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SOME APPLICATIONS FOR GENERALIZED FRACTIONAL OPERATORS IN ANALYTIC FUNCTIONS SPACES

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Abstract. In this study a new generalization for operators of two parameters type of fractional in the unit disk is proposed. The fractional operators in this generalization are in the Srivastava-Owa sense. Concerning with the related applications, the generalized Gauss hypergeometric function is introduced. Further, some boundedness properties on Bloch space are also discussed.

1. Introduction

The subject of fractional integrals of any arbitrary complex order has achieved popularity and attention many of researchers during the past five decades, this mainly due to its applications in various diverse and comprehensive fields of science and engineering. In fact, the most common classical fractional integral operator and its generalizations are due to Srivastava-Owa on the domain of open unit disk $\mathbb{U}$ in the complex plane $\mathbb{C}$.

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DEFINITION 1.1. For $0 < \alpha < 1$ and $z \in \mathbb{U}$, the Srivstava-Owa fractional integral operator of $f(z)$ of order α is defined by

$$I_z^{\alpha} f(z) := \frac{1}{\Gamma(\alpha)} \int_0^z f(\xi)(z-\xi)^{\alpha-1} d\xi, \tag{1}$$

where $f(z)$ is analytic in simply-connected region of the complex z-plane containing the origin and the multiplicity of $(z-\xi)^{-\beta}$ is removed by requiring $\log(z-\xi)$ to be real when $(z-\xi) > 0$ (see [1,2]).

In 2011, for α and ρ real numbers, the generalization of the integral operator (1) was defined by Ibrahim [3] and expressed by the following form

$$I_z^{\alpha,\rho} f(z) := \frac{(\rho+1)^{1-\alpha}}{\Gamma(\alpha)} \int_0^z (z^{\rho+1} - \xi^{\rho+1})^{\alpha-1} \xi^{\rho} f(\xi) d\xi. \tag{2}$$

After that, for $0 < \alpha \leq 1$ and $0 < \beta \leq 1$, such that $0 < \alpha - \beta < 1$, Kılıçman *et al.* [4] applied Srivastava-Owa integral operator type fractional (1) to define a new generalized fractional integral operator presented as follows

$$\mathcal{L}_z^{\alpha,\beta} f(z) := \frac{\Gamma(\alpha)}{\Gamma(\beta)\Gamma(\alpha-\beta)}\, z^{1-\alpha} \int_0^z (z-\xi)^{\alpha-\beta-1} \xi^{\beta-1} f(\xi) d\xi. \tag{3}$$

Moreover, based on the equation (3), Kılıçman *et al.* [4] realized some of the univalence applications that by utilization the Gauss hypergeometric function and its properties on the unit disk were investigated. While, Srivastava *et at.* [5] established generating functions covering the Gauss hypergeometric function $F_p^{(\omega,\gamma,\kappa,\tau)}(a,b;c;z)$ and its some special cases

$$F_p^{(\omega,\gamma,\kappa,\tau)}(a,b;c;z) := \sum_{m=0}^{\infty} \left(\frac{B_p^{(\omega,\gamma,\kappa,\tau)}(b+m, c-b)}{B(b,c-b)} \right) (a)_m \frac{z^m}{m!},\ |z| < 1 \tag{4}$$

$$(\min\{\Re(\kappa),\ \Re(\tau),\ \Re(\omega),\ \Re(\gamma)\} > 0,\ \Re(c) > \Re(b) > 0,\ \Re(p) \geq 0)$$

where the generalized beta function $B_p^{(\omega,\gamma,\kappa,\tau)}(.,.)$ is defined by

$$B_p^{(\omega,\gamma,\kappa,\tau)}(\mu,\upsilon) = \int_0^1 t^{\mu-1}(1-t)^{\upsilon-1}\, {}_1F_1\left(\omega,\gamma; \frac{-p}{t^{\kappa}(1-t)^{\tau}}\right) dt, \tag{5}$$

where $(\Re(\mu), \Re(\upsilon), \Re(\omega), \Re(\gamma)) > 0$, $\Re(\kappa) > 0, \Re(\tau) > 0$ and $\Re(p) \geq 0$. As special cases, when $\kappa = \tau$, the functions in (4) and (5) would be immediately reduced to those functions introduced by [6] as follows:

$$F_p^{(\omega,\gamma,\tau)}(a,b;c;z) := \sum_{m=0}^{\infty}\left(\frac{B_p^{(\omega,\gamma,\tau)}(b+m,c-b)}{B(b,c-b)}\right)(a)_m\frac{z^m}{m!}, \tag{6}$$

$$(\min\{\Re(\tau),\ \Re(\omega),\ \Re(\gamma)\} > 0,\ \Re(c) > \Re(b) > 0,\ \Re(p) \geq 0,\ |z| < 1)$$

where $B_p^{(\omega,\gamma,\tau)}(.,.)$ is the generalized beta function defined by

