Korean J. Math. **27** (2019), No. 2, pp. 357–374
https://doi.org/10.11568/kjm.2019.27.2.357

$k-$FRACTIONAL INTEGRAL INEQUALITIES FOR $(h-m)-$CONVEX FUNCTIONS VIA CAPUTO $k-$FRACTIONAL DERIVATIVES

Lakshmi Narayan Mishra*, Qurat Ul Ain, Ghulam Farid, and Atiq Ur Rehman

Abstract. In this paper, first we obtain some inequalities of Hadamard type for $(h-m)-$convex functions via Caputo $k-$fractional derivatives. Secondly, two integral identities including the $(n+1)$ and $(n+2)$ order derivatives of a given function via Caputo $k-$fractional derivatives have been established. Using these identities estimations of Hadamard type integral inequalities for the Caputo $k-$fractional derivatives have been proved.

1. Introduction and preliminaries

Fractional Calculus does not mean the calculus of fractions nor does it mean a fraction of any calculus differentiation and integration. The Fractional Calculus is a name of theory of integration and derivatives of arbitrary order. From historical point of view, development of Fractional Calculus begins with step by step introduction of fractional derivative. Fractional calculus may be describe as extension of the concept of derivative operator from integer order n to arbitrary order α. Fractional derivative has been considered as the inverse of a fractional integral.

Received November 26, 2018. Revised May 16, 2019. Accepted June 3, 2019.
2010 Mathematics Subject Classification: 26B15, 26A51, 34L15.
Key words and phrases: Caputo fractional derivatives; Caputo $k-$fractional derivatives; $(h-m)-$convex functions.
$*$ Corresponding author.

Systems containing fractional derivatives and integrals have been studied by many science and engineering areas. There are growing number of physical systems whose behavior can be compactly describe by using fractional calculus systems. A particular subgroup of nonlocal fields theories play an increasingly important role and may be described with the operators of fractional nature and is specified within the framework of Fractional Calculus. Despite the fact that the concept is being discussed since the day of Leibniz (1695) and since then has occupied great number of mathematicians of their time [4–6, 8, 10].
As in this paper we have to prove the Hadamard type inequalities for $(h-m)-$convex functions via Caputo $k-$fractional derivatives, we recall following definition which is well known in the literature [5].

Definition 1.1. Let $\alpha > 0$ and $\alpha \notin \{1,2,3,...\}$, $n = [\alpha]+1$, $f \in AC^n[a,b]$. The Caputo fractional derivatives of order α are defined as follows:

$$ {}^{C}D^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(n-\alpha)}\int_a^x \frac{f^{(n)}(t)}{(x-t)^{\alpha-n+1}}dt, x > a \tag{1}$$

and

$$ {}^{C}D^{\alpha}_{b-}f(x) = \frac{(-1)^n}{\Gamma(n-\alpha)}\int_x^b \frac{f^{(n)}(t)}{(t-x)^{\alpha-n+1}}dt, x < b. \tag{2}$$

If $\alpha = n \in \{1,2,3,...\}$ and usual derivative of order n exists, then Caputo fractional derivative $({}^{C}D^{\alpha}_{a+}f)(x)$ coincides with $f^{(n)}(x)$, whereas $({}^{C}D^{\alpha}_{b-}f)(x)$ coincides with $f^{(n)}(x)$ with exactness to a constant multiplier $(-1)^n$. In particular we have

$$ ({}^{C}D^{0}_{a+}f)(x) = ({}^{C}D^{0}_{b-}f)(x) = f(x) \tag{3}$$

where $n = 1$ and $\alpha = 0$.

In [2], Farid *et al.* defined Caputo $k-$fractional integrals as follows:

Definition 1.2. Let $\alpha > 0, k \geq 1$ and $\alpha \notin \{1,2,3,...\}$, $n = [\alpha]+1$, $f \in AC^n[a,b]$. The Caputo k-fractional derivatives of order α are defined as follows:

$$ {}^{C}D^{\alpha,k}_{a+}f(x) = \frac{1}{k\Gamma_k(n-\frac{\alpha}{k})}\int_a^x \frac{f^{(n)}(t)}{(x-t)^{\frac{\alpha}{k}-n+1}}dt, x > a \tag{4}$$

and

$$ {}^{C}D_{b-}^{\alpha,k}f(x)=\frac{(-1)^{n}}{k\Gamma_{k}(n-\frac{\alpha}{k})}\int_{x}^{b}\frac{f^{(n)}(t)}{(t-x)^{\frac{\alpha}{k}-n+1}}dt,x<b \tag{5}$$

where $\Gamma_k(\alpha)$ is the k-Gamma function defined as

$$\Gamma_k(\alpha)=\int_0^\infty t^{\alpha-1}e^{\frac{-t^k}{k}}dt,$$

also

$$\Gamma_k(\alpha+k)=\alpha\Gamma_k(\alpha).$$

If $\alpha=n\in\{1,2,3,...\}$ and usual derivative $f^{(n)}(x)$ of order n exists, then Caputo k-fractional derivative $({}^{C}D_{a+}^{\alpha,1}f)(x)$ coincides with $f^{(n)}(x)$, whereas $({}^{C}D_{b-}^{\alpha,1}f)(x)$ coincides with $f^{(n)}(x)$ with exactness to a constant multiplier $(-1)^n$.

In the following we give definitions of convex and $(h-m)-$convex function.

Definition 1.3. A real-valued function f defined on an interval is called convex, if for any two points x and y in its domain and any $\alpha\in[0,1]$, we have

$$f(\alpha x+(1-\alpha)y)\leq\alpha f(x)+(1-\alpha)f(y).$$

In other words, a function is convex if and only if its epigraph (the set of points lying on or above the graph) is a convex set.

In [9], $(h-m)-$convexity introduced by Özdemir *et al.* as a generalization of convex functions.

Definition 1.4. Let $J\subseteq\mathbb{R}$ be an interval containing $(0,1)$ and let $h:J\to\mathbb{R}$ be a non-negative function. We say that $f:[0,b]\to\mathbb{R}$ is a $(h-m)-$convex function, if f is non-negative and for all $x,y\in[0,b],m\in[0,1]$ and $\alpha\in(0,1)$, one has

$$f(\alpha x+m(1-\alpha)y)\leq h(\alpha)f(x)+mh(1-\alpha)f(y). \tag{6}$$

If the inequality (6) is reversed, then f is said to be an $(h-m)-$concave function.

