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PSEUDO-METRIC ON KU-ALGEBRAS

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Abstract. In this paper we have introduced the concept of pseudo-metric which we induced from a pseudo-valuation on KU-algebras and investigated the relationship between pseudo-valuations and ideals of KU-algebras. Conditions for a real-valued function to be a pseudo-valuation on KU-algebras are provided.

1. Introduction

Pseudo-metric induce by pseudo-valuations on Hilbert algebras was initially introduced by Busneag [2]. Further Busneag [3] proved many results on extensions of pseudo-valuations. Pseudo-valuations in residuated lattices was introduced by Busneag [4] where many theorems based on pseudo-valuations in lattice terms and their extension for residuated lattices to pseudo-valuation from valuations has been shown using the model of Hilbert algebras [3].
Logical algebras have become the keen interest for researchers in recent years and intensively studied under the influence of different mathematical concepts. Doh and Kang [5] introduced the concept of pseudo-valuation on BCK/BCI algebras and studied results based on them. Ghorbani [6] defined congruence relations and gave quotient structure

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of BCI-algebras based on pseudo-valuation. Zhan and Jun [12] studied pseudo valuation on R_0-algebras. Based on the concept of pseudo-valuation in R_0-algebras, Yang and Xin [10] characterized pseudo pre-valuations on EQ-algebras.
KU-algebras were introduced by Prabpayak and Leerawat [8] in 2009. Further Prabpayak and Leerawat [9] studied homomorphisms and related properties with KU-algebras. Naveed et. al [11] introduced the concept of cubic KU-ideals of KU-algebras. Recently Ansari and Koam [1] gave the concept of roughness in KU-Algebras.
We define a pseudo-valuations on KU-algebras using the model of Busneag and introduce a pseudo-metric on KU-algebras. We also prove that the binary operation defined on KU-algebras is uniformly continuous under the induced pseudo-metric.

2. Preliminaries

In this section, we shall consider concepts based on KU-algebras, KU-ideals and other important terminologies with examples and some related results.

Definition 1. [8] By a KU-algebra we mean an algebra $(X, \circ, 1)$ of type $(2, 0)$ with a single binary operation $\circ$ that satisfies the following identities: for any $x, y, z \in X$,
(ku1) $(x \circ y) \circ [(y \circ z) \circ (x \circ z)] = 1$,
(ku2) $x \circ 1 = 1$,
(ku3) $1 \circ x = x$,
(ku4) $x \circ y = y \circ x = 1$ implies $x = y$.

In what follows, let $(X, \circ, 1)$ denote a KU-algebra unless otherwise specified. For brevity we also call X a KU-algebra. In X we can define a binary relation $\leq$ by : $x \leq y$ if and only if $x \circ y = 1$.

Lemma 1. [8] $(X, \circ, 1)$ *is a KU-algebra if and only if it satisfies:*
(ku5) $x \circ y \leq (y \circ z) \circ (x \circ z)$,
(ku6) $x \leq 1$,
(ku7) $x \leq y$, $y \leq x$ *implies* $x = y$,

Lemma 2. *In a KU-algebra, the following identities are true [7]:*
(1) $z \circ z = 1$,
(2) $z \circ (x \circ z) = 1$,

(3) $x \leq y$ *imply* $y \circ z \leq x \circ z$,
(4) $z \circ (y \circ x) = y \circ (z \circ x)$,
(5) $y \circ [(y \circ x) \circ x] = 1$, *for all* $x, y, z \in X$,

Example 1. [7] Let $X = \{1, 2, 3, 4, 5\}$ in which $\circ$ is defined by the following table

$\circ$	1	2	3	4	5
1	1	2	3	4	5
2	1	1	3	4	5
3	1	2	1	4	4
4	1	1	3	1	3
5	1	1	1	1	1

It is easy to see that X is a KU-algebra.