$$B_p^{(\omega,\gamma,\tau)}(\mu,\upsilon) = \int_0^1 t^{\mu-1}(1-t)^{\upsilon-1}{}_1F_1\left(\omega,\gamma;\frac{-p}{t^{\tau}(1-t)^{\tau}}\right)dt, \tag{7}$$

where $(\Re(\mu),\Re(\upsilon),\Re(\omega),\Re(\gamma),\Re(\tau)) > 0$ and $\Re(p) > 0$. It is clearly that, when $\tau = 1$, the equation in (6) reduces to the functions defined by [7] as follows

$$F_p^{(\omega,\gamma)}(a,b;c;z) := \sum_{m=0}^{\infty}(a)_m\left(\frac{B_p^{(\omega,\gamma)}(b+m,c-b)}{B(b,c-b)}\right)\frac{z^m}{m!},\ z\in\mathbb{U}, \tag{8}$$

$$(\min\{\Re(\omega),\ \Re(\gamma)\} > 0, \Re(c) > \Re(b) > 0,\ \Re(p) \geq 0)$$

and $B_p^{(\omega,\gamma)}(.,.)$ is the generalized beta function defined by [7,8]:

$$B_p^{(\omega,\gamma)}(\mu,\upsilon) = \int_0^1 t^{\mu-1}(1-t)^{\upsilon-1}{}_1F_1\left(\omega,\gamma;\frac{-p}{t(1-t)}\right)dt, \tag{9}$$

where $(\Re(\mu),\Re(\upsilon),\Re(\omega),\Re(\gamma)) > 0$ and $\Re(p) > 0$. Moreover, when $\omega = \gamma$, the equation in (9) reduces to the extended beta function $B_p(\mu,\upsilon)$ due to Chaudhry *el at.* [9],

$$B_p(\mu,\upsilon) = \int_0^1 t^{\mu-1}(1-t)^{\upsilon-1}\exp\left(\frac{-p}{t(1-t)}\right)dt,\ \Re(\mu) > 0, \Re(\upsilon) > 0. \tag{10}$$

Obviously $B_0^{(\omega,\gamma)}(\mu,\upsilon) = B_0^{(\gamma,\gamma)}(\mu,\upsilon) = B(\mu,\upsilon)$ is the Euler's beta function given by

$$B(\mu,\upsilon) = \int_0^1 t^{\mu-1}(1-t)^{\upsilon-1}\,dt, \qquad \Re(\mu) > 0, \Re(\upsilon) > 0 \tag{11}$$

and $F_0^{(\omega,\gamma)} = F_0^{(\gamma,\gamma)} = {}_2F_1$ is the Gauss hypergeometric function given by [10]:

$${}_2F_1(a,b;c;z) := \sum_{m=0}^{\infty}\frac{(a)_m(b)_m}{(c)_m}\frac{z^m}{m!}, \tag{12}$$

where $(\varrho)_m$ is the Pochhammer symbol defined by:

$$(\varrho)_m = \frac{\Gamma(\varrho+m)}{\Gamma(\varrho)} = \begin{cases} 1 & ;(m=0,\ \varrho \in \mathbb{C}\setminus\{0\}) \\ \varrho(\varrho+1)\dots(\varrho+m-1) & ;\ (m=n\in\mathbb{N};\varrho\in\mathbb{C}), \end{cases}$$

where $\mathbb{N}$ denotes the set of the positive integers numbers and it is known that $(0)_0 = 1$.

Many applications demonstrated the influence and importance of the generalized Gauss hypergeometric function $F_p^{(\omega,\gamma,\kappa,\tau)}$ and their special cases in various areas of mathematical, physical, engineering, and statistical sciences. In the current sequel, we define new generalized integral and differential operators of two parameters type fractional and then investigate some of their geometric and boundedness properties on analytic spaces. The family of generalized Gauss hypergeometric functions $F_p^{(\omega,\gamma,\kappa,\tau)}$ is also considered to provide some applications at this point.

2. Fractional integral operator

This section aims to define a new generalized fractional integral operator of two parameters in the integral setting.

Definition 2.1. For $0 \le \rho,\ 0 \le \alpha < 1$ and $0 \le \beta < 1$ such that $0 < \alpha - \beta \le 1$, let $f(z)$ be analytic in a simple-connected region containing the origin. The generalized fractional integral operator of two parameters $\mathcal{L}_z^{\alpha,\beta}$ is defined by:

$$\mathcal{L}_z^{\alpha,\beta,\rho} f(z) := \frac{(1+\rho)^{1-\alpha+\beta}\Gamma(\alpha)}{\Gamma(\beta)\Gamma(\alpha-\beta)} z^{1-\alpha} \int_0^z \frac{t^{\rho+\beta-1} f(t)}{(z^{\rho+1} - t^{\rho+1})^{1-\alpha+\beta}} dt,\ z \in \mathbb{U}. \tag{13}$$

Note that, in case $\rho = 0$, the equation in (13) reduces to the fractional integral operator defined in (3).