Fractional integral inequalities are in the study of several researchers (see, [1–3,9] and references there in).In this paper, in Section 2 we present some inequalities of Hadamard type for $(h-m)-$convex functions via Caputo $k-$fractional derivatives. In Section 3 we prove the integral identities including the $(n+1)$-order and $(n+2)$-order derivative of f to establish interesting Hadamard type inequalities for $(h-m)-$convexity via Caputo $k-$fractional derivatives.

2. Hadamard type inequalites for $(h-m)-$convex functions via Caputo $k-$fractional derivatives

Here we give the Caputo $k-$fractional derivatives inequality of Hadamarad type for a functions whose nth derivatives are $(h-m)-$convex.

THEOREM 2.1. *Let $f:[0,\infty)\to\mathbb{R}$ be a function such that $f\in C^n[a,b]$. Also, let $f^{(n)}$ be $(h-m)-$convex with $m\in(0,1]$. Then the following inequality for Caputo $k-$fractional derivatives holds*

(7)

$$f^{(n)}\left(\frac{bm+a}{2}\right)$$
$$\leq h\left(\frac{1}{2}\right)\frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{(mb-a)^{n-\frac{\alpha}{k}}}\left[m^{\alpha+1}(-1)^{(n)}({}^CD^{\alpha,k}_{b^-}f)\left(\frac{a}{m}\right)+({}^CD^{\alpha,k}_{a^+}f)(mb)\right]$$
$$\leq\left(n-\frac{\alpha}{k}\right)h\left(\frac{1}{2}\right)\left\{\left[m^2f^{(n)}\left(\frac{a}{m^2}\right)+mf^{(n)}(b)\right]\int_0^1t^{n-\frac{\alpha}{k}-1}h(1-t)dt\right.$$
$$\left.+\left[mf^{(n)}(b)+f^{(n)}(a)\right]\int_0^1t^{n-\frac{\alpha}{k}-1}h(t)dt\right\}.$$

Proof. Since $f^{(n)}$ is $(h-m)-$convex, therefore one can have

$$f^{(n)}\left(\frac{um+v}{2}\right)\leq h\left(\frac{1}{2}\right)(mf^{(n)}(u)+f^{(n)}(v)).$$

Putting $u=(1-t)\frac{a}{m}+tb$ and $v=m(1-t)b+ta$ in the above inequality where $t\in[0,1]$, and multiplying with $t^{n-\frac{\alpha}{k}-1}$, then integrating over $[0,1]$ we get

$$f^{(n)}\left(\frac{bm+a}{2}\right)\int_0^1t^{n-\frac{\alpha}{k}-1}dt$$
$$\leq h\left(\frac{1}{2}\right)\left[\int_0^1t^{n-\frac{\alpha}{k}-1}mf^{(n)}\left((1-t)\frac{a}{m}+tb\right)dt\right.$$
$$\left.+\int_0^1t^{n-\frac{\alpha}{k}-1}f^{(n)}(m(1-t)b+ta)dt\right].$$

By change of variables, we get

$$f^{(n)}\left(\frac{bm+a}{2}\right) \tag{8}$$
$$\leq h\left(\frac{1}{2}\right)\frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{(mb-a)^{n-\frac{\alpha}{K}}}\left[m^{\alpha+1}(-1)^{(n)}({}^CD^{\alpha,k}_{b^-}f)\left(\frac{a}{m}\right)+({}^CD^{\alpha,k}_{a^+}f)(mb)\right].$$

Now using the $(h-m)-$convexity of $f^{(n)}$ one can also have

$$mf^{(n)}\left((1-t)\frac{a}{m}+tb\right)+f^{(n)}(m(1-t)b+ta)$$
$$\leq m^2h(1-t)f^{(n)}\left(\frac{a}{m^2}\right)+mh(t)f^{(n)}(b)+mh(1-t)f^{(n)}(b)+h(t)f^{(n)}(a).$$

Multiplying both sides of above inequality with $(n-\frac{\alpha}{k})h\left(\frac{1}{2}\right)t^{\frac{\alpha}{k}-1}$ and integrating over $[0,1]$, then with the change of variables one can have

$$h\left(\frac{1}{2}\right)\frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{(mb-a)^{n-\frac{\alpha}{k}}}\left[m^{\alpha+1}(-1)^{(n)}({}^CD^{\alpha,k}_{b^-}f)\left(\frac{a}{m}\right)+({}^CD^{\alpha,k}_{a^+}f)(mb)\right]$$
$$\leq\left(n-\frac{\alpha}{k}\right)h\left(\frac{1}{2}\right)\left\{\left[m^2f^{(n)}\left(\frac{a}{m^2}\right)+mf^{(n)}(b)\right]\int_0^1 t^{n-\frac{\alpha}{k}-1}h(1-t)dt\right.$$
$$\left.+\left[mf^{(n)}(b)+f^{(n)}(a)\right]\int_0^1 t^{n-\frac{\alpha}{k}-1}h(t)dt\right\}.$$

Combining it with (8) we get (7). □

Corollary 2.2. *In Theorem 2.1 if we take $k=1$, then we get the following inequality for $(h-m)-$convex functions via Caputo fractional derivatives*

$$f^{(n)}\left(\frac{bm+a}{2}\right)$$
$$\leq h\left(\frac{1}{2}\right)\frac{\Gamma(n-\alpha+1)}{(mb-a)^{n-\alpha}}\left[m^{\alpha+1}(-1)^{(n)}({}^CD^{\alpha}_{b^-}f)\left(\frac{a}{m}\right)+({}^CD^{\alpha}_{a^+}f)(mb)\right]$$
$$\leq(n-\alpha)\,h\left(\frac{1}{2}\right)\left\{\left[m^2f^{(n)}\left(\frac{a}{m^2}\right)+mf^{(n)}(b)\right]\int_0^1 t^{n-\alpha-1}h(1-t)dt\right.$$
$$\left.+\left[mf^{(n)}(b)+f^{(n)}(a)\right]\int_0^1 t^{n-\alpha-1}h(t)dt\right\}.$$

Corollary 2.3. *In Theorem 2.1 if we take $k=1$, $h(t)=t$, and $m=1$, then we get the following inequality for convex functions via*

Caputo fractional derivatives

$$\begin{aligned}&f^{(n)}\left(\frac{a+b}{2}\right)\\&\leq \frac{k\Gamma_k(n-\frac{\alpha}{k}+1)}{2(b-a)^{n-\frac{\alpha}{k}}}\left[({}^CD^{\alpha,k}_{a^+}f)(b)+(-1)^n({}^CD^{\alpha,k}_{b^-}f)\,(a)\right]\\&\leq \frac{f^{(n)}(a)+f^{(n)}(b)}{2}.\end{aligned}$$

In the following we generalize the fractional Hardamard type inequalities for $(h-m)-$convex function via Caputo k-fractional derivative.