Definition 2. [8] A non-empty subset A of a KU-algebra X is called a KU-ideal of X if it satisfies the following conditions:
(1) $1 \in A$,
(2) $x \circ (y \circ z) \in A$, $y \in A$ imply $x \circ z \in A$, for all $x, y, z \in X$.

Example 2. [1] Let $X = \{1, 2, 3, 4, 5, 6\}$ in which $\circ$ is defined by the following table:

$\circ$	1	2	3	4	5	6
1	1	2	3	4	5	6
2	1	1	3	3	5	6
3	1	1	1	2	5	6
4	1	1	1	1	5	6
5	1	1	1	2	1	6
6	1	1	2	1	1	1

Clearly $(X, \circ, 1)$ is a KU-algebra. It is easy to show that $A = \{1, 2\}$ and $B = \{1, 2, 3, 4, 5\}$ are KU-ideals of X.

3. Pseudo-valuations on KU-algebras

Definition 3. A real-valued function ζ on a KU-algebra X is called a pseudo-valuation on X if it satisfies the following two conditions:
(1) $\zeta(1) = 0$
(2) $\zeta(x \circ z) \leq \zeta(x \circ (y \circ z)) + \zeta(y)$ $\forall x, y, z \in X$
A pseudo-valuation ζ on a KU-algebra X satisfying the following condition:

$\zeta(x) = 0 \Rightarrow x = 1\ \forall x \in X$ is called a valuation on X.

Example 3. Let $X = \{1, 2, 3, 4\}$ be a set with operation $\circ$. A table for such X is defined by following table

$\circ$	1	2	3	4
1	1	2	3	4
2	1	1	3	4
3	1	1	1	1
4	1	2	3	1

Here X is a KU-algebra. We find that a real valued function defined on X by

$\zeta(1) = 0$, $\zeta(2) = 1$, $\zeta(3) = \zeta(4) = 3$, is a pseudo-valuation on X.

Proposition 1. *Let ζ be a pseudo-valuation on a KU-algebra X. Then we have*

(1) $x \leq y \Rightarrow \zeta(y) \leq \zeta(x)$.
(2) $\zeta(x \circ y) \leq \zeta(y)\ \forall x, y \in X$.
(3) $\zeta((x \circ (y \circ z)) \circ z) \leq \zeta(x) + \zeta(y)\ \forall x, y, z \in X$.

Proof. (1) Let $x, y \in X$ be such that $x \leq y$. Now choosing $x = 1$, $y = x$, $z = y$, in Definition 3(1),(2) and using (ku3) we get
$\zeta(y) = \zeta(1 \circ y) \leq \zeta(1 \circ (x \circ y)) + \zeta(x) = \zeta(1 \circ 1) + \zeta(x) = \zeta(1) + \zeta(x) = \zeta(x)$.

(2) If we choose $z = y$ in Definition 3(2), then we get $\zeta(x \circ y) \leq \zeta(x \circ (y \circ y)) + \zeta(y) = \zeta(x \circ 1) + \zeta(y) = \zeta(1) + \zeta(y) = \zeta(y)\ \forall x, y \in X$.

(3) If we choose $x = x \circ (y \circ z)$ in Definition 3(2) then we get

$$\zeta((x \circ (y \circ z)) \circ z) \leq \zeta((x \circ (y \circ z)) \circ (y \circ z)) + \zeta(y) \tag{3.1}$$

Now using the relation $\leq$ and Lemma 2 (5), we get $x \leq (x \circ (y \circ z)) \circ (y \circ z)$. By Proposition 1, it follows that $\zeta((x \circ (y \circ z)) \circ (y \circ z)) \leq \zeta(x)$ using this relation in Equation 3.1, we get $\zeta((x \circ (y \circ z)) \circ z) \leq \zeta(x) + \zeta(y)\ \forall x, y, z \in X$. □

Corollary 1. *Every pseudo-valuation ζ on a KU-algebra X satisfies the following inequality $\zeta(x) \geq 0\ \forall x \in X$.*

Proposition 2. *If ζ is a pseudo-valuation on a KU-algebra X, then we have*

$\zeta((x \circ y) \circ y) \leq \zeta(x)\ \forall x, y \in X$.