Example 2.2. Let $f(z) = z^m$, $m \in \mathbb{R}$, then

$$\mathcal{L}_z^{\alpha,\beta,\rho}\{z^m\} = \frac{(\rho+1)^{\alpha-\beta}\Gamma(\frac{m+\beta+\rho}{\rho+1})\Gamma(\alpha)}{\Gamma(\beta)\Gamma(\frac{m+\beta+\rho}{\rho+1} + \alpha - \beta)} z^{(\alpha-\beta)\rho+m}, \tag{14}$$

for some $\Re(\rho) \ge 0$, $\Re(m) > 0$, $0 \le \alpha < 1$, $0 \le \beta < 1$, and for all $|z| < 1$.

Solution. Let

$$\mathcal{L}_z^{\alpha,\beta,\rho} z^m = \frac{(1+\rho)^{1-\alpha+\beta}\Gamma(\alpha)}{\Gamma(\beta)\Gamma(\alpha-\beta)} z^{1-\alpha} \int_0^z t^{\rho+\beta+m-1}(z^{\rho+1}-t^{\rho+1})^{\alpha-\beta-1}dt,$$

then by using the substitution $s = \left(\frac{t}{z}\right)^{\rho+1}$, we have

$$\begin{aligned} \mathcal{L}_z^{\alpha,\beta,\rho} z^m &= \tfrac{(1+\rho)^{\beta-\alpha}\Gamma(\alpha) z^{1-\alpha}}{\Gamma(\beta)\Gamma(\alpha-\beta)} \int_0^1 z^{(\rho+1)(\alpha-\beta-1)}(1-s)^{\alpha-\beta-1}(zs^{\frac{1}{\rho+1}})^{\beta+m-1} z^{\rho+1} ds, \\ &= \tfrac{(1+\rho)^{\beta-\alpha}\Gamma(\alpha)}{\Gamma(\beta)\Gamma(\alpha-\beta)} z^{\rho(\alpha-\beta)+m} \int_0^1 s^{\frac{\beta+m-1}{\rho+1}}(1-s)^{\alpha-\beta-1}ds. \end{aligned}$$

Now by applying the definition of beta function $B(.,.)$, we obtain

$$\begin{aligned} \mathcal{L}_z^{\alpha,\beta,\rho} z^m &= \frac{(1+\rho)^{\beta-\alpha}\Gamma(\alpha)}{\Gamma(\beta)\Gamma(\alpha-\beta)} z^{\rho(\alpha-\beta)+m} B\left(\frac{\beta+m-1}{\rho+1}+1, \alpha-\beta\right) \\ &= \frac{(1+\rho)^{\beta-\alpha}\Gamma(\alpha)}{\Gamma(\beta)} \frac{\Gamma\left(\frac{\beta+m+\rho}{\rho+1}\right)}{\Gamma\left(\alpha-\beta+\frac{\beta+m+\rho}{\rho+1}\right)} z^{\rho(\alpha-\beta)+m}. \end{aligned}$$

In fractional calculus, the fractional differential operators are derived by making use fractional integral operators for example see [11–13]. Here we define a new fractional differential operator by using fractional integral operator given by (13) in $\mathbb{U}$. A new generalization of the fractional differential operator is based on the observation that, for $0 \le \rho$, $0 \le \beta < 1$ and $0 \le \alpha < 1$, such that $0 < \alpha - \beta < 1$, we obtain

$$f(z) = \frac{(1+\rho)^{1-\alpha+\beta}\Gamma(\alpha)}{\Gamma(\beta)\Gamma(\alpha-\beta)} z^{1-\alpha} \int_0^z \frac{\xi^{\rho+\beta-1}\varphi(\xi)}{(z^{\rho+1}-\xi^{\rho+1})^{1-\alpha+\beta}} d\xi, \quad |z| < 1. \tag{15}$$

Formally, equation (15) can be solved by changing the variables z into ξ and ξ into s respectively. In addition, multiplying both sides of equation (15) by the factor $\xi^\rho(z^{\rho+1}-\xi^{\rho+1})^{\beta-\alpha}$ and integrating, we have

$$\begin{aligned} &\int_0^z \frac{\xi^\rho d\xi}{(z^{\rho+1}-\xi^{\rho+1})^{\alpha-\beta}} \int_0^\xi \frac{s^{\rho+\beta-1}\varphi(s)ds}{(\xi^{\rho+1}-s^{\rho+1})^{1-\alpha+\beta}} \\ &\qquad = \frac{(1+\rho)^{\alpha-\beta-1}\Gamma(\beta)\Gamma(\alpha-\beta)}{\Gamma(\alpha)} \int_0^z \frac{\xi^{\rho+\alpha-1}f(\xi)d\xi}{(z^{\rho+1}-\xi^{\rho+1})^{\alpha-\beta}}, \end{aligned}$$

thus the Dirichlet formula, implies that

$$\begin{aligned} &\int_0^z s^{\rho+\beta-1}\varphi(s)ds \int_s^z \frac{\xi^\rho d\xi}{(z^{\rho+1}-\xi^{\rho+1})^{\alpha-\beta}(\xi^{\rho+1}-s^{\rho+1})^{1-\alpha+\beta}} \\ &\qquad = \frac{(1+\rho)^{\alpha-\beta-1}\Gamma(\beta)\Gamma(\alpha-\beta)}{\Gamma(\alpha)} \int_0^z \frac{\xi^{\rho+\alpha-1}f(\xi)d\xi}{(z^{\rho+1}-\xi^{\rho+1})^{\alpha-\beta}}. \end{aligned}$$