THEOREM 2.4. *Let $f:[0,\infty)\to\mathbb{R}$ be a function such that $f\in C^n[a,b]$. Also, let $f^{(n)}$ be $(h-m)-$convex with $m\in(0,1]$. Then the following inequality for Caputo $k-$fractional derivatives holds*

(9)

$$\begin{aligned}&f^{(n)}\left(\frac{a+bm}{2}\right)\\&\leq 2^{(n-\frac{\alpha}{k})}h\left(\frac{1}{2}\right)\frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{(bm-a)^{n-\frac{\alpha}{K}}}\\&\times\left[({}^CD^{\alpha,k}_{(\frac{a+bm}{2})^+}f)(mb)+m^{\alpha+1}(-1)^{(n)}({}^CD^{\alpha,k}_{(\frac{a+bm}{2m})^-}f)\left(\frac{a}{m}\right)\right]\\&\leq\left(n-\frac{\alpha}{k}\right)h\left(\frac{1}{2}\right)\left\{m^2f^{(n)}\left(\frac{a}{m^2}\right)\int_0^1 t^{n-\frac{\alpha}{k}-1}h\left(\frac{2-t}{2}\right)dt\right.\\&\left.+mf^{(n)}(b)\int_0^1 t^{n-\frac{\alpha}{k}-1}h\left(\frac{2-t}{2}\right)dt+\left[mf^{(n)}(b)+f^{(n)}(a)\right]\int_0^1 t^{n-\frac{\alpha}{k}-1}h\left(\frac{t}{2}\right)dt\right\}.\end{aligned}$$

Proof. From $(h-m)-$convexity of $f^{(n)}$ one can have

$$f^{(n)}\left(\frac{u+vm}{2}\right)\leq h\left(\frac{1}{2}\right)(f^{(n)}(u)+mf^{(n)}(v)).$$

Putting $u=\frac{t}{2}b+\frac{(2-t)}{2}\frac{a}{m}$ and $u=\frac{t}{2}a+m\frac{(2-t)}{2}b$ in the above inequality where $t\in[0,1]$, and multiplying with $t^{n-\frac{\alpha}{k}-1}$, then integrating over $[0,1]$

one can have

$$f^{(n)}\left(\frac{a+bm}{2}\right)\int_0^1 t^{n-\frac{\alpha}{k}-1}dt$$
$$\leq h\left(\frac{1}{2}\right)\left[\int_0^1 t^{n-\frac{\alpha}{k}-1}mf^{(n)}\left(\frac{t}{2}a+m\frac{(2-t)}{2}b\right)dt\right.$$
$$\left.+\int_0^1 t^{n-\frac{\alpha}{k}-1}f^{(n)}\left(\frac{t}{2}b+\frac{(2-t)}{2}\frac{a}{m}\right)dt\right].$$

By change of variables, we get

$$f^{(n)}\left(\frac{a+bm}{2}\right) \tag{10}$$
$$\leq 2^{(n-\frac{\alpha}{k})}h\left(\frac{1}{2}\right)\frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{(bm-a)^{n-\frac{\alpha}{K}}}$$
$$\times\left[({}^C D^{\alpha,k}_{(\frac{a+bm}{2})^+}f)(mb)+m^{\alpha+1}(-1)^{(n)}({}^C D^{\alpha,k}_{(\frac{a+bm}{2m})^-}f)\left(\frac{a}{m}\right)\right].$$

Now, using the $(h-m)-$convexity of $f^{(n)}$, we can write

$$f^{(n)}\left(\frac{t}{2}a+m\frac{(2-t)}{2}b\right)+mf^{(n)}\left(\frac{t}{2}b+\frac{(2-t)}{2}\frac{a}{m}\right)$$
$$\leq h\left(\frac{t}{2}\right)f^{(n)}(a)+mh\left(\frac{2-t}{2}\right)f^{(n)}(b)+mh\left(\frac{t}{2}\right)f^{(n)}(b)$$
$$+m^2h\left(\frac{2-t}{2}\right)f^{(n)}\left(\frac{a}{m^2}\right)$$

Multiplying both sides of above inequality with $(n-\frac{\alpha}{k})h\left(\frac{1}{2}\right)t^{n-\frac{\alpha}{k}-1}$ and integrating over $[0,1]$, then by change of variables we have

$$2^{(n-\frac{\alpha}{k})}h\left(\frac{1}{2}\right)\frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{(bm-a)^{n-\frac{\alpha}{K}}}$$
$$\times\left[({}^C D^{\alpha,k}_{(\frac{a+bm}{2})^+}f)(mb)+m^{\alpha+1}(-1)^{(n)}({}^C D^{\alpha,k}_{(\frac{a+bm}{2m})^-}f)\left(\frac{a}{m}\right)\right]$$
$$\leq\left(n-\frac{\alpha}{k}\right)h\left(\frac{1}{2}\right)\left\{m^2f^{(n)}\left(\frac{a}{m^2}\right)\int_0^1 t^{n-\frac{\alpha}{k}-1}h\left(\frac{2-t}{2}\right)dt\right.$$
$$\left.+mf^{(n)}(b)\int_0^1 t^{n-\frac{\alpha}{k}-1}h\left(\frac{2-t}{2}\right)dt+\left[mf^{(n)}(b)+f^{(n)}(a)\right]\int_0^1 t^{n-\frac{\alpha}{k}-1}h\left(\frac{t}{2}\right)dt\right\}.$$