Proof. Choosing $y = 1$ and $z = y$ in Proposition 1, using (ku3) and Definition 3(1) we get that
$\zeta((x \circ y) \circ y) = \zeta((x \circ (1 \circ y)) \circ y) \leq \zeta(x) + \zeta(1) = \zeta(x)\ \forall x, y \in X.$ □

The following theorem provides conditions for a real valued function on a KU-algebra X to be a pseudo-valuation on X.

Theorem 1. *Let ζ be a real valued function on a KU-algebra X satisfying the following conditions.*
(1) If $\zeta(a) \leq \zeta(x)\ \forall x \in X$, then $\zeta(a) = 1$.
(2) $\zeta(x \circ y) \leq \zeta(y)\ \forall x, y \in X$.
(3) $\zeta((x \circ (y \circ z)) \circ z) \leq \zeta(x) + \zeta(y)$.
Then ζ is a pseudo-valuation on X

Proof. From Lemma 2 (1) and given condition (2), we have $\zeta(1) = \zeta(x \circ x) \leq \zeta(x)\ \forall x \in X$ and hence $\zeta(1) = 0$, using given condition (1). Now, from (ku3), Lemma 2 (1) and given condition (3), we get $\zeta(y) = \zeta(1 \circ y) = \zeta(((x \circ y) \circ (x \circ y)) \circ y) \leq \zeta(x \circ y) + \zeta(x)\ \forall x, y \in X$. It follows from Lemma 2 (4) that $\zeta(x \circ z) \leq \zeta(y \circ (x \circ z)) + \zeta(y) = \zeta(x \circ (y \circ z)) + \zeta(y)$ $\forall x, y, z \in X$. Therefore ζ is a pseudo-valuation on X. □

Corollary 2. *Let ζ be a real-valued function on a KU-algebra X satisfying the following conditions:*
(1) $\zeta(1) = 0$
(2) $\zeta(x \circ y) \leq \zeta(y)$, $\forall x, y \in X$).
(3) $\zeta((x \circ (y \circ z) \circ z) \leq \zeta(x) + \zeta(y))$, $\forall x, y, z \in X$.
Then ζ is a pseudo-valuation on X.

Theorem 2. *If ζ is a pseudo-valuation on a KU-algebra X, then $\zeta(y) \leq \zeta(x \circ y) + \zeta(x))$, $\forall x, y \in X$.*

Proof. Let $m = (x \circ y) \circ y$ for any $x, y \in X$, and $n = x \circ y$.
Then $y = 1 \circ y = (((x \circ y) \circ y) \circ ((x \circ y) \circ y)) \circ y = (m \circ (n \circ y)) \circ y$. It follows from Theorem 2, Propositions 1 and Propositions 2 that $\zeta(y) = \zeta((m \circ (n \circ y)) \circ y) \leq \zeta(m) + \zeta(n) = \zeta((x \circ y) \circ y) + \zeta(x \circ y) \leq \zeta(x) + \zeta(x \circ y)$. This completes the proof. □

Theorem 3. *Let ζ be a real-valued function on a KU-algebra X satisfying the following conditions.*
(1) $\zeta(1) = 0$
(2) $\zeta(y) \leq \zeta(x \circ y) + \zeta(x)$, $\forall x, y \in X$.
Then ζ is a pseudo-valuation on X.

Proof. By Lemma 2 (4), Lemma 2 (5) and given condition (2), we have

$\zeta[(b \circ (a \circ x) \circ x)] \leq \zeta[b \circ ((b \circ (a \circ x)) \circ x] + \zeta(b)$ (by given condition (2))

$\leq \zeta[(b \circ (a \circ x)) \circ (b \circ x)] + \zeta(b)$ (by Lemma 2 (4))

$= \zeta[(a \circ (b \circ x)) \circ (b \circ x))] + \zeta(b)$ (by Lemma 2 (4)).

