begin{aligned} &F_{Au,Bx_n}(\phi(t)) \\ &> min\{F_{Su,Tx_n}(\phi(t)), \psi(F_{Au,Su}(\phi(\tfrac{t_1}{a_1})), F_{Bx_n,Tx_n}(\phi(\tfrac{t_2}{b_1}))), \\ &\quad \psi(F_{Au,Tx_n}(\phi(\tfrac{t_3}{a_2})), F_{Bx_n,Su}(\phi(\tfrac{t_4}{b_2})))\}. \end{aligned} \tag{2.14}$$

Now as (2.3) and (2.4) the same choice of t_1, t_2, t_3, t_4 and c_1, c_2, c gives us,

$$\begin{aligned} &F_{Au,Bx_n}(\phi(t)) \\ &> min\{F_{Su,Tx_n}(\phi(t)), \psi(F_{Au,Su}(\phi(\tfrac{t}{c})), F_{Bx_n,Tx_n}(\phi(\tfrac{t}{c}))), \\ &\quad \psi(F_{Au,Tx_n}(\phi(\tfrac{t}{c})), F_{Bx_n,Su}(\phi(\tfrac{t}{c})))\}. \\ &\geq min\{F_{Su,Tx_n}(\phi(t)), \psi(F_{Au,Su}(\phi(t)), F_{Bx_n,Tx_n}(\phi(t))), \\ &\quad \psi(F_{Au,Tx_n}(\phi(t)), F_{Bx_n,Su}(\phi(t)))\}. \end{aligned}$$

Therefore taking limit as $n \to \infty$ we get,

$$\begin{aligned} F_{Au,Su}(\phi(t)) \geq min\{&F_{Su,Su}(\phi(t)), \psi(F_{Au,Su}(\phi(t)), F_{Su,Su}(\phi(t))), \\ &\psi(F_{Au,Su}(\phi(t)), F_{z,z}(\phi(t)))\}. \end{aligned} \tag{2.15}$$

If for all $t > 0,\ F_{Au,Su}(\phi(t)) = 1$ then as above we have $Au = Su$. If not then there exists some $t > 0$ such that $0 < F_{Au,Su}(\phi(t)) < 1$.

Therefore,

$$\begin{aligned} F_{Au,Su}(\phi(t)) &\geq min\{1, \psi(F_{Au,Su}(\phi(t)), 1), \psi(F_{Au,Su}(\phi(t)), 1)\} \\ &= \psi(F_{Au,Su}(\phi(t)), 1) \\ &\geq \psi(F_{Au,Su}(\phi(t)), F_{Au,Su}(\phi(t))) \\ &> F_{Au,Su}(\phi(t)) \end{aligned}$$

which is a contradiction.

Therefore $Au = Su$.

Now pointwise R-weak commutativity of A and S implies that there exist a $R_1 > 0$ such that for all $t > 0$,

$$F_{ASu,SAu}(\phi(t)) \geq F_{Au,Su}(\phi(\tfrac{t}{R_1})) = 1,$$

that is, $ASu = SAu$ and that of $AAu = ASu = SAu = SSu$. Since $AM \subset TM$, there exists a w in M such that $Au = Tw$. We next show that $Tw = Bw$. If not then we have,

$$\begin{aligned} &F_{Au,Bw}(\phi(t)) \\ &> min\{F_{Su,Tw}(\phi(t)), \psi(F_{Au,Su}(\phi(\tfrac{t_1}{a_1})), F_{Bw,Tw}(\phi(\tfrac{t_2}{b_1}))), \\ &\qquad \psi(F_{Au,Tw}(\phi(\tfrac{t_3}{a_2})), F_{Bw,Su}(\phi(\tfrac{t_4}{b_2})))\}. \end{aligned} \tag{2.16}$$