Combining it with (10), we get (9). □

Corollary 2.5. *In Theorem 2.4 if we take $k = 1$, then we get the following inequality for $(h-m)-$convex function via Caputo fractional derivative*

$$
\begin{aligned}
& f^{(n)}\left(\frac{a+bm}{2}\right) \\
& \leq 2^{(n-\alpha)} h\left(\frac{1}{2}\right) \frac{\Gamma(n-\alpha+1)}{(bm-a)^{n-\alpha}} \\
& \times \left[({}^{C}D^{\alpha}_{(\frac{a+bm}{2})^{+}} f)(mb) + m^{\alpha+1}(-1)^{(n)}({}^{C}D^{\alpha}_{(\frac{a+bm}{2m})^{-}} f)\left(\frac{a}{m}\right)\right] \\
& \leq (n-\alpha)\, h\left(\frac{1}{2}\right)\left\{m^{2} f^{(n)}\left(\frac{a}{m^{2}}\right)\int_{0}^{1} t^{n-\alpha-1} h\left(\frac{2-t}{2}\right) dt\right. \\
& \left. + m f^{(n)}(b)\int_{0}^{1} t^{n-\alpha-1} h\left(\frac{2-t}{2}\right) dt + \left[m f^{(n)}(b) + f^{(n)}(a)\right]\int_{0}^{1} t^{n-\alpha-1} h\left(\frac{t}{2}\right) dt\right\}.
\end{aligned}
$$

Corollary 2.6. *In Theorem 2.4 if we take $k = 1$, $h(t) = t$ and $m = 1$, then we get the following inequality for convex function via Caputo fractional derivative*

$$
\begin{aligned}
& f^{(n)}\left(\frac{a+b}{2}\right) \\
& \leq 2^{(n-\alpha-1)} \frac{\Gamma(n-\alpha+1)}{(b-a)^{n-\alpha}} \left[({}^{C}D^{\alpha}_{(\frac{a+b}{2})^{+}} f)(b) + (-1)^{(n)}({}^{C}D^{\alpha}_{(\frac{a+b}{2})^{-}} f)\,(a)\right] \\
& \leq \frac{f^{(n)}(a) + f^{(n)}(b)}{2}.
\end{aligned}
$$

In the next, some other inequalities of Hadamard type for $(h-m)-$convex function via Caputo $k-$fractional derivative are given.

Theorem 2.7. *Let $f : [0,\infty) \to \mathbb{R}$ be a function such that $f \in C^{n}[a,b]$. Also, let $f^{(n)}$ be $(h-m)-$convex with $m \in (0,1]$. Then the* *following inequality for Caputo $k-$fractional derivatives holds*

$$
\begin{aligned}
(11)\qquad & \frac{k\Gamma_k(n-\frac{\alpha}{k})}{(b-a)^{n-\frac{\alpha}{k}}}[({}^C D_{a^+}^{\alpha,k}f)(b)+(-1)^n({}^C D_{b^-}^{\alpha,k}f)(a)] \\
& \le [f^{(n)}(a)+f^{(n)}(b)]\int_0^1 t^{n-\frac{\alpha}{k}-1}h(t)dt \\
& + m\left[f^{(n)}\left(\frac{b}{m}\right)+f^{(n)}\left(\frac{a}{m}\right)\right]\int_0^1 t^{n-\frac{\alpha}{k}-1}h(1-t)dt \\
& \le \frac{1}{(np-\frac{\alpha}{k}p-p+1)^{\frac{1}{p}}}\left(\int_0^1 (h(t))^q dt\right)^{\frac{1}{q}} \\
& \times\left[f^{(n)}(a)+f^{(n)}(b)+m\left(f^{(n)}\left(\frac{b}{m}\right)+f^{(n)}\left(\frac{a}{m}\right)\right)\right],
\end{aligned}
$$

where $p^{-1}+q^{-1}=1$ and $p>1$.

Proof. Since $f^{(n)}$ is $(h-m)-$convex on $[a,b]$, then for $m\in(0,1]$ and $t\in[0,1]$, we have

$$
\begin{aligned}
& f^{(n)}(ta+(1-t)b)+f^{(n)}((1-t)a+tb) \\
& \le h(t)[f^{(n)}(a)+f^{(n)}(b)]+mh(1-t)\left[f^{(n)}\left(\frac{b}{m}\right)+f^{(n)}\left(\frac{a}{m}\right)\right].
\end{aligned}
$$

Multiplying both sides of above inequality with $t^{n-\frac{\alpha}{k}-1}$ and integrating the above inequality with respect to t on $[0,1]$, we have

$$
\begin{aligned}
& \int_0^1 t^{n-\frac{\alpha}{k}-1}[f^{(n)}(ta+(1-t)b)+f^{(n)}((1-t)a+tb)]dt \\
& \le [f^{(n)}(a)+f^{(n)}(b)]\int_0^1 t^{n-\frac{\alpha}{k}-1}h(t)dt \\
& + m\left[f^{(n)}\left(\frac{b}{m}\right)+f^{(n)}\left(\frac{a}{m}\right)\right]\int_0^1 t^{n-\frac{\alpha}{k}-1}h(1-t)dt.
\end{aligned}
$$

If we set $x = ta + (1-t)b$ in the left side of above inequality, we get the following inequality

(12)
$$\begin{aligned}
&\frac{k\Gamma_k(n-\frac{\alpha}{k})}{(b-a)^{n-\frac{\alpha}{k}}}[({}^C D^{\alpha,k}_{a^+} f)(b) + (-1)^n ({}^C D^{\alpha,k}_{b^-} f)(a)] \\
&\leq [f^{(n)}(a) + f^{(n)}(b)] \int_0^1 t^{n-\frac{\alpha}{k}-1} h(t)dt + m\left[f^{(n)}\left(\frac{b}{m}\right) + f^{(n)}\left(\frac{a}{m}\right)\right] \\
&\times \int_0^1 t^{n-\frac{\alpha}{k}-1} h(1-t)dt.
\end{aligned}$$

We get the first inequality of (11). For the second inequality of (11) follows from the fact by using the Hölder's inequality

$$\int_0^1 t^{n-\frac{\alpha}{k}-1} h(t)dt \leq \frac{1}{(np - \frac{\alpha}{k}p - p + 1)^{\frac{1}{p}}} \left(\int_0^1 (h(t))^q dt\right)^{\frac{1}{q}}$$