By setting $\xi^{\rho+1} = s^{\rho+1}+u(z^{\rho+1}-s^{\rho+1})$ and $(\rho+1)\xi^{\rho}d\xi = (z^{\rho+1}-s^{\rho+1})du$, we obtain

$$\int_s^z \xi^{\rho}(z^{\rho+1}-\xi^{\rho+1})^{\beta-\alpha}(\xi^{\rho+1}-s^{\rho+1})^{\alpha-\beta-1}d\xi = (1+\rho)^{-1}\int_0^1 \frac{u^{\alpha-\beta-1}}{(1-u)^{\beta-\alpha}}du$$
$$=(1+\rho)^{-1}B(\alpha-\beta, 1-\alpha+\beta)$$
$$= (1+\rho)^{-1}\Gamma(\alpha-\beta)\Gamma(1-\alpha+\beta),$$

where $B(\cdot,\cdot)$ is the beta function. Consequently, we have

$$\int_0^z s^{\rho+\beta-1}\varphi(s)ds := \frac{(1+\rho)^{\alpha-\beta}\Gamma(\beta)}{\Gamma(\alpha)\Gamma(1-\alpha+\beta)}\int_0^z \frac{\xi^{\rho+\alpha-1}f(\xi)d\xi}{(z^{\rho+1}-\xi^{\rho+1})^{\alpha-\beta}}. \tag{16}$$

Hence after differentiation equation (16), we obtain

$$\varphi(z) := \frac{(1+\rho)^{\alpha-\beta}\Gamma(\beta)}{\Gamma(\alpha)\Gamma(1-\alpha+\beta)} z^{1-\beta-\rho}\frac{\mathrm{d}}{\mathrm{dz}}\int_0^z \frac{\xi^{\rho+\alpha-1}f(\xi)d\xi}{(z-\xi)^{\alpha-\beta}}. \tag{17}$$

From (17), we aim to present a new generalized fractional differential operator which is very closed to operator defined by [14] as follows:

Definition 2.3. For $0 \le \rho$, $0 < \alpha \le 1$ and $0 < \beta \le 1$ such that $0 \le \alpha - \beta < 1$, let $f(z)$ be analytic in a simple-connected region containing the origin. The generalized fractional differential operator of two parameters $\mathcal{T}_z^{\alpha,\beta,\rho}$ is defined by:

(18)

$$\mathcal{T}_z^{\alpha,\beta,\rho}f(z) := \frac{(1+\rho)^{\alpha-\beta}\Gamma(\beta)}{\Gamma(\alpha)\Gamma(1-\alpha+\beta)}\left(z^{1-\beta-\rho}\frac{\mathrm{d}}{\mathrm{dz}}\right)\int_0^z \frac{t^{\rho+\alpha-1}f(t)}{(z^{\rho+1}-t^{\rho+1})^{\alpha-\beta}}dt,\ z \in \mathbb{U}.$$

Note that, in the case $\rho = 0$, the equation (18) undeviating reduces to fractional differential operator studied in [12, 13].

Example 2.4. Let $f(z) = z^m$, $m \in \mathbb{R}$ and $|z| < 1$, then

$$\mathcal{T}_z^{\alpha,\beta,\rho}z^m = \frac{(\rho+1)^{\alpha-\beta}\Gamma(\frac{m+\alpha+\rho}{\rho+1})\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\frac{m+\alpha+\rho}{\rho+1}-\alpha+\beta)}z^{(1-\alpha+\beta)\rho+m}. \tag{19}$$

Now, we are ready to assert and demonstrate the following properties for the functions f type of analytic in the open unit disk $\mathbb{U}$.

Theorem 2.5. *(Inverse property) Let $0 < \alpha \le 1$ and $0 < \beta \le 1$ such that $0 \le \alpha - \beta < 1$, then*

$$\left(\mathcal{T}_z^{\alpha,\beta,\rho}\mathcal{L}_z^{\alpha,\beta,\rho}\right)f(z) = f(z), \tag{20}$$

for $0 \le \rho$ and $|z| < 1$.

