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CERTAIN RESULTS INVOLVING FRACTIONAL OPERATORS AND SPECIAL FUNCTIONS

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Abstract. In this study, the author provided a discussion on one dimensional Laplace and Fourier transforms with their applications. It is shown that the combined use of exponential operators and integral transforms provides a powerful tool to solve space fractional partial differential equation with non - constant coefficients. The object of the present article is to extend the application of the joint Fourier - Laplace transform to derive an analytical solution for a variety of time fractional non - homogeneous KdV. Numerous exercises and examples presented throughout the paper.

1. Introduction and Definitions

In this section, we introduce here a method which is free of disadvantages and suitable for a wide range of boundary value problems for fractional differential equations. The method uses the Laplace transform technique and is based on the properties of the Laplace transforms.

Definition 1.1. The Laplace transform of the function $f(t)$ is defined as follows

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt := F(s).$$

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If $\mathcal{L}\{f(t)\} = F(s)$, then $\mathcal{L}^{-1}\{F(s)\}$ is given by [3]

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}F(s)ds,$$

where $F(s)$ is analytic in the region $Re(s) > c$.

In the literature, there are many different definitions of the fractional derivative, all of which generalize on the usual integer order derivative. We will consider here the so called Riemann - Liouville and Caputo derivatives.

DEFINITION 1.2. If the function $\phi(t)$ belongs to $C[a,b]$ and $a < t < b$, then the left Riemann-Liouville fractional integral of order $\alpha > 0$ is defined as

$$I_a^{RL,\alpha}\{\phi(t)\} = \frac{1}{\Gamma(\alpha)}\int_a^t \frac{\phi(\xi)}{(t-\xi)^{1-\alpha}}d\xi. \qquad (1.3)$$

DEFINITION 1.3. The left Riemann-Liouville fractional derivative of order $\alpha > 0$ is defined as following [8,9].

$$D_a^{RL,\alpha}\phi(x) = \tfrac{1}{\Gamma(1-\alpha)}\tfrac{d}{dx}\int_a^x \tfrac{\Phi(\xi)}{(x-\xi)^{\alpha}}d\xi, \qquad (1.4)$$

it follows that $D_a^{RL,\alpha}\phi(x)$ exists for all $\phi(x)$ belongs to $C[a,b]$,and $a < x < b$.

NOTE: A very useful fact about the R- L operators is that they satisfy semi-group properties of fractional integrals. The special case of the fractional derivative when
$\alpha = 0.5$ is called semi-derivative.

DEFINITION 1.4. The left Caputo fractional derivative of order α $(0 < \alpha < 1)$ of $\phi(x)$ is defined as follows [8,9]

$$D_a^{C,\alpha}\phi(x) = \tfrac{1}{\Gamma(1-\alpha)}\int_a^x \tfrac{1}{(x-\xi)^{\alpha}}\phi'(\xi)d\xi. \qquad (1.5)$$

In recent years, a growing number of works by many researchers from various fields of sciences and engineering deal with fractional differential and integral equations which means equations involving derivatives and integrals of non - integer order. These new models are more adequate than the previously used integer order models. One of the most interesting applications of the Laplace transforms is solving linear differential equations with discontinuous forcing functions which are common place in mechanical systems and circuit analysis problems. Recently [10], the authors used Yang- Laplace transform method to solve Volterra and Abels integro-differential equations of fractional order.

EXAMPLE 1.1 Let us consider the following fractional differential equation under non-zero initial condition

$$D_t^{C,\alpha} y(t) + \beta y(t) = \lambda \chi_{[0,a]}, \qquad y(0) = 1, t > 0, \quad \alpha = 0.5.$$

NOTE. Observe that the characteristic function $\chi_{[0,a]}$ is discontinuous but not differentiable at $t = a$.
The above fractional equation has the following formal solution.

$$y(t) = e^{\beta^2 t} Erfc(\beta\sqrt{t}) + \frac{\lambda}{\beta} - \frac{\lambda e^{\beta^2 t}}{\beta} Erfc(\beta\sqrt{t}) - .....$$

$$.. - \lambda \int_a^t (\frac{1}{\sqrt{\pi(t-\xi)}} - a e^{a^2(t-\xi)} Erfc(a(\sqrt{t-\xi})))d\xi.$$