By the similar choice of t_1, t_2, t_3, t_4 and c_1, c_2, c as in (2.3) and (2.4) respectively gives us,

$$\begin{aligned} &F_{Au,Bw}(\phi(t)) \\ &> min\{F_{Su,Tw}(\phi(t)), \psi(F_{Au,Su}(\phi(\tfrac{t}{c})), F_{Bw,Tw}(\phi(\tfrac{t}{c}))), \\ &\qquad \psi(F_{Au,Tw}(\phi(\tfrac{t}{c})), F_{Bw,Su}(\phi(\tfrac{t}{c})))\} \\ &= min\{F_{Au,Au}(\phi(t)), \psi(F_{Au,Au}(\phi(\tfrac{t}{c})), F_{Bw,Au}(\phi(\tfrac{t}{c}))), \\ &\qquad \psi(F_{Au,Au}(\phi(\tfrac{t}{c})), F_{Bw,Au}(\phi(\tfrac{t}{c})))\} \\ &= min\{1, \psi(1, F_{Bw,Au}(\phi(\tfrac{t}{c}))), \psi(1, F_{Bw,Au}(\phi(\tfrac{t}{c})))\} \\ &\geq min\{1, \psi(F_{Bw,Au}(\phi(\tfrac{t}{c}))), F_{Bw,Au}(\phi(\tfrac{t}{c}))), \\ &\qquad \psi(F_{Bw,Au}(\phi(\tfrac{t}{c}))), F_{Bw,Au}(\phi(\tfrac{t}{c})))\} \\ &\geq min\{1, F_{Bw,Au}(\phi(\tfrac{t}{c})), F_{Bw,Au}(\phi(\tfrac{t}{c}))\} \\ &\geq F_{Bw,Au}(\phi(\tfrac{t}{c})) \\ &\geq F_{Bw,Au}(\phi(t)), \text{ which is a contradiction.} \end{aligned}$$

Hence $Au = Bw = Tw = Su$. Pointwise R-weakly commutativity of B and T implies that $BTw = TBw$ and $BBw = BTw = TBw = TTw$. Now if $Au \neq AAu$ then applying the same procedure we have,

$$\begin{aligned} &F_{Au,AAu}(\phi(t)) \\ &= F_{AAu,Au}(\phi(t)) = F_{AAu,Bw}(\phi(t)) \\ &> min\{F_{SAu,Tw}(\phi(t)), \psi(F_{AAu,SAu}(\phi(\tfrac{t}{c})), F_{Bw,Tw}(\phi(\tfrac{t}{c}))), \\ &\qquad \psi(F_{AAu,Tw}(\phi(\tfrac{t}{c})), F_{Bw,SAu}(\phi(\tfrac{t}{c})))\} \\ &= min\{F_{AAu,Au}(\phi(t)), \psi(F_{AAu,AAu}(\phi(\tfrac{t}{c})), F_{Au,Au}(\phi(\tfrac{t}{c}))), \\ &\qquad \psi(F_{AAu,Au}(\phi(\tfrac{t}{c})), F_{Au,AAu}(\phi(\tfrac{t}{c})))\} \\ &\geq min\{F_{AAu,Au}(\phi(t)), \psi(1,1), F_{AAu,Au}(\phi(\tfrac{t}{c}))\} \\ &\geq min\{F_{AAu,Au}(\phi(t)), 1, F_{AAu,Au}(\phi(\tfrac{t}{c}))\} \end{aligned}$$

$\geq min\{F_{AAu,Au}(\phi(t)), F_{AAu,Au}(\phi(t))\}$
$= F_{AAu,Au}(\phi(t))$, which is a contradiction.

Therefore, $Au = AAu = SAu$ and Au is a common fixed point of A and S. Similarly $Au = Bw$ is fixed point of B and T. The proof is similar when TM is assumed to be a complete subspace of M. When AM or BM is a complete subspace of M the proof will be similar to the case when TM or SM is complete respectively as $AM \subset TM$ and $BM \subset SM$.

For uniqueness if possible let u and v be two distinct common fixed points of A, B, S and T that is, $Au = Bu = Su = Tu = u$ and $Av = Bv = Sv = Tv = v$. Since u and v be two distinct points we can find a $t > 0$ such that, $0 < F_{u,v}(\phi(t)) < 1$ and we have

$$\begin{aligned}
&F_{u,v}(\phi(t)) \\
&= F_{Au,Bv}(\phi(t)) \\
&> min\{F_{Su,Tv}(\phi(t)), \psi(F_{Au,Su}(\phi(\tfrac{t_1}{a_1})), F_{Bv,Tv}(\phi(\tfrac{t_2}{b_1}))), \psi(F_{Au,Tv}(\phi(\tfrac{t_3}{a_2})), F_{Bv,Su}(\phi(\tfrac{t_4}{b_2})))\} \\
&= min\{F_{u,v}(\phi(t)), \psi(F_{u,u}(\phi(\tfrac{t_1}{a_1})), F_{v,v}(\phi(\tfrac{t_2}{b_1}))), \psi(F_{u,v}(\phi(\tfrac{t_3}{a_2})), F_{v,u}(\phi(\tfrac{t_4}{b_2})))\} \\
&= min\{F_{u,v}(\phi(t)), \psi(1,1), \psi(F_{u,v}(\phi(\tfrac{t_3}{a_2})), F_{v,u}(\phi(\tfrac{t_4}{b_2})))\} \qquad (2.17)
\end{aligned}$$

