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# IHARA ZETA FUNCTION OF DUMBBELL GRAPHS

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Abstract. We study the Ihara zeta function of the dumbbell graph $D_{1,1,n}$ of type $(1,1,n)$ and $D_{1,2,n}$ of type $(1,2,n)$. Explicit formulas of the zeta functions of the graphs, their radius of convergence, and the connection with the number of closed cycles are given.

## 1. Introduction

Let $G$ be a finite connected undirected graph with no degree 1 vertices. Let $VG$ and $EG$ be the set of vertices and the set of edges of $G$, respectively. In addition, we denote by $E^{\pm}G$ the set of all oriented edges of $G$. Thus, we have $|E^{\pm}G| = 2|EG|$.

Let $P = (e_1, e_2, \dots, e_{l(P)-1}, e_{l(P)})$ be a *primitive closed cycle* without backtracking. That is, $o(e_1) = t(e_{l(P)})$, $e_{i+1} \neq e_i^{-1}$ (mod $l(P)$) for all $i$ and $P \neq D^m$ for any integer $m \geqslant 2$ and a path $D$ in $A$. If a closed cycle $Q$ is obtained by changing the cyclic order of $P$, then we say $P$ and $Q$ are euqivalent. A *prime* $[P]$ in $G$ is an equivalence class of primitive closed cylce without backtracking.

The *Ihara zeta function* of $G$ is defined at $u \in \mathbb{C}$, for which $|u|$ is sufficiently small, by

$$Z_G(u) = \prod_{[P]} (1 - u^{l(P)})^{-1}$$

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where $[P]$ runs over the primes of $G$.

The Ihara determinant formula [1] gives that $Z_G(u)$ is a rational function, given by

$$Z_G(u) = \frac{1}{(1-u^2)^{\chi(X)-1}\det(I - A(u) + Bu^2)}$$

where $\chi = |EG| - |VG| + 1$, $A$ is the vertex adjacency matrix of $G$ and $B$ is diagonal matrix whose $j$-th diagonal entry is $\deg(v_j) - 1$. Let $R_G$ be the radius of convergence of $Z_G(u)$.

We denote by $D_{a,b,n}$ the *dumbbell* graph of type $(a, b, n)$ defined as a graph consisting of two vertex-disjoint cycles $C_a$, $C_b$ and a path $P_n$ $(a, b \geqslant 1, n \geqslant 2)$ joining them having only its end-vertices in common with the two cycles. It has $a+b+n-2$ number of vertices and $a+b+n-1$ number of edges. Below is the figure of the graph $D_{a,b,n}$.

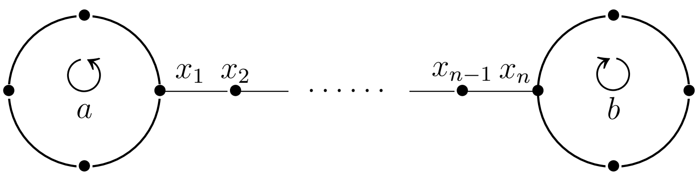


FIGURE 1. Dumbbell graph $D_{a,b,n}$

In this article, we study the Ihara-zeta function of the graph $D_{1,1,n}$ and $D_{1,2,n}$. If $G = D_{1,1,n}$, then we have $|VG| = n$ and $|EG| = n + 1$. If $G = D_{1,2,n}$, then we have $|VG| = n + 1$ and $|EG| = n + 2$.

The following two theorems are main results of the paper.