Solution. Let us take the Laplace transform of the above fractional differential equation term wise and using boundary condition, we get the following

$$s^\alpha Y(s) - s^{\alpha-1} + \beta Y(s) = \frac{\lambda(1-e^{-as})}{s},$$

let us put $\alpha = 0.5$ we arrive at

$$(\sqrt{s} + \beta) Y(s) = \frac{1}{\sqrt{s}} + \frac{\lambda}{s}(1 - e^{-as}).$$

Solving the above equation, yields

$$Y(s) = \frac{1}{\sqrt{s}(\sqrt{s}+\beta)} + \frac{\lambda}{s(\sqrt{s}+\beta)} - \frac{\lambda e^{-as}}{s(\sqrt{s}+\beta)}.$$

At this stage, taking the inverse Laplace transform of the above relation, leads to

$$y(t) = e^{\beta^2 t} Erfc(\beta\sqrt{t}) + \frac{\lambda}{\beta} - \frac{\lambda e^{\beta^2 t}}{\beta} Erfc(\beta\sqrt{t}) - .....$$

$$.. - \lambda \int_a^t (\frac{1}{\sqrt{\pi(t-\xi)}} - a e^{a^2(t-\xi)} Erfc(a(\sqrt{t-\xi})))d\xi.$$

Let us evaluate the Laplace transform of the Krätzel function $Z^{\nu}_{\rho}(\xi)$.

Definition 1.5. By definition, Krätzel function $Z^{\nu}_{\rho}(\xi)$ is as follows [9]

$$Z^{\nu}_{\rho}(\xi) = \int_0^{\infty} \frac{e^{-u^{\rho}-\frac{\xi}{u}}}{u^{1-\nu}} du.$$

Krätzel function was introduced by E.Krätzel as a kernel of the integral transform

$$(K^{\nu}_{\rho}\phi)(\xi) = \int_0^{\infty} Z^{\nu}_{\rho}(\xi t)\phi(t)dt.$$

We note that the Krätzel function occurs in the study of astrophysical thermonuclear functions, which are derived on the basis of Boltzmann-Gibbs statistical mechanics. It is also important to note that, the Krätzel function $Z^{\nu}_{1}(\xi)$ is related to the modified Bessel function of second kind K_{ν} or Macdonald's function. Note that this function is useful in chemical physics. Some authors, deduced explicit forms of Krätzel function in terms of the generalized Wright function,[8].

The Laplace transform of the Krätzel function is defined as follows

$$\mathcal{L}\{Z^{\nu}_{\rho}(\xi)\} = \int_0^{\infty} e^{-s\xi}(\int_0^{\infty} \frac{e^{-u^{\rho}-\frac{\xi}{u}}}{u^{1-\nu}} du)d\xi,$$

by changing the order of integration, the following relationship will be obtained

$$\mathcal{L}\{Z^{\nu}_{\rho}(\xi)\} = \int_0^{\infty} \frac{e^{-u^{\rho}}}{u^{1-\nu}}(\frac{u}{su+1})du = \int_0^{\infty} (\frac{u^{\nu}e^{-u^{\rho}}}{su+1})du.$$

Let us take $s = 0$, after integration and simplifying, we arrive at

$$\mathcal{L}\{Z^{\nu}_{\rho}(\xi)\}_{s=0} = \int_0^{\infty} Z^{\nu}_{\rho}(\xi)d\xi = \int_0^{\infty} (u^{\nu}e^{-u^{\rho}})du = \frac{1}{\rho}\Gamma(\frac{\nu+1}{\rho}).$$

At this point, if we differentiate the above relation with respect to ν, we get

$$\int_0^{\infty} Z^{*,\nu}_{\rho}(\xi)d\xi = \frac{1}{\rho^2}\Gamma'(\frac{\nu+1}{\rho}).$$

1. In special case $\rho = 1$ after simplifying,

$$\int_0^\infty Z_1^{*,\nu}(\xi)d\xi = \Gamma'(1+\nu).$$

2. In special case $\nu = 0$ and $\rho = 1$ after simplifying, we get the following integral representation for the Euler constant γ

$$\int_0^\infty Z_1^{*,0}(\xi)d\xi = \Gamma'(1) = -\gamma.$$

Note: In the above relation $\gamma = 0.57721566....$, stands for the Euler - Mascheroni constant.