By the same choice of t_3, t_4 and c as in (2.3) and (2.4) respectively we have from (2.17),

$$\begin{aligned}
F_{u,v}(\phi(t)) &> min\{F_{u,v}(\phi(t)), \psi(1,1), \psi(F_{u,v}(\phi(\tfrac{t}{c})), F_{v,u}(\phi(\tfrac{t}{c})))\} \\
&\geq min\{F_{u,v}(\phi(t)), 1, F_{u,v}(\phi(\tfrac{t}{c}))\} \\
&\geq min\{F_{u,v}(\phi(t)), F_{u,v}(\phi(t))\} \\
&= F_{u,v}(\phi(t)), \text{ which is a contradiction.}
\end{aligned}$$

Hence the uniqueness of the common fixed point is proved.

This completes the proof of the theorem. □

Example 3.4. Let (M, F) be a probabilistic semimetric space, where $M = [2, 20]$ and $F_{x,y}(t) = e^{\frac{-(x-y)^2}{t}}$. Define $f, g : M \to M$ as follows:

$$fx = \begin{cases} 2, & \text{if } \ x = 2, \\ 6, & \text{if } \ 2 < x \leq 5, \\ 2, & \text{if } \ x > 5, \end{cases}$$

$$gx = \begin{cases} 2, & \text{if} \quad x = 2, \\ 12, & \text{if} \quad 2 < x \leq 5, \\ \frac{x+1}{3}, & \text{if} \quad x > 5. \end{cases}$$

If we take $\phi(t) = \begin{cases} t^2, & \text{if} \quad 0 \leq x \leq 1, \\ \frac{t^2+1}{2}, & \text{if} \quad x > 1. \end{cases}$ and $\psi(x, y) = \frac{\sqrt{x}+\sqrt{y}}{2}$ then f and g satisfies all the conditions of Theorem 3.1 have a unique common fixed point at $x = 2$.

EXAMPLE 3.5. Let (M, F) be a probabilistic semimetric space, where $M = [2, 20]$ and $F_{x,y}(t) = e^{\frac{-(x-y)^2}{t}}$. Define $A, B, S, T : M \to M$ as follows:

$$Ax = 2 \qquad \text{if} \quad 2 \leq x \leq 20,$$

$$Bx = \begin{cases} 2, & \text{if} \quad x = 2, \\ 8, & \text{if} \quad 2 < x \leq 5, \\ 2, & \text{if} \quad x > 5, \end{cases}$$

$$Sx = \begin{cases} x, & \text{if} \quad 2 \leq x \leq 8, \\ 8, & \text{if} \quad x > 8, \end{cases}$$

$$Tx = \begin{cases} 2, & \text{if} \quad x = 2, \\ 12 + x, & \text{if} \quad 2 < x \leq 5, \\ x - 3, & \text{if} \quad x \geq 5. \end{cases}$$

If we take $\phi(t) = \begin{cases} t^2, & \text{if} \quad 0 \leq x \leq 1, \\ \frac{t^2+1}{2}, & \text{if} \quad x > 1. \end{cases}$ and $\psi(x, y) = \frac{\sqrt{x}+\sqrt{y}}{2}$ then A, B, S and T satisfies all the conditions of Theorem 3.3 have a unique common fixed point at $x = 2$.

4. Conclusion

The structure of the probabilistic metric space allows us to develop fixed point theory in several ways not always available in ordinary metric space. In this paper we have proved some fixed point results for non-compatible R-weakly commuting mappings. Our paper is also an instance of the use of control functions in fixed point theory in probabilistic metric spaces. This control function appears to be helpful in exploring the geometric aspects of Menger spaces which is also relevant to the study of geometry at the quantum level.

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