THEOREM 1.1. *Let $G$ be the dumbbell graph $D_{1,1,n}$ of type $(1, 1, n)$. If $n$ is odd, then*

$$Z_G(u) = \frac{1}{(1-u^2)(u-1)(2u^{n-1} - 2u^{n-2} + - \cdots - 2u + 1)(2u^n + u - 1)}$$

*and if $n$ is even, then*

$$Z_G(u) = \frac{1}{(1-u^2)(u-1)(2u^{n-1} - 2u^{n-2} + - \cdots + 2u - 1)(2u^n - u + 1)}.$$

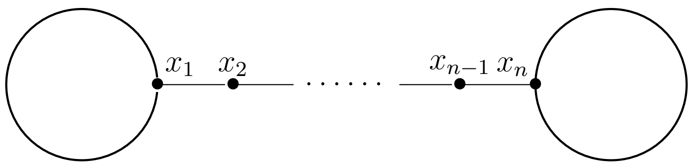


FIGURE 2. Dumbbell graph $D_{1,1,n}$

THEOREM 1.2. *Let $G$ be the dumbbell graph $D_{1,2,n}$ of type $(1, 2, n)$. Then, we have*

$$Z_G(u) = \frac{1}{(1-u^2)(u-1)(4u^{2n-1} - u^3 + u^2 + u - 1)}.$$

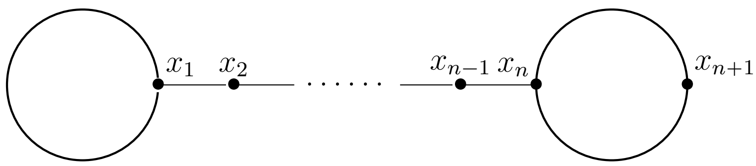


FIGURE 3. Dumbbell graph $D_{1,2,n}$

From the formula of the zeta function $Z_G(u)$ and the Perron-Frobenius theory, we get the irreducible polynomial of the radius of convergence of $Z_G(u)$.

COROLLARY 1.3. *Let $G$ be the dumbbell graph $D_{1,1,n}$. The radius $R_G$ of convergence of the rational function $Z_G(u)$ the unique real root of $P(u)$ where $P(u)$ is given by*

$$P(u) = \begin{cases} 2u^n + u - 1, & n \text{ is odd} \\ 2u^{n-1} - 2u^{n-2} + - \cdots - 2u^2 + 2u - 1, & n \text{ is even.} \end{cases}$$

Finally, the prime geodesic theorem gives the following.

COROLLARY 1.4. *Let $G$ be the dumbbell graph $D_{1,1,n}$. If we denote by $\pi(m)$ the number of prime cycles of length $m$, then we have*

$$\lim_{m\to\infty} \frac{m\pi(m)}{\lambda_G^m} = 1.$$

*Here, $\lambda_G$ is the unique real root of $Q(u)$ where $Q(u)$ is given by*

$$Q(u) = \begin{cases} u^n - u^{n-1} - 2, & n \text{ is odd} \\ u^{n-1} - 2u^{n-2} + - \cdots - 2u^2 + 2u - 2, & n \text{ is even.} \end{cases}$$

The analogous results for $D_{1,2,n}$ hold.

## 2. Proof of the results

In this section, we prove Theorem 1.1 and Theorem 1.2. First, we consider the case when the graph $G$ is $D_{1,1,n}$.

*Proof of Theorem 1.1.* By the Ihara determinant formula, we have

$$Z_G(u)^{-1} = (1-u^2)\det(B)$$

where $B$ is an $n \times n$ tri-diagonal matrix given by

$$B = \begin{pmatrix} 1-2u+2u^2 & -u & & & \\ -u & 1+u^2 & -u & & \\ & -u & \cdots & -u & \\ & & -u & 1+u^2 & -u \\ & & & -u & 1-2u+2u^2 \end{pmatrix}.$$

Let us denote by $f(k)$ the determinant of the $k \times k$ matrix given by

$$\begin{pmatrix} 1+u^2 & -u & & \\ -u & 1+u^2 & -u & \\ & -u & \cdots & -u \\ & & -u & 1+u^2 \end{pmatrix}.$$