2. Evaluation of Integrals by Means of The Laplace Transform

The integral transform technique is one of the most useful tools of applied mathematics employed in many branches of engineering and science.

Problem 2.1. Let us evaluate the following integrals

$$(1) \quad I_1 = \int_0^\infty \frac{e^{-\beta x}\cos mx}{x^\rho}dx \quad , I_2 = \int_0^\infty \frac{e^{-\beta x}\sin mx}{x^\rho}dx \quad , 0 < \rho < 1.$$

Solution. Let us define the following integral

$$I = I_1 - iI_2 = \int_0^\infty e^{-(\beta+im)x}x^{-\rho}dx.$$

Recall the definition of the Laplace transform, it is easily verified that

$$I = \mathcal{L}\{x^{-\rho}; s = \beta + im\} = \frac{\Gamma(1-\rho)}{(\beta+im)^{1-\rho}} = \Gamma(1-\rho)e^{(\rho-1)log(\beta+im)}.$$

The last equation holds because $\xi^b = e^{b\ln\xi}$, on the other hand we know that
$\mathcal{L}og(\beta + im) = \ln(\sqrt{m^2+\beta^2}) + i\arctan(\frac{m}{\beta})$ then we obtain

$$I = \Gamma(1-\rho)e^{(\rho-1)\ln(\sqrt{m^2+\beta^2})+i(\rho-1)\arctan(\frac{m}{\beta})},$$

consequently

$$I_1 = \Gamma(1-\rho)e^{(\rho-1)\ln(\sqrt{m^2+\beta^2})}\cos((\rho-1)\arctan(\frac{m}{\beta})),$$

and

$$I_2 = \Gamma(1-\rho)e^{(\rho-1)\ln(\sqrt{m^2+\beta^2})}\sin((\rho-1)\arctan(\frac{m}{\beta})).$$

Let us consider the special case, $\rho = \frac{1}{n}$, $\beta = 0$, we have

$$I_1 = \int_0^\infty \frac{\cos mx}{x^{\frac{1}{n}}}dx = \Gamma(1-\frac{1}{n})m^{-1+\frac{1}{n}}\cos((1-\frac{1}{n})\frac{\pi}{2}) = \Gamma(1-\frac{1}{n})m^{-1+\frac{1}{n}}sin(\frac{\pi}{2n}),$$

and

$$I_2 = \int_0^\infty \frac{\sin mx}{x^{\frac{1}{n}}}dx = -\Gamma(1-\frac{1}{n})m^{-1+\frac{1}{n}}\sin((1-\frac{1}{n})\frac{\pi}{2}) = \Gamma(1-\frac{1}{n})m^{-1+\frac{1}{n}}cos(\frac{\pi}{2n}).$$

Kelvin functions: Kelvin functions $ber(x)$ and $bei(x)$, are defined as [3]

$$ber(x) = \mathcal{R}eJ_0(i\sqrt{i}x)\ ,\ bei(x) = \mathcal{I}mJ_0(i\sqrt{i}x).$$

We know that $\mathcal{L}\{J_0(2\sqrt{at}); t \to s\} = \frac{e^{-\frac{a}{s}}}{s}$ let $a = i^3$ to get $\mathcal{L}\{bei(2\sqrt{t})\} = Im\left(\frac{e^{-\frac{i^3}{s}}}{s}\right) = \frac{1}{s}\sin\frac{1}{s}$, $\qquad \mathcal{L}\{ber(2\sqrt{t})\} = Re\left(\frac{e^{-\frac{i^3}{s}}}{s}\right) = \frac{1}{s}\cos\frac{1}{s}$.

Example 2.1. The following integral identity holds true

$$I = \int_0^\infty e^{-\beta t^2} ber(2\sqrt{t})dt = \frac{1}{2}\sqrt{\frac{\pi}{\beta}}cos(\frac{1}{2\sqrt{\beta}}). \tag{2}$$