$= \zeta[a \circ [(a \circ (b \circ x)) \circ (b \circ x)]] + \zeta(a) + \zeta(b)$ (by given condition (2))

$= \zeta(1) + \zeta(a) + \zeta(b)$ (by Lemma 2(5))

$= \zeta(a) + \zeta(b)$.

Also $\zeta(x \circ y) \leq \zeta(y)$ by Lemma 2(2) and Proposition 1(1). Using Corollary 2 we get that ζ is a pseudo-valuation on X. □

PROPOSITION 3. *If ζ is a pseudo-valuation on a KU-algebra X, then*

$$a \leq b \circ x \Rightarrow \zeta(x) \leq \zeta(a) + \zeta(b)\ \forall a, b, x \in X. \tag{3.2}$$

Proof. Suppose that $a, b, x \in X$ such that $a \leq b \circ x$. Then by Proposition 1 (3) and Theorem 2, we have that

$\zeta(x) \leq \zeta((a \circ (b \circ x)) \circ x) + \zeta(a \circ (b \circ x)) = \zeta((a \circ (b \circ x)) \circ x) + \zeta(1) = \zeta((a \circ (b \circ x)) \circ x)$

$\leq \zeta(a) + \zeta(b)$. □

THEOREM 4. *Let ζ be a real-valued function on a KU-algebra X. If ζ satisfies $\zeta(1) = 0$ and condition (3.2), then ζ is a pseudo-valuation on X.*

Proof. From Lemma 2 (5), we have $a \circ ((a \circ x) \circ x) = 1$, which implies from $x \leq y \iff x \circ y = 1$ that $a \leq (a \circ x) \circ x$, $\forall a, x \in X$. Thus it follows from Proposition 3 that $\zeta(x) \leq \zeta(a \circ x) + \zeta(a)$, $\forall a, x \in X$. Hence from Theorem 3, we conclude that ζ is a pseudo-valuation on X. □

PROPOSITION 4. *Suppose that X is a KU-algebra. Then every pseudo-valuation ζ on X satisfies the following inequality:*

$\zeta(x \circ z) \leq \zeta(x \circ y) + \zeta(y \circ z)$, $\forall x, y, z \in X$.

Proof. It follows from (ku1) and Theorem 4. □

THEOREM 5. *If ζ is a pseudo-valuation on a KU-algebra X, then the set $I := \{x \in X | \zeta(x) = 0\}$ is an ideal of X.*

Proof. We have $\zeta(1) = 0$ and hence $1 \in I$. Next $x, y, z \in X$ be such that $y \in I$ and $x \circ (y \circ z) \in I$. Then $\zeta(y) = 0$ and $\zeta(x \circ (y \circ z)) = 0$. By Definition 3(2) we get that $\zeta(x \circ z) \leq \zeta(x \circ (y \circ z)) + \zeta(y) = 0$ so that $\zeta(x \circ z) = 0$. Hence $x \circ z \in I$, and therefore I is an ideal of X. □

EXAMPLE 4. Let $X = \{1, 2, 3, 4, 5, 6\}$ in which $\circ$ is defined by the following table:

$\circ$	1	2	3	4	5	6
1	1	2	3	4	5	6
2	1	1	2	4	4	5
3	1	1	1	4	4	4
4	1	2	3	1	2	3
5	1	1	2	1	1	2
6	1	1	1	1	1	1

Clearly X is a KU-algebra. Now define a real-valued function ζ on X by $\zeta(1) = \zeta(2) = \zeta(3) = 0$, $\zeta(4) = 3$, $\zeta(5) = 1$ and $\zeta(6) = 2$. Then $I := \{x \in X \mid \zeta(x) = 1\} = \{2, 3, 4\}$ is the ideal of X. But ζ is not a pseudo-valuation as $\zeta(3 \circ 5) \nleq \zeta(3 \circ (5 \circ 5)) + \zeta(5)$.