Combining it with (12) we get (11). □

COROLLARY 2.8. *In Theorem 2.7 if we take $k = 1$, then we get the following inequality for $(h-m)-$convex function via Caputo fractional derivative*

$$\begin{aligned}
&\frac{\Gamma(n-\alpha)}{(b-a)^{n-\alpha}}[({}^C D^{\alpha}_{a^+} f)(b) + (-1)^n ({}^C D^{\alpha}_{b^-} f)(a)] \\
&\leq [f^{(n)}(a) + f^{(n)}(b)] \int_0^1 t^{n-\alpha-1} h(t)dt \\
&+ m\left[f^{(n)}\left(\frac{b}{m}\right) + f^{(n)}\left(\frac{a}{m}\right)\right] \int_0^1 t^{n-\alpha-1} h(1-t)dt \\
&\leq \frac{1}{(np - \alpha p - p + 1)^{\frac{1}{p}}} \left(\int_0^1 (h(t))^q dt\right)^{\frac{1}{q}} \\
&\times \left[f^{(n)}(a) + f^{(n)}(b) + m\left(f^{(n)}\left(\frac{b}{m}\right) + f^{(n)}\left(\frac{a}{m}\right)\right)\right].
\end{aligned}$$

THEOREM 2.9. *Let $f : [0, \infty) \to \mathbb{R}$ be a function such that $f \in C^n[a, b]$. Also, let $f^{(n)}$ be $(h-m)-$convex with $m \in (0, 1]$ and h be superaddtive. Then the following inequality for Caputo $k-$fractional* *derivatives holds*

$$\frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{(b-a)^{n-\frac{\alpha}{k}}}[({}^C D_{a^+}^{\alpha,k} f)(b)+(-1)^n({}^C D_{b^-}^{\alpha,k} f)(a)] \leq h(1)\left[\frac{f(a)+f(b)}{2}+m\left(\frac{f\left(\frac{a}{m}\right)+f\left(\frac{b}{m}\right)}{2}\right)\right]. \tag{13}$$

Proof. Since $f^{(n)}$ is $(h-m)-$convex on $[a,b]$, then for $t\in[0,1]$, we get

$$f^{(n)}(ta+(1-t)b)+f^{(n)}((1-t)a+tb) \leq [h(t)+h(1-t)]\left[\frac{f^{(n)}(a)+f^{(n)}(b)}{2}+m\left(\frac{f^{(n)}\left(\frac{a}{m}\right)+f^{(n)}\left(\frac{b}{m}\right)}{2}\right)\right].$$

Since h is superadditive, therefore

$$f^{(n)}(ta+(1-t)b)+f^{(n)}((1-t)a+tb) \leq h(1)\left[\frac{f^{(n)}(a)+f^{(n)}(b)}{2}+m\left(\frac{f^{(n)}\left(\frac{a}{m}\right)+f^{(n)}\left(\frac{b}{m}\right)}{2}\right)\right].$$

Multiplying both sides of above inequality with $t^{n-\frac{\alpha}{k}-1}$ and integrating over $[0,1]$, yield the following

$$\int_0^1 t^{n-\frac{\alpha}{k}-1}[f^{(n)}(ta+(1-t)b)+f^{(n)}((1-t)a+tb)]dt \leq h(1)\left[\frac{f^{(n)}(a)+f^{(n)}(b)}{2}+m\left(\frac{f^{(n)}\left(\frac{a}{m}\right)+f^{(n)}\left(\frac{b}{m}\right)}{2}\right)\right]\int_0^1 t^{n-\frac{\alpha}{k}-1}dt.$$

By change of variable, we get the required result. □

Corollary 2.10. *In Theorem 2.9 if we take $k=1$, then we get the following inequality for $(h-m)-$convex function via Caputo fractional derivative*

$$\frac{\Gamma(n-\alpha+1)}{(b-a)^{n-\alpha}}[({}^C D_{a^+}^{\alpha} f)(b)+(-1)^n({}^C D_{b^-}^{\alpha} f)(a)] \leq h(1)\left[\frac{f(a)+f(b)}{2}+m\left(\frac{f\left(\frac{a}{m}\right)+f\left(\frac{b}{m}\right)}{2}\right)\right].$$

3. Caputo $k-$fractional derivative inequalities of Hadamard type for functions whose nth derivatives in absolute values are $(h-m)-$convex

We need following lemma to prove next results.

LEMMA 3.1. *Let $f:[a,mb]\to\mathbb{R}$ be a differentiable mapping on interval (a,mb) with $a<mb$. If $f\in C^{n+1}[a,mb]$, then the following equality for Caputo $k-$fractional integrals holds*

$$\frac{f^{(n)}(mb)+f^{(n)}(b)}{2}-\frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{2(mb-a)^{n-\frac{\alpha}{k}}}\left[({}^CD^{\alpha,k}_{a^+}f)(mb)+({}^CD^{\alpha,k}_{mb^-}f)(a)\right]$$
$$=\frac{mb-a}{2}\int_0^1[(1-t)^{n-\frac{\alpha}{k}}-t^{n-\frac{\alpha}{k}}]f^{(n+1)}(m(1-t)b+ta)dt.$$

Proof. Since

$$\frac{mb-a}{2}\int_0^1[(1-t)^{n-\frac{\alpha}{k}}-t^{n-\frac{\alpha}{k}}]f^{(n+1)}(m(1-t)b+ta)$$
$$=\frac{mb-a}{2}\left[\int_0^1(1-t)^{n+\frac{\alpha}{k}}f^{(n+1)}(m(1-t)b+ta)dt\right.$$
$$\left.-\int_0^1 t^{n+\frac{\alpha}{k}}f^{(n+1)}(m(1-t)b+ta)dt\right].$$

One have

$$\frac{mb-a}{2}\int_0^1(1-t)^{n-\frac{\alpha}{k}}f^{(n+1)}(m(1-t)b+ta)dt$$
$$=\frac{f(mb)}{2}-\frac{k\Gamma_k(\frac{\alpha}{k}+k)}{2(mb-a)^{n-\frac{\alpha}{k}}}({}^CD^{\alpha,k}_{mb^-}f)(a),$$

and

$$\frac{mb-a}{2}\int_0^1(t)^{n-\frac{\alpha}{k}}f^{(n)}(m(1-t)b+ta)dt$$
$$=\frac{f(a)}{2}-\frac{k\Gamma_k(\frac{\alpha}{k}+k)}{2(mb-a)^{n-\frac{\alpha}{k}}}({}^CD^{\alpha,k}_{a^+}f)(mb).$$

Hence the required equality can be established. □

Caputo $k-$fractional derivative inequalities of Hadamard type for $(h-m)-$convex function in terms of the $(n+1)$th derivatives in absolute, is obtained in the following by using above lemma.