Solution. Let us define the following integral

$$I(x) = \int_0^\infty e^{-\beta t^2} ber(2\sqrt{xt})dt.$$

Taking the Laplace transform w.r.t x, and using table of the Laplace transform leads to

$$\mathcal{L}\{I(x); x \to s\} = \frac{1}{s}\int_0^\infty e^{-\beta t^2}\cos\frac{t}{s}dt,$$

consequently, by evaluation of the above integral, we get $\mathcal{L}\{I(x)\} = \frac{1}{2s}\sqrt{\frac{\pi}{\beta}}e^{-\frac{1}{4\beta s^2}}$, using the fact that $\mathcal{L}\{x^{p-1}\} = \frac{\Gamma(p)}{s^p}$, we arrive at

$$I(x) = \mathcal{L}^{-1}\{\frac{1}{2}\sqrt{\frac{\pi}{\beta}}\sum_{k=0}^{+\infty}\frac{(-1)^k}{s(4\beta s^2)^k}\} = \frac{1}{2}\sqrt{\frac{\pi}{\beta}}\sum_{k=0}^{+\infty}\frac{(-1)^k x^{2k}}{(4\beta)^k\Gamma(2k+1)} = \frac{1}{2}\sqrt{\frac{\pi}{\beta}}cos(\frac{x}{2\sqrt{\beta}}).$$

Now, let us choose $x = 1$ to get the desired identity as below

$$I = \frac{1}{2}\sqrt{\frac{\pi}{\beta}}\sum_{k=0}^{+\infty}\frac{(-1)^k}{(4\beta)^k\Gamma(2k+1)} = \frac{1}{2}\sqrt{\frac{\pi}{\beta}}cos(\frac{1}{2\sqrt{\beta}}).$$

LEMMA 2.1. *The following exponential operator relations hold true*

1. $\exp(\pm\lambda\frac{d}{dt})\Psi(t) = \Psi(t\pm\lambda).$
2. $\exp(\pm\lambda t\frac{d}{dt})\Psi(t) = \Psi(te^{\pm\lambda}).$
3. $\exp(\pm\lambda\frac{d}{tdt}\Psi(t)) = \Psi(\sqrt{t^2 \pm \lambda^2}).$
4. $\exp(\lambda q(t)\frac{d}{dt})\Psi(t) = \Psi(Q(F(t)+\lambda)).$

Where $F(t)$ is a primitive function of $(q(t))^{-1}$ and $Q(t)$ is the inverse function of $F(t)$.

Proof. See[4,5,6] □

Lemma 2.2. Let us assume that $\mathcal{L}(\phi(t)) = \Phi(s)$, then we have the following relations

1. $e^{-\omega s^\beta} = \frac{1}{\pi}\int_0^\infty e^{-r^\beta(\omega cos\beta\pi)}sin(\omega r^\beta sin\beta\pi)(\int_0^\infty e^{-s\tau - r\tau}d\tau)dr.$
2. $\mathcal{L}^{-1}(\Phi(\sqrt{s}) = \frac{1}{2t\sqrt{\pi t}}\int_0^\infty ue^{-\frac{u^2}{4t}}\phi(u)du.$
3. $e^{-k\sqrt{s}} = \int_0^\infty \frac{k}{2\xi\sqrt{\pi\xi}}e^{-s\xi - \frac{k^2}{4\xi}}d\xi.$

Proof. See [1,2]. □

Lemma 2.3. *The following exponential identity holds true.*
1. $\frac{1}{\sqrt{\frac{1}{2\xi}\frac{d}{d\xi}}}\exp(\frac{-\lambda^2}{2\xi}\frac{d}{d\xi})\phi(\xi) = \frac{1}{\sqrt{\pi}}\int_{\lambda^2}^{\infty}\frac{1}{\sqrt{x-\lambda^2}}\phi(\sqrt{\xi^2-x}))dx.$

Proof. Let us consider the following elementary integral

$$\int_{\lambda^2}^{\infty}\frac{e^{-qx}}{\sqrt{x-\lambda^2}}dx = \frac{\sqrt{\pi}}{\sqrt{q}}e^{-\lambda^2 q}, \tag{3}$$

in the above integral, let us choose $q = \frac{1}{2\xi}\frac{d}{d\xi}$, we get the following relation

$$\frac{\sqrt{\pi}}{\sqrt{\frac{1}{2\xi}\frac{d}{d\xi}}}e^{\frac{-\lambda^2}{2\xi}\frac{d}{d\xi}}\phi(\xi) = \int_{\lambda^2}^{\infty}dx\frac{e^{-\frac{x}{2\xi}\frac{d}{d\xi}}}{\sqrt{x-\lambda^2}}\phi(\xi), \tag{4}$$

after simplifying, we arrive at

$$\frac{1}{\sqrt{\frac{1}{2\xi}\frac{d}{d\xi}}}\exp(\frac{-\lambda^2}{2\xi}\frac{d}{d\xi})\phi(\xi) = \frac{1}{\sqrt{\pi}}\int_{\lambda^2}^{\infty}\frac{1}{\sqrt{x-\lambda^2}}\phi(\sqrt{\xi^2-x}))dx. \tag{5}$$