Proof. By using Fubini's theorem and Dirichlet technique, adding to the direct integration, we have

$$\begin{aligned}
&\left(\mathcal{T}_z^{\alpha,\beta,\rho}\mathcal{L}_z^{\alpha,\beta,\rho}\right)f(z)\\
&= \frac{(\rho+1)z^{1-\beta-\rho}}{\Gamma(\alpha-\beta)\Gamma(1-\alpha+\beta)}\frac{\mathrm{d}}{\mathrm{dz}}\int_0^z (z^{\rho+1}-t^{\rho+1})^{\alpha-\beta}t^\rho \int_s^t \frac{s^{\rho+\beta-1}f(s)dsdt}{(t^{\rho+1}-s^{\rho+1})^{\alpha-\beta}}\\
&= \frac{(\rho+1)z^{1-\beta-\rho}}{\Gamma(\alpha-\beta)\Gamma(1-\alpha+\beta)}\frac{\mathrm{d}}{\mathrm{dz}}\int_0^z s^{\rho+\beta-1}f(s)\int_s^t \frac{(z^{\rho+1}-t^{\rho+1})^{\alpha-\beta}}{(t^{\rho+1}-s^{\rho+1})^{\alpha-\beta}}t^\rho dtds\\
&= \frac{(z^{1-\beta-\rho}}{\Gamma(\alpha-\beta)\Gamma(1-\alpha+\beta)}\frac{\mathrm{d}}{\mathrm{dz}}\int_0^z s^{\rho+\beta-1}f(s)\cdot\Gamma(1-\alpha+\beta)\Gamma(\alpha-\beta).
\end{aligned}$$

It is clear here that the inner integral is evaluated by the change of variable $u = \dfrac{(t^{\rho+1}-s^{\rho+1})}{z^{\rho+1}-s^{\rho+1})}$, and using beat function

$$B(\alpha,\beta) := \int_0^1 u^{\alpha-1}(1-u)^{\beta-1}du = \tfrac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}. \tag{21}$$

Thus supplementing the proof of Theorem 2.5. □

THEOREM 2.6. *For $0 \le \rho$ and $0 < \mu \le 1$, let $0 < \alpha \le 1$ and $0 < \beta \le 1$ such that $0 < \alpha - \beta \le 1$ and $0 < \beta - \mu \le 1$, then*

$$\left(\mathcal{L}_z^{\alpha,\beta,\rho}\mathcal{L}_z^{\beta,\mu,\rho}\right)f(z) = \mathcal{L}_z^{\alpha,\mu,\rho}f(z), \quad |z| < 1.$$

Proof. The proof is clear as in the proof of Theorem 2.5. □

3. Analytical applications

Recently, many authors have been presented various extensions of the fractional operators in the open unit disk $\mathbb{U}$, so here we aim to define a new extended fractional integral operator involving the generalized hypergeometric function type $F_p^{(\omega,\gamma,\kappa,\tau)}$ and introduce some of its properties. For this propose, we apply generalized integral operator type fractional $\mathcal{L}_z^{\alpha,\beta,\rho}$ defined by (13) as follows:

THEOREM 3.1. *For $0 \le \rho$, $0 < \alpha \le 1$ and $0 < \beta \le 1$, such that $0 < \alpha - \beta \le 1$, let $\vartheta_{p,\omega,\gamma,\kappa,\tau}(z)$ be a function defined by*

$$\vartheta_{p,\omega,\gamma,\kappa,\tau}(z) = z\,F_p^{(\omega,\gamma,\kappa,\tau)}(a,b;c;z)$$

$$(\min\{\Re(\kappa), \Re(\tau), \Re(\omega), \Re(\gamma)\} > 0, \Re(c) > \Re(b) > 0, p \ge 0, |z| < 1)$$

then

$$\mathcal{L}_z^{\alpha,\beta,\rho}\vartheta_{p,\omega,\gamma,\kappa,\tau}(z) =(\rho+1)^{\alpha-\beta}z^{(\alpha-\beta)\rho+1} \times \sum_{m=0}^{\infty}\frac{B_p^{(\omega,\gamma,\kappa,\tau)}(b+m,\, c-b)B(\frac{m+\beta+\rho}{\rho+1},\, \alpha-\beta)}{B(b,\, c-b)\,B(\beta,\, \alpha-\beta)}\frac{(a)_m}{m!}z^m. \tag{22}$$

Proof. Let

$$\begin{aligned}\mathcal{L}_z^{\alpha,\beta,\rho}\vartheta_{p,\omega,\gamma,\kappa,\tau}(z) &= \mathcal{L}_z^{\alpha,\beta,\rho}\left(\sum_{m=0}^{\infty}\left(\frac{B_p^{(\omega,\gamma,\kappa,\tau)}(b+m,\, c-b)}{B(b,\, c-b)}\right)(a)_m\frac{z^{m+1}}{m!}\right)\\ &= \sum_{m=0}^{\infty}\frac{B_p^{(\omega,\gamma,\kappa,\tau)}(b+m,\, c-b)}{B(b,\, c-b)}\frac{(a)_m}{m!}\left(\mathcal{L}_z^{\alpha,\beta,\rho}\left[z^{m+1}\right]\right).\end{aligned}$$