THEOREM 3.2. *Let $f:[0,\infty)\to\mathbb{R}$ be a function such that $[a,b]\subset[0,\infty)$ and $f\in C^{n+1}[a,b]$. If $\left|f^{(n+1)}\right|$ is an $(h-m)-$convex with $m\in(0,1]$ and $h^q\in[0,1]$, $q>1$. Then the following inequality for Caputo $k-$fractional derivatives holds*

(14)

$$\left|\frac{f(mb)+f(a)}{2}-\frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{2(mb-a)^{n-\frac{\alpha}{k}}}({}^CD^{\alpha,k}_{b^+}f)(mb)+({}^CD^{\alpha,k}_{mb^-}f)(a)\right|$$
$$\leq\frac{(mb-a)\left[|f'(a)|+m\left|f'(b)\right|\right]}{2}\left[\left[\frac{2^{n-\frac{\alpha}{k}p+1}-1}{2^{n-\frac{\alpha}{k}p+1}(np-\frac{\alpha}{k}p+1)}\right]^{\frac{1}{p}}\right.$$
$$\left.-\left[\frac{1}{2^{n-\frac{\alpha}{k}p+1}(np-\frac{\alpha}{k}p+1)}\right]^{\frac{1}{p}}\right]\left[\int_0^{\frac{1}{2}}(h(t))^qdt\right]^{\frac{1}{q}}+\left[\int_{\frac{1}{2}}^1(h(t))^qdt\right]^{\frac{1}{q}},$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From Lemma 3.1 and by using the property of modulus, we get

$$\left|\frac{f^{(n)}(mb)+f^{(n)}(a)}{2}-\frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{2(mb-a)^{n-\frac{\alpha}{k}}}(I^{\alpha,k}_{a^+}f(mb)+I^{\alpha,k}_{mb^-}f(a))\right|$$
$$\leq\frac{mb-a}{2}\int_0^1\left|(1-t)^{n-\frac{\alpha}{k}}-t^{n-\frac{\alpha}{k}}\right|\left|f^{(n+1)}(m(1-t)b+ta)\right|dt.$$

By $(h-m)-$convexity of $\left|f^{(n+1)}\right|$, we have

(15)

$$\left|\frac{f^{(n)}(mb)+f^{(n)}(a)}{2}-\frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{2(mb-a)^{n-\frac{\alpha}{k}}}({}^CD^{\alpha,k}_{a^+}f)(mb)+({}^CD^{\alpha,k}_{mb^-}f)(a)\right|$$
$$\leq\frac{mb-a}{2}\int_0^{\frac{1}{2}}\left[(1-t)^{n-\frac{\alpha}{k}}-t^{n-\frac{\alpha}{k}}\right]\left[mh(1-t)\left|f^{(n+1)}(b)\right|+h(t)\left|f^{(n+1)}(a)\right|\right]dt$$
$$+\int_{\frac{1}{2}}^1\left[(1-t)^{n-\frac{\alpha}{k}}-t^{n-\frac{\alpha}{k}}\right]\left[mh(1-t)\left|f^{(n+1)}(b)\right|+h(t)\left|f^{(n+1)}(a)\right|\right]dt$$
$$=\frac{mb-a}{2}\left\{\left|f^{(n)}(a)\right|\left[\int_0^{\frac{1}{2}}(1-t)^{n-\frac{\alpha}{k}}h(1-t)dt-\int_0^{\frac{1}{2}}t^{n-\frac{\alpha}{k}}h(1-t)dt\right]\right.$$
$$+m\left|f^{(n)}(b)\right|\left[\int_0^{\frac{1}{2}}(1-t)^{n-\frac{\alpha}{k}}h(t)dt-\int_0^{\frac{1}{2}}t^{n-\frac{\alpha}{k}}h(t)dt\right]$$
$$+\left|f^{(n)}(a)\right|\left[\int_{\frac{1}{2}}^1 t^{n-\frac{\alpha}{k}}h(1-t)dt-\int_{\frac{1}{2}}^1(1-t)^{n-\frac{\alpha}{k}}h(1-t)dt\right]$$
$$\left.+m\left|f^{(n)}(b)\right|\left[\int_{\frac{1}{2}}^1 t^{n-\frac{\alpha}{k}}h(1-t)dt-\int_{\frac{1}{2}}^1(1-t)^{n-\frac{\alpha}{k}}h(1-t)dt\right]\right\}.$$

Now, using the Hölder's inequality in the right hand side of (15), we get

$$\left|\frac{f^{(n)}(mb)+f^{(n)}(b)}{2}-\frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{2(mb-a)^{n-\frac{\alpha}{k}}}({}^CD^{\alpha,k}_{b^+}f)(mb)+({}^CD^{\alpha,k}_{mb^-}f)(a)\right|\leq\frac{mb-a}{2}$$
$$\left\{|f'(a)|\left[\left(\left[\frac{2^{n-\frac{\alpha}{k}p+1}-1}{2^{n-\frac{\alpha}{k}p+1}(np-\frac{\alpha}{k}p+1)}\right]^{\frac{1}{p}}-\left[\frac{1}{2^{n-\frac{\alpha}{k}p+1}(np-\frac{\alpha}{k}p+1)}\right]^{\frac{1}{p}}\right)\left(\int_0^{\frac{1}{2}}[h(t)]^q dt\right)^{\frac{1}{q}}\right.\right.$$
$$\left.+\left(\left[\frac{2^{n-\frac{\alpha}{k}p+1}-1}{2^{n-\frac{\alpha}{k}p+1}(np-\frac{\alpha}{k}p+1)}\right]^{\frac{1}{p}}-\left[\frac{1}{2^{n-\frac{\alpha}{k}p+1}(np-\frac{\alpha}{k}p+1)}\right]^{\frac{1}{p}}\right)\left(\int_{\frac{1}{2}}^1[h(t)]^q dt\right)^{\frac{1}{q}}\right]$$
$$+m|f'(b)|\left[\left(\left[\frac{2^{n-\frac{\alpha}{k}p+1}-1}{2^{n-\frac{\alpha}{k}p+1}(np-\frac{\alpha}{k}p+1)}\right]^{\frac{1}{p}}-\left[\frac{1}{2^{n-\frac{\alpha}{k}p+1}(np-\frac{\alpha}{k}p+1)}\right]^{\frac{1}{p}}\right)\left(\int_{\frac{1}{2}}^1[h(t)]^q dt\right)^{\frac{1}{q}}\right.$$
$$\left.\left.+\left(\left[\frac{2^{n-\frac{\alpha}{k}p+1}-1}{2^{n-\frac{\alpha}{k}p+1}(np-\frac{\alpha}{k}p+1)}\right]^{\frac{1}{p}}-\left[\frac{1}{2^{n-\frac{\alpha}{k}p+1}(np-\frac{\alpha}{k}p+1)}\right]^{\frac{1}{p}}\right)\left(\int_0^{\frac{1}{2}}[h(t)]^q dt\right)^{\frac{1}{q}}\right]\right\}.$$