□

Problem 2.2. Let us solve the following space fractional linear PDE with non - constant coefficients by means of the fractional exponential operator method

$$\sqrt{x}D_x^{R.L,0.5}u - \beta t^{\beta-1}u + u_t = 0,$$
$$u(x,0) = \phi(x).$$

Solution. In order to solve the above space fractional PDE, by solving the first order FPDE with respect to t, we get the formal solution as follows

$$u(x,t) = e^{t^\beta}e^{-t\sqrt{x}D_x^{R.L,0.5}}\phi(x),$$

the right hand side of the above equation can be simplified using the third identity of the Lemma 2.2, then, by replacing $s = D_x$, $k = t\sqrt{x}$, we get the solution as below

$$u(x,t) = e^{t^\beta}\int_0^{\infty}\frac{t\sqrt{x}}{2\xi\sqrt{\pi\xi}}e^{-\xi\partial_x-\frac{(t\sqrt{x})^2}{4\xi}}d\xi\phi(x),$$

after simplifiying, we arrive at

$$u(x,t) = e^{t^\beta}\int_0^{\infty}\frac{t\sqrt{x}}{2\xi\sqrt{\pi\xi}}e^{-\frac{xt^2}{4\xi}}\phi(x-\xi)d\xi.$$

LEMMA 2.4. The following exponential identity holds true

1. $\exp(-\lambda\sqrt{\frac{d}{dx}})\Psi(x) = \frac{\lambda}{\sqrt{\pi}}\int_0^\infty rJ_0(\lambda r)(\int_0^\infty \frac{e^{-\eta r^2}}{\sqrt{\eta}}\Psi(x-\eta)d\eta)dr.$

Proof. Let us start with the following Laplace transform identity

$$\mathcal{L}(J_0(rt)) = \int_0^\infty e^{-pt}J_0(rt)dt = \frac{1}{\sqrt{p^2+r^2}}.$$

The left hand side can be rewritten as Hankel transform of the exponential function of order zero as below

$$\int_0^\infty \frac{e^{-pt}}{t}tJ_0(rt)dt = \mathcal{H}_0(\frac{e^{-pt}}{t}) = \frac{1}{\sqrt{p^2+r^2}},$$

upon inversion of the Hankel transforms of order zero, we get the following

$$\int_0^\infty \frac{1}{\sqrt{r^2+p^2}}rJ_0(tr)dr = \frac{e^{-pt}}{t}.$$

At this stage, let us put $p^2 = s$ in the above relation to obtain

$$e^{-t\sqrt{s}} = t\int_0^\infty rJ_0(tr)(r^2+s)^{-\frac{1}{2}}dr,$$

in the above identity, let us choose $s = \frac{d}{dx}$ and $t = \lambda$, then we get the following exponential identity

$$e^{-\lambda\sqrt{\frac{d}{dx}}}\Psi(x) = \lambda\int_0^\infty rJ_0(\lambda r)((r^2+\frac{d}{dx})^{-\frac{1}{2}}\Psi(x))dr,$$

in order to find the result of the action of the operator over the function, we use the following well known elementary identity

$$\xi^{-\nu} = \frac{1}{\Gamma(\nu)}\int_0^\infty e^{-\xi u}u^{\nu-1}du, \qquad \nu > 0,$$

by choosing $\nu = \frac{1}{2}$ and $\xi = r^2 + \frac{d}{dx}$, and in view of the Lemma 2.1. we get the following relation

$$\exp(-\lambda\sqrt{\frac{d}{dx}})\Psi(x) = \frac{\lambda}{\sqrt{\pi}}\int_0^\infty rJ_0(\lambda r)(\int_0^\infty \frac{e^{-\eta r^2}}{\sqrt{\eta}}\Psi(x-\eta)d\eta)dr.$$