Now, by taking advantage of Example 2.2 with m replacing $m+1$, we have

$$\begin{aligned}&\mathcal{L}_z^{\alpha,\beta,\rho}\vartheta_{p,\omega,\gamma,\kappa,\tau}(z)\\ &= \frac{(\rho+1)^{\alpha-\beta}\Gamma(\alpha)}{\Gamma(\beta)}\sum_{m=0}^{\infty}\frac{B_p^{(\omega,\gamma,\kappa,\tau)}(b+m,\, c-b)}{B(b,\, c-b)}\\ &\qquad\times\frac{\Gamma(\frac{m+\beta-1}{\rho+1}+1)}{\Gamma(\frac{m+\beta-1}{\rho+1}+\alpha-\beta+1)}\frac{(a)_m}{m!}z^{(\alpha-\beta)\rho+m+1}\\ &= (\rho+1)^{\alpha-\beta}z^{(\alpha-\beta)\rho+1}\sum_{m=0}^{\infty}\frac{B_p^{(\omega,\gamma,\kappa,\tau)}(b+m,c-b)B(\frac{m+\beta+\rho}{\rho+1},\alpha-\beta)}{B(b,c-b)\,B(\beta,\alpha-\beta)}\frac{(a)_m}{m!}z^m.\end{aligned}$$

$\square$

Remark 3.2. If $\kappa=\tau$ in (22), then we have the following result.

Corollary 3.3. *For $0\le\rho$, $0<\alpha\le 1$ and $0<\beta\le 1$ such that $0<\alpha-\beta\le 1$,*

$$\mathcal{L}_z^{\alpha,\beta,\rho}\vartheta_{p,\omega,\gamma,\tau}(z) =(\rho+1)^{\alpha-\beta}z^{(\alpha-\beta)\rho+1} \times \sum_{m=0}^{\infty}\frac{B_p^{(\omega,\gamma,\tau)}(b+m,\, c-b)B(\frac{m+\beta+\rho}{\rho+1},\, \alpha-\beta)}{B(b,\, c-b)\,B(\beta,\, \alpha-\beta)}\frac{(a)_m}{m!}z^m. \tag{23}$$

REMARK 3.4. Letting $\tau = 1$ in equation (23), we have the following consequence.

COROLLARY 3.5. *For* $0 \leq \rho$, $0 < \alpha \leq 1$ *and* $0 < \beta \leq 1$ *such that* $0 \leq \alpha - \beta < 1$,

$$\mathcal{L}_z^{\alpha,\beta,\rho}\vartheta_{p,\omega,\gamma}(z) =(\rho+1)^{\alpha-\beta}z^{(\alpha-\beta)\rho+1} \times \sum_{m=0}^{\infty}\frac{B_p^{(\omega,\gamma)}(b+m,\, c-b)B(\frac{m+\beta+\rho}{\rho+1},\, \alpha-\beta)}{B(b,\, c-b)\,B(\beta,\, \alpha-\beta)}\frac{(a)_m}{m!}z^m. \tag{24}$$

REMARK 3.6. Obviously, upon setting $\omega = \gamma$ in equation (24), we get the following outcome.

COROLLARY 3.7. *For* $0 \leq \rho$, $0 < \alpha \leq 1$, *and* $0 < \beta \leq 1$ *such that* $0 \leq \alpha - \beta < 1$,

$$\mathcal{L}_z^{\alpha,\beta,\rho}\vartheta_{p}(z) =(\rho+1)^{\alpha-\beta}z^{(\alpha-\beta)\rho+1} \times \sum_{m=0}^{\infty}\frac{B_p(b+m,\, c-b)B(\frac{m+\beta+\rho}{\rho+1},\, \alpha-\beta)}{B(b,\, c-b)\,B(\beta,\, \alpha-\beta)}\frac{(a)_m}{m!}z^m. \tag{25}$$

REMARK 3.8. In the case of $p = 0$ in the equation (25), we have the following conclusion corollary for Theorem 3.1.

COROLLARY 3.9. *For* $0 \leq \rho$, $0 < \alpha \leq 1$ *and* $0 < \beta \leq 1$ *such that* $0 \leq \alpha - \beta < 1$,

$$\mathcal{L}_z^{\alpha,\beta,\rho}\vartheta_{0}(z) =(\rho+1)^{\alpha-\beta}z^{(\alpha-\beta)\rho+1} \times \sum_{m=0}^{\infty}\frac{(a)_m\,(b)_m\,B(\frac{m+\beta+\rho}{\rho+1},\, \alpha-\beta)}{(c)_m\,B(\beta,\, \alpha-\beta)}\frac{z^m}{m!}. \tag{26}$$

It is clear that the results become comparatively more substantial from the applications in viewpoint this happens whenever a generalized Gauss hypergeometric function reduced to the Gauss hypergeometric function and its special cases.