After a little computation one can get inequality (14). □

COROLLARY 3.3. *In Theorem 3.2 if we take $k=1$, then we get the following inequality for $(h-m)-$convex function via Caputo fractional* *integrals*

$$\left|\frac{f^{(n)}(mb)+f^{(n)}(a)}{2}-\frac{\Gamma(\alpha+1)}{2(mb-a)^{n-\alpha}}({}^{C}D^{\alpha}_{a^+}f)(mb)+({}^{C}D^{\alpha}_{mb^-}f)(a)\right|$$
$$\leq \frac{(mb-a)\left[|f'(a)|+m\left|f'\left(b\right)\right|\right]}{2}\left[\left[\frac{2^{n-\alpha p+1}-1}{2^{n-\alpha p+1}(np-\alpha p+1)}\right]^{\frac{1}{p}}\right.$$
$$\left.-\left[\frac{1}{2^{n-\alpha p+1}(np-\alpha p+1)}\right]^{\frac{1}{p}}\right]\left[\left(\int_0^{\frac{1}{2}}[h(t)]^q dt\right)^{\frac{1}{q}}+\left(\int_{\frac{1}{2}}^1[h(t)]^q dt\right)^{\frac{1}{q}}\right].$$

Lemma 3.4. *Let $f:[a,mb]\to\mathbb{R}$ be a differentiable mapping on interval (a,mb) with $a<mb$. If $f\in C^{n+2}[a,mb]$, then the following equality for Caputo $k-$fractional integrals holds*

$$\frac{f^{(n)}(a)+f^{(n)}(mb)}{2}-\frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{2(mb-a)^{n-\frac{\alpha}{k}}}\left[({}^{C}D^{\alpha,k}_{a^+}f)(mb)+({}^{C}D^{\alpha,k}_{mb^-}f)(a)\right]$$
$$=\frac{(mb-a)^2}{2}\int_0^1\frac{1-(1-t)^{n-\frac{\alpha}{k}+1}-t^{n-\frac{\alpha}{k}+1}}{n-\frac{\alpha}{k}+1}f^{(n+2)}(ta+m(1-t)b)dt.$$

Proof. It suffices to note that

(16)
$$\int_0^1[(1-t)^{n-\frac{\alpha}{k}}-t^{n-\frac{\alpha}{k}}]f^{(n+1)}(ta++m(1-t)b)dt$$
$$=(mb-a)\int_0^1\frac{1-(1-t)^{n-\frac{\alpha}{k}+1}-t^{n-\frac{\alpha}{k}+1}}{n-\frac{\alpha}{k}+1}f^{(n+2)}(ta+m(1-t)b)dt.$$

One have

(17)

$$\begin{aligned}
&\int_0^1 [(1-t)^{n-\frac{\alpha}{k}} - t^{n-\frac{\alpha}{k}}] f^{(n+1)}(ta+m(1-t)b)dt \\
&= -\int_0^1 f^{(n+1)}(ta++m(1-t)b)d\frac{1-(1-t)^{n-\frac{\alpha}{k}+1} - t^{n-\frac{\alpha}{k}+1}}{n-\frac{\alpha}{k}+1} \\
&= -\frac{1-(1-t)^{n-\frac{\alpha}{k}+1} - t^{n-\frac{\alpha}{k}+1}}{n-\frac{\alpha}{k}+1} f^{(n+1)}(ta+m(1-t)b)dt\Big|_0^1 \\
&- (bm-a)\int_0^1 \frac{1-(1-t)^{n-\frac{\alpha}{k}+1} - t^{n-\frac{\alpha}{k}+1}}{n-\frac{\alpha}{k}+1} f^{(n+2)}(ta+m(1-t)b)dt \\
&= \frac{f^{(n+1)}(b) - f^{(n+1)}(a)}{n-\frac{\alpha}{k}+1} - (bm-a)\int_0^1 \frac{1-(1-t)^{n-\frac{\alpha}{k}+1} - t^{n-\frac{\alpha}{k}+1}}{n-\frac{\alpha}{k}+1} \\
&f^{(n+2)}(ta+m(1-t)b)dt.
\end{aligned}$$

and

$$\begin{aligned}
(18)\quad f^{(n+1)}(b) - f^{(n+1)}(a) &= \int_a^b f^{(n+2)}(x)dx \\
&= (bm-a)\int_0^1 f^{(n+2)}(ta++m(1-t)b)dt.
\end{aligned}$$

From (18) to (17), one obtain (16). □

Another Caputo $k-$fractional derivative inequalities of Hadamard type for $(h-m)-$convex function in terms of the $(n+2)$th derivatives in absolute is obtained as following by using above lemma.

THEOREM 3.5. *Let $f : [0,\infty) \to \mathbb{R}$ be a function such that $[a,b] \subset [0,\infty)$ and $f \in C^{n+2}[a,b]$. If $\left|f^{(n+2)}\right|$ is an $(h-m)-$convex with $m \in (0,1]$ and $h^q \in [0,1]$, $q > 1$. Then the following inequality for Caputo $k-$fractional derivatives holds*

$$\begin{aligned}
&\left|\frac{f^{(n)}(a)+f^{(n)}(mb)}{2} - \frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{2(mb-a)^{n-\frac{\alpha}{k}}}\left[({}^C D_{a^+}^{\alpha,k} f)(mb) + ({}^C D_{mb^-}^{\alpha,k} f)(a)\right]\right| \\
&\le \frac{(mb-a)^2}{2\left(n-\frac{\alpha}{k}+1\right)}\left(1-\frac{2}{p(\alpha+1)+1}\right)^{\frac{1}{p}}\left(\int_0^1 (h(t))^q dt\right)^{\frac{1}{q}}\left[\left|f^{(n+2)}(a)\right| + m\left|f^{(n+2)}(b)\right|\right].
\end{aligned}$$