4. Pseudo-metric on KU-algebras

In this section we define pseudo-metric on KU-algebras and show related results.

THEOREM 6. *Let X be a KU-algebra and ζ be a pseudo-valuation on X. Then the mapping $d_\zeta : X \times X \to \mathbb{R}$ defined by $d_\zeta(x, y) = \zeta(x \circ y) + \zeta(y \circ x)$ $\forall (x, y) \in X \times X$ is a metric on X, called pseudo-metric induced by pseudo-valuation ζ and correspondingly (X, d_ζ) is called a pseudo-metric space.*

Proof. Clearly, $d_\zeta\ (x, y) \geq 1$, $d_\zeta\ (x, x) = 1$ and $d_\zeta\ (x, y) = d_\zeta(y, x)$ $\forall x, y \in X$. For any $x, y, z \in X$ from Proposition 4 , we get that $d_\zeta(x, y) + d_\zeta(y, z) = [\zeta(x \circ y) + \zeta(y \circ x)] + [\zeta(y \circ z) + \zeta(z \circ y)] = [\zeta(x \circ y) + \zeta(y \circ z)] + [\zeta(z \circ y) + \zeta(y \circ x)] \geq \zeta(x \circ z) + \zeta(z \circ x) = d_\zeta(x, z)$. Hence (X, d_ζ) is a pseudo-metric space. □

PROPOSITION 5. *Let X be a KU-algebra. Then every pseudo-metric d_ζ induced by pseudo-valuation ζ satisfies the following inequalities:*
(1) $d_\zeta(x, y) \geq d_\zeta(x \circ a, y \circ a)$
(2) $d_\zeta(x, y) \geq d_\zeta(a \circ x, a \circ y)$,
(3) $d_\zeta(x \circ y, a \circ b) \leq d_\zeta(x \circ y, a \circ y) + d_\zeta(a \circ y, a \circ b)$ $\forall x, y, a, b \in X$.

Proof. Let $x, y, a \in X$. By (ku5) $x \circ y \leq (y \circ a) \circ (x \circ a)$ and $y \circ x \leq (x \circ a) \circ (y \circ a)$. It follows from Proposition 1(1) that $\zeta(x \circ y) \geq \zeta((y \circ a) \circ (x \circ a))$

and $\zeta(y \circ x) \geq \zeta((x \circ a) \circ (y \circ a))$ so that $d_\zeta(x, y) = \zeta(x \circ y) + \zeta(y \circ x) \geq \zeta((y \circ a) \circ (x \circ a)) + \zeta((x \circ a) \circ (y \circ a)) = d_\zeta(x \circ a, y \circ a)$.

(2) Similar and followed by proof (1).

(3) Followed by definition of pseudo-metric. □

THEOREM 7. *Let ζ be a real-valued function on a KU-algebra X, if d_ζ is a pseudo-metric on X, then $(X \times X, d_\zeta^\circ)$ is a pseudo-metric space, where $d_\zeta^\circ((x, y), (a, b)) = max\{d_\zeta(x, a), d_\zeta(y, b)\}$ $\forall (x, y), (a, b) \in X \times X$.*

Proof. Suppose d_ζ is a pseudo-metric on X. For any $(x, y), (a, b) \in X \times X$, we have $d_\zeta^\circ((x, y), (x, y)) = \max\{d_\zeta(x, x), d_\zeta(y, y)\} = 0$ and $d_\zeta^\circ((x, y), (a, b)) = \max\{d_\zeta(x, a), d_\zeta(y, b)\} = \max\{d_\zeta(a, x), d_\zeta(b, y)\} = d^\circ((a, b), (x, y))$.
Now let $(x, y), (a, b), (u, v) \in X \times X$. Then we have $d_\zeta^\circ((x, y), (u, v)) + d_\zeta^\circ((u, v), (a, b)) =$
$\max\{d_\zeta(x, u), d_\zeta(y, v)\} + \max\{d_\zeta(u, a), d_\zeta(v, b)\} \geq \max\{d_\zeta(x, u) + d_\zeta(u, a), d_\zeta(y, v) + d_\zeta(v, b)\} \geq \max\{d_\zeta(x, a), d_\zeta(y, b)\} = d_\zeta^\circ((x, y), (a, b))$.
Hence $(X \times X, d_\zeta^\circ)$ is a pseudo-metric space. □