4. Univalency applications

Let $\mathcal{A}$ be the classes of all analytic functions f given by the following form

$$f(z) = z + \sum_{m=2}^{\infty} a_m z^m \tag{27}$$

and defined on the unit disk $\mathbb{U} := \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}$. For two analytic functions $f(z)$ given by (27) and $g(z) = z + \sum_{m=2}^{\infty} b_m z^m$ in $\mathcal{A}$, the binary operation $(*)$ denoted by the convolution (or Hadamard product) of two analytic functions and defined by

$$(f * g)(z) = z + \sum_{m=2}^{\infty} a_m b_m z^m = (g * f)(z) \quad (z \in \mathbb{U}).$$

LEMMA 4.1. [15] *For two analytic functions f and g in $\mathcal{A}$, we obtain*

$$z(g * f)'(z) := g(z) * zf'(z) \Leftrightarrow (g * f)'(z) = \frac{g(z)}{z} * f'(z).$$

In this section, first, we proceed to define extension operator of fractional order in terms of power series by using operator given in (13). For $0 \leq \alpha < 1$ and $0 \leq \beta < 1$, let $\mathfrak{L}^{\alpha,\beta,\rho} : \mathcal{A} \to \mathcal{A}$ be defined

$$\begin{aligned} \mathfrak{L}^{\alpha,\beta,\rho} f(z) :&= \frac{\Gamma(\beta)\Gamma(\frac{1+\beta+\rho}{\rho+1} + \alpha - \beta)}{(\rho+1)^{\alpha-\beta}\Gamma(\frac{1+\beta+\rho}{\rho+1})\Gamma(\alpha)} z^{(\beta-\alpha)\rho} \mathcal{L}_z^{\alpha,\beta,\rho} f(z) \\ &= z + \sum_{m=2}^{\infty} \frac{\Gamma(\frac{1+\beta+\rho}{\rho+1} + \alpha - \beta)\Gamma(\frac{m+\beta+\rho}{\rho+1})}{\Gamma(\frac{1+\beta+\rho}{\rho+1})\Gamma(\frac{m+\beta+\rho}{\rho+1} + \alpha - \beta)} a_m z^m \end{aligned} \tag{28}$$

for $\rho \geq 0$ and $z \in \mathbb{U}$. Additionally in term of the Fox-Wright function we obtain the following fractional integral normalized operator:

$$\begin{aligned} \mathfrak{L}^{\alpha,\beta,\rho} f(z) &= z + \sum_{m=2}^{\infty} \frac{\Gamma(\frac{1+\beta+\rho}{\rho+1} + \alpha - \beta)\Gamma(\frac{m+\beta+\rho}{\rho+1})}{\Gamma(\frac{1+\beta+\rho}{\rho+1})\Gamma(\frac{m+\beta+\rho}{\rho+1} + \alpha - \beta)} a_m z^m \\ &= \frac{\Gamma(\frac{1+\beta+\rho}{\rho+1} + \alpha - \beta)}{\Gamma(\frac{1+\beta+\rho}{\rho+1})} \sum_{m=0}^{\infty} \frac{\Gamma(m+1)\Gamma(\frac{m+\beta}{\rho+1} + 1)}{\Gamma(\frac{m+\beta}{\rho+1} + 1 + \alpha - \beta)} \frac{a_{m+1}}{m!} z^{m+1} \\ &= \frac{\Gamma(\frac{1+\beta+\rho}{\rho+1} + \alpha - \beta)}{\Gamma(\frac{1+\beta+\rho}{\rho+1})} z\, {}_2\Psi_1 \left[\begin{matrix} (1,1), \, (1 + \frac{\beta}{\rho+1}, \frac{1}{1+\rho}) \\ (1 + \alpha - \beta + \frac{\beta}{\rho+1}, \frac{1}{1+\rho}) \end{matrix} ; z \right] * f(z) \\ &= \left(\frac{\Gamma(\frac{1+\beta+\rho}{\rho+1} + \alpha - \beta)}{\Gamma(\frac{1+\beta+\rho}{\rho+1})} \right) z\, {}_2\Psi_1(z) * f(z) \end{aligned} \tag{29}$$

where ${}_2\Psi_1$ is the Fox-Wright generalized function defined by [16]:

$$ {}_p\Psi_q(z) = {}_p\Psi_q\left[\begin{array}{c} (a_i, A_i)_{1,p} \\ (b_j, B_j)_{1,q} \end{array} ; z \right] = \sum_{m=0}^{\infty} \frac{\Pi_{i=1}^{p}\Gamma(a_i + mA_i)}{\Pi_{j=1}^{q}\Gamma(b_j + mB_j)(1)_m} z^m, $$

and a_i, b_j are two parameters in complex plane $\mathbb{C}$. $A_i > 0, B_j > 0$ for all $j = 1, \ldots, q$, $i = 1, \ldots, p$ and $|z| < 1$, satisfying

$$ 0 \leq 1 + \sum_{j=1}^{q} B_j - \sum_{i=1}^{p} A_i. $$

As well as, we present a new normalized fractional differential operator in the open unit disk as follows: for $0 \leq \alpha < 1$ and $0 \leq \beta < 1$, let $\mathfrak{T}^{\alpha,\beta,\rho} : \mathcal{A} \to \mathcal{A}$ be defined