Proof. Using Lemma 3.4 and $(h-m)-$ convexity of $|f^{(n+2)}|$, we find

$$\left|\frac{f^{(n)}(a)+f^{(n)}(mb)}{2}-\frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{2(mb-a)^{n-\frac{\alpha}{k}}}\left[({}^CD^{\alpha,k}_{a^+}f)(mb)+({}^CD^{\alpha,k}_{mb^-}f)(a)\right]\right|$$
$$\leq\frac{(mb-a)^2}{2}\int_0^1\frac{1-(1-t)^{n-\frac{\alpha}{k}+1}-t^{n-\frac{\alpha}{k}+1}}{n-\frac{\alpha}{k}+1}\left|f^{(n+2)}(ta+m(1-t)b)\right|dt$$
$$\leq\frac{(mb-a)^2}{2}\int_0^1\frac{1-(1-t)^{n-\frac{\alpha}{k}+1}-t^{n-\frac{\alpha}{k}+1}}{n-\frac{\alpha}{k}+1}\left[h(t)\left|f^{(n+2)}(a)\right|+mh(1-t)\left|f^{(n+2)}(b)\right|\right]dt$$

$$=\frac{(mb-a)^2}{2\left(n-\frac{\alpha}{k}+1\right)}\left\{\left|f^{(n+2)}(a)\right|\int_0^1(1-(1-t)^{n-\frac{\alpha}{k}+1}-t^{n-\frac{\alpha}{k}+1})h(t)dt\right.$$
$$\left.+\,m\left|f^{(n+2)}(b)\right|\int_0^1(1-(1-t)^{n-\frac{\alpha}{k}+1}-t^{n-\frac{\alpha}{k}+1})h(1-t)dt\right\}.$$

Now, using the Hölder inequality, we obtain

$$\left|\frac{f^{(n)}(a)+f^{(n)}(mb)}{2}-\frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{2(mb-a)^{n-\frac{\alpha}{k}}}\left[({}^CD^{\alpha,k}_{a^+}f)(mb)+({}^CD^{\alpha,k}_{mb^-}f)(a)\right]\right|$$
$$=\frac{(mb-a)^2}{2\left(n-\frac{\alpha}{k}+1\right)}\left\{\left|f^{(n+2)}(a)\right|\left(1-\frac{2}{p(\alpha+1)+1}\right)^{\frac{1}{p}}\left(\int_0^1(h(t))^qdt\right)^{\frac{1}{q}}\right.$$
$$\left.+\,m\left|f^{(n+2)}(b)\right|\left(1-\frac{2}{p(\alpha+1)+1}\right)^{\frac{1}{p}}\left(\int_0^1(h(1-t))^qdt\right)^{\frac{1}{q}}\right\}.$$

From which after a little computation we get the required result. □

Corollary 3.6. *In Theorem 3.5 if we take $k=1$, then we get the following inequality for $(h-m)-$convex function via Caputo fractional derivative*

$$\left|\frac{f^{(n)}(a)+f^{(n)}(mb)}{2}-\frac{\Gamma(n-\alpha+1)}{2(mb-a)^{n-\frac{\alpha}{k}}}\left[({}^CD^{\alpha}_{a^+}f)(mb)+({}^CD^{\alpha}_{mb^-}f)(a)\right]\right|$$
$$\leq\frac{(mb-a)^2}{2\left(n-\alpha+1\right)}\left(1-\frac{2}{p(\alpha+1)+1}\right)^{\frac{1}{p}}\left(\int_0^1(h(t))^qdt\right)^{\frac{1}{q}}\left[\left|f^{(n+2)}(a)\right|+m\left|f^{(n+2)}(b)\right|\right].$$

Acknowledgment: The authors thank the referees for their careful reading, valuable comments and suggestions for the improvement of the manuscript.

References

[1] F. Chen, *On Hermite-Hadamard type inequalities for Riemann-Liouville fractional integrals via two kinds of convexity*, Chin. J. Math. (2014) Article ID 173293, pp 7.

[2] G. Farid, A. Javed and A. U. Rehman, *On Hadamard inequalities for $n-$times differentiable functions which are relative convex via Caputo k-fractional derivatives*, Nonlinear Anal. Forum. **22** (2) (2017), 17–28.

[3] G. Farid, A. U. Rehman and M. Zahra, *On Hadamard inequalities for k-fractional integrals*, Nonlinear Funct. Anal. Appl. **21** (3) (2016), 463–478.

[4] R. Gorenflo and F. Mainardi, *Fractional calculus: Integral and differential equations of fractional order*, Springer Verlag, Wien, (1997).

[5] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and applications of fractional differential equations*, Elsevier, Amsterdam, (2006).

[6] M. Lazarević, *Advanced topics on applications of fractional calculus on control problems, System stability and modeling*, WSEAS Press, Belgrade, Serbia, (2012).

[7] S. Mubeen and G. M. Habibullah, *k-Fractional integrals and applications*, Int. J. Contemp. Math. Sci. **7** (2012), 89–94.

[8] K. Oldham, J. Spanier, *The fractional calculus theory and applications of differentiation and integration to arbitrary order*, Academic Press, New York-London (1974).

[9] M. E. Özdemir, A. O. Akdemri and E. Set, *On $(h-m)-$convexity and Hadamard-type inequalities*, Transylv. J. Math. Mech. **8** (1) (2016), 51–58.

[10] I. Podlubni, *Fractional differential equations*, Academic press, San Diego, (1999).

Lakshmi Narayan Mishra
Department of Mathematics, School of Advanced Sciences
Vellore Institute of Technology (VIT) University
Vellore 632 014, Tamil Nadu, India
E-mail: lakshminarayanmishra04@gmail.com, l_n_mishra@yahoo.co.in, lakshminarayan.mishra@vit.ac.in

Qurat Ul Ain
COMSATS University Islamabad, Attock Campus, Pakistan.
E-mail: math.annie2594@gmail.com

Ghulam Farid
COMSATS University Islamabad, Attock Campus, Pakistan.
E-mail: faridphdsms@hotmail.com, ghlmfarid@cuiatk.edu.pk

Atiq Ur Rehman
COMSATS University Islamabad, Attock Campus, Pakistan.
E-mail: atiq@mathcity.org