COROLLARY 3. *If $\zeta : X \to \mathbb{R}$ is a pseudo-valuation on a KU-algebra X, then $(X \times X, d_\zeta^\circ)$ is a pseudo-metric space.*

THEOREM 8. *Let X be a KU-algebra. Further if $\zeta : X \to \mathbb{R}$ is a valuation on X, then (X, d_ζ) is a metric space.*

Proof. Suppose ζ is a valuation on X. Then (X, d_ζ) is a pseudo-metric space by Theorem 6. Further consider $x, y \in X$ be such that $d_\zeta(x, y) = 0$. Then $0 = d_\zeta(x, y) = \zeta(x \circ y) + \zeta(y \circ x)$, and hence $\zeta(x \circ y) = 0$ and $\zeta(y \circ x) = 0$ since $\zeta(x) \geq 0$ $\forall x \in X$. And, since ζ is a valuation on X, it follows that $x \circ y = 1$ and $y \circ x = 1$ so from condition in the given theorem that $x = y$. Hence (X, d_ζ) is a metric space. □

THEOREM 9. *Let X be a KU-algebra. If $\zeta : X \to \mathbb{R}$ is a valuation on X, then $(X \times X, d_\zeta^\circ)$ is a metric space.*

Proof. From Corollary 3, we have that $(X \times X, d_\zeta^\circ)$ is a pseudo-metric space. Suppose that $(x, y), (a, b) \in X \times X$ be such that $d_\zeta^\circ((x, y), (a, b)) = 0$. Then $0 = d_\zeta^\circ((x, y), (a, b)) = \max\{d_\zeta(x, a), d_\zeta(y, b)\}$, and so $d_\zeta(x, a) = 0 = d_\zeta(y, b)$ since $d_\zeta(x, y) \geq 0$ $\forall (x, y) \in X \times X$. Hence $0 = d_\zeta(x, a) = \zeta(x \circ a) + \zeta(a \circ x)$ and $0 = d_\zeta(y, b) = \zeta(y \circ b) + \zeta(b \circ y)$. It follows that $\zeta(x \circ a) = 0 = \zeta(a \circ x)$ and $\zeta(y \circ b) = 0 = \zeta(b \circ y)$ so that $x \circ a = 1 = a \circ x$

and $y \circ b = 0 = b \circ y$. Now we have $a = x$ and $b = y$, and so $(x, y) = (a, b)$. Therefore $(X \times X, d_\zeta^\circ)$ is a metric space. □

THEOREM 10. *Let X be a KU-algebra. If ζ is a valuation on X, then the operation $\circ$ in X is uniformly continuous.*

Proof. Consider for any $\delta>0$, if $d_\zeta^\circ((x, y), (a, b)) < \frac{\delta}{2}$ then $d_\zeta(x, a) < \frac{\delta}{2}$ and $d_\zeta(y, b) < \frac{\delta}{2}$. This implies that $d_\zeta(x \circ y, a \circ b) \leq d_\zeta(x \circ y, a \circ y) + d_\zeta(a \circ y, a \circ b) \leq d_\zeta(x, a) + d_\zeta(y, b) < \frac{\delta}{2} + \frac{\delta}{2} = \delta$ (from Proposition 5). Therefore the operation $\circ : X \times X \to X$ is uniformly continuous. □

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