$$ \mathfrak{T}^{\alpha,\beta,\rho} f(z) := z + \sum_{m=2}^{\infty} \frac{\Gamma(\frac{\alpha+\rho+1}{\rho+1} - \alpha + \beta)\Gamma(\frac{m+\alpha+\rho}{\rho+1})}{\Gamma(\frac{m+\alpha+\rho}{\rho+1} - \alpha + \beta)\Gamma(\frac{\alpha+\rho+1}{\rho+1})} a_m z^m \tag{30} $$

where $\rho \geq 0$ and $z \in \mathbb{U}$. Moreover, in term of the Fox-Wright function we obtain the following result

$$ \mathfrak{T}^{\alpha,\beta,\rho} f(z) := \frac{\Gamma(\frac{\alpha+\rho+1}{\rho+1} - \alpha + \beta)}{\Gamma(\frac{\alpha+\rho+1}{\rho+1})} z\, {}_2\Psi_1 * f(z). \tag{31} $$

where ${}_2\Psi_1 := {}_2\Psi_1\left[\begin{array}{c} (1,1),\ (1 + \frac{\alpha}{\rho+1}, \frac{1}{1+\rho}) \\ (1 - \alpha + \beta + \frac{\alpha}{\rho+1}, \frac{1}{1+\rho}) \end{array} ; z \right]$.

Ultimately study of the univalence properties, we determine the upper bounded of the normalized operators given by (28) and (30) respectively to be bounded in the Bloch space $\mathcal{B}$ of those analytic functions $f \in \mathcal{A}$ on the unit disk $\mathbb{U}$ in the complex plane $\mathbb{C}$, with the norm defined by

$$ ||f||_{\mathcal{B}} = \sup_{z \in \mathbb{U}} \big\{ (1 - |z|^2)|f'(z)| \big\}. $$

Theorem 4.2. *Let $0 \leq \rho,\ 0 \leq \alpha < 1,\ 0 \leq \beta < 1$, and $f \in \mathcal{A}$. Then*

$$ ||\mathfrak{L}^{\alpha,\beta,\rho} f(z)||_{\mathcal{B}} \leq M\, ||f||_{\mathcal{B}}, $$

where $\left| \frac{\Gamma(\frac{1+\beta+\rho}{\rho+1} + \alpha - \beta)}{\Gamma(\frac{1+\beta+\rho}{\rho+1})} {}_2\Psi_1(z) \right| \leq M < \infty$ *and* ${}_2\Psi_1(z)$ *given in* (29).

Proof. Assume that $f(z) \in \mathcal{A}$. It is easy to see that if $q \in \mathcal{B}$, then

$$\sup_{|z|<1} \left\{ (1-|z|^2)|q(z)| \right\} \le M < \infty$$

and let $\mathfrak{L}^{\alpha,\beta,\rho} \in \mathcal{B}$ and by applying (29) we have

$$\begin{aligned} ||\mathfrak{L}^{\alpha,\beta,\rho} f(z)||_{\mathcal{B}} &= \sup_{|z|<1} \left\{ (1-|z|^2) \left| \left(\mathfrak{L}^{\alpha,\beta,\rho} f(z) \right)' \right| \right\} \\ &\le \sup_{|z|<1} \left\{ (1-|z|^2) \left| \left(\frac{\Gamma(\frac{1+\beta+\rho}{\rho+1} + \alpha - \beta)}{\Gamma(\frac{1+\beta+\rho}{\rho+1})} z \, {}_2\Psi_1(z) * f(z) \right)' \right| \right\} \end{aligned}$$

and by employing Lemma 4.1, we obtain

$$\begin{aligned} ||\mathfrak{L}^{\alpha,\beta,\rho} f(z)||_{\mathcal{B}} &= \sup_{|z|<1} \left\{ (1-|z|^2) \left| \frac{\Gamma(\frac{1+\beta+\rho}{\rho+1} + \alpha - \beta)}{\Gamma(\frac{1+\beta+\rho}{\rho+1})} {}_2\Psi_1(z) * f'(z) \right| \right\} \\ &\le M \, ||f||_{\mathcal{B}}. \end{aligned}$$

□

THEOREM 4.3. *Let* $0 \le \rho,\ 0 \le \alpha < 1,\ 0 \le \beta < 1,$ *and* $f \in \mathcal{A}$. *Then*

$$||\mathfrak{T}^{\alpha,\beta,\rho} f(z)||_{\mathcal{B}} \le M \, ||f||_{\mathcal{B}}$$

where $\left| \frac{\Gamma(\frac{1+\alpha+\rho}{\rho+1} + \alpha - \beta)}{\Gamma(\frac{1+\alpha+\rho}{\rho+1})} {}_2\Psi_1(z) \right| \le M < \infty$ *and* ${}_2\Psi_1(z)$ *is defined in* (31).

5. Conclusion

In spaces of analytic functions, we defined new type of generalization of fractional integral and differential operators of two parameters by considering the generalized Srivastava-Owa operators. Definitions new normalized fractional operators are also derived. Some analytic applications with generalized Gauss hypergeometric function are introduced. In addition, the upper bounded of generalized fractional normalized operators on Bloch space are determined.

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