□

THEOREM 2.1. The following exponential identity holds true

1. $$\exp(-\lambda\sqrt{\frac{d}{dx}})\Psi(x) = \frac{\lambda^{1-\nu}2^{\nu}}{\sqrt{\pi}}\int_0^\infty r^{\nu+1}J_\nu(\lambda r)(\int_0^\infty \frac{e^{-\eta r^2}}{\eta^{0.5-\nu}}\Psi(x-\eta)d\eta)dr.$$

Proof. Let us start with the following Laplace transform identity

$$\mathcal{L}(t^\nu J_\nu(rt)) = \int_0^\infty e^{-pt}t^\nu J_\nu(rt)dt = \frac{(2r)^\nu\Gamma(\nu+0.5)}{\sqrt{\pi}(p^2+r^2)^{\nu+0.5}}.$$

The left hand side can be rewritten as Hankel transform of exponential function of order ν as below

$$\int_0^\infty \frac{e^{-pt}}{t^{1-\nu}}tJ_\nu(rt)dt = \mathcal{H}_\nu(\frac{e^{-pt}}{t^{1-\nu}}) = \frac{2^\nu r^\nu\Gamma(\nu+0.5)}{\sqrt{\pi}(p^2+r^2)^{\nu+0.5}},$$

upon inversion of the Hankel transform of order ν, we get the following

$$\int_0^\infty \frac{2^\nu r^\nu\Gamma(\nu+0.5)}{\sqrt{\pi}(r^2+p^2)^{\nu+0.5}}rJ_\nu(tr)dr = \frac{e^{-pt}}{t^{1-\nu}}.$$

At this point, let us put $p^2 = s$ in the above relation to get

$$e^{-t\sqrt{s}} = t^{1-\nu}\frac{2^\nu\Gamma(\nu+0.5)}{\sqrt{\pi}}\int_0^\infty r^{\nu+1}J_\nu(tr)(r^2+s)^{-(\nu+\frac{1}{2})}dr,$$

in the above identity, let us choose $s = \frac{d}{dx}$ and $t = \lambda$, then we get the following exponential identity

$$e^{-\lambda\sqrt{\frac{d}{dx}}}\Psi(x) = \lambda^{1-\nu}\frac{2^\nu\Gamma(\nu+0.5)}{\sqrt{\pi}}\int_0^\infty r^{\nu+1}J_\nu(\lambda r)((r^2+\frac{d}{dx})^{-(\nu+\frac{1}{2})}\Psi(x))dr,$$

in order to find the result of the action of the operator over the function, we use the following well known elementary identity

$$\xi^{-\delta} = \frac{1}{\Gamma(\delta)}\int_0^\infty e^{-\xi\eta}\eta^{\delta-1}d\eta, \qquad \delta > 0,$$

by choosing $\delta = \nu + \frac{1}{2}$ and $\xi = r^2 + \frac{d}{dx}$, in view of the Lemma 2.1. we have the following

$$\exp(-\lambda\sqrt{\frac{d}{dx}})\Psi(x) = \frac{\lambda^{1-\nu}2^{\nu}}{\sqrt{\pi}}\int_0^\infty r^{\nu+1}J_\nu(\lambda r)(\int_0^\infty \frac{e^{-\eta r^2}}{\eta^{0.5-\nu}}\Psi(x-\eta)d\eta)dr.$$

$\square$

Note. The Hankel transforms arise naturally in solving boundary value problems formulated in cylindrical coordinates. They also occur in other applications such as determining the oscillations of heavy chain suspended from one end.

Solution to Singular Integral Equation with Kernel of Kelvin's Functions.

Problem 2.3. Let us consider the following singular integral equation

$$\sqrt{\frac{2}{\pi}} \int_0^{+\infty} \phi(t)\; bei(2\sqrt{\xi t})dt = J_0(2\sqrt{k\xi}),$$

the formal solution is as follows

$$\phi(t) = \sqrt{\frac{2}{\pi}} \frac{t}{t^2 + k^2}.$$

Solution. By taking the Laplace transform of each term in the above equation, we find

$$\sqrt{\frac{2}{\pi}} \int_0^{+\infty} \phi(t)\; \mathcal{L}\{bei(2\sqrt{\xi t}); \xi -> s\}dt = \mathcal{L