Then $f(k) = (1+u^2)f(k-1) - u^2 f(k-2)$. Since $f(1) = 1+u^2$ and $f(2) = 1+u^2+u^4$, it follows that $f(k) = 1+u^2+\cdots+u^{2k}$. Hence, this yields

$$\begin{aligned} \det(B) =&(1-2u+2u^2)[(1-2u+2u^2)f(n-2) - u^2 f(n-3)] \\ &- u^2[(1-2u+2u^2)f(n-3) - u^2 f(n-4)] \\ =&(1-2u+2u^2)^2 f(n-2) - 2(1-2u+2u^2)u^2 f(n-3) + u^4 f(n-4) \\ =&4u^{2n} - 8u^{2n-1} + 8u^{2n-2} - + \cdots - 8u^3 + 7u^2 - 4u + 1 \\ =&(u-1)(2u^{n-1} - 2u^{n-2} + 2u^{n-3} - + \cdots \pm 1)(2u^n \pm u \mp 1). \end{aligned}$$

Therefore, we have

$$Z_G(u)^{-1} = (1-u^2)(u-1)(2u^{n-1} - 2u^{n-2} + \cdots \pm 1)(2u^n \pm u \mp 1)$$

which completes the proof of the Theorem 1.1. □

*Proof of Theorem 1.2.* Similarly, if $G$ is the graph $D_{1,2,n}$, then the Ihara determinant formula implies that

$$Z_G(u)^{-1} = (1-u^2)\det(C)$$

where

$$C = \begin{pmatrix} 1-2u+2u^2 & -u & & & \\ -u & 1+u^2 & -u & & \\ & -u & \cdots & -u & \\ & & -u & 1+2u^2 & -2u \\ & & & -2u & 1+u^2 \end{pmatrix}.$$

Let us denote by $g(k)$ the determinant of the $k \times k$ matrix given by

$$\begin{pmatrix} 1+u^2 & -u & & \\ -u & 1+u^2 & -u & \\ & -u & \cdots & -u \\ & & -u & 1+2u^2 \end{pmatrix}.$$

Then $g(k) = (1+u^2)g(k-1) - u^2 g(k-2)$. Since $g(1) = 1+2u^2$ and $g(2) = 1+2u^2+2u^4$, it follows that $g(k) = 1+2u^2+\cdots+2u^{2k}$. This yields

$$\begin{aligned} \det(C) =& (1-2u+2u^2)[(1+u^2)g(n-2) - 4u^2 f(n-3)] \\ & - u^2[(1+u^2)g(n-3) - 4u^2 f(n-4)] \\ =& 4u^{2n} - 4u^{2n-1} - u^4 + 2u^3 - 2u + 1 \\ =& (u-1)(4u^{2n-1} - u^3 + u^2 + u - 1). \end{aligned}$$

Thus, we have

$$Z_G(u)^{-1} = (1-u^2)(u-1)(4u^{2n-1} - u^3 + u^2 + u - 1)$$

which completes the proof of the Theorem 1.2. □

Let us now prove Corollary 1.3 and 1.4. Let $L(G)$ be the vertex adjacency matrix of the oriented line graph of $G$ (see Section 3 of [4]). According to the determinant formula for the edge zeta function (Theorem 3.3 of [2]), we also have

$$Z_G(u) = \frac{1}{\det(I - uL(G))}.$$

Since $G$ is connected, it follows that $L(G)$ is a non-negative irreducible matrix. The Perron-Frobenius theorem of the non-negative matrices

(Section 4 of [4]) implies that the Perron-Frobenius eigenvalue $\lambda_G$ of $L(G)$ is simple and real. It follows that $0 < R_G < 1$ and $\lambda_G = R_G^{-1}$ is an algebraic integer. This gives us that $R_G$ is the unique real root of $P(u)$ where

$$P(u) = \begin{cases} 2u^n + u - 1, & n \text{ is odd} \\ 2u^{n-1} - 2u^{n-2} + - \cdots - 2u^2 + 2u - 1, & n \text{ is even.} \end{cases}$$

Corollary 1.4 directly follows from the prime geodesic theorem in graphs (Theorem 2.10 of [2]) since $\Delta_G = 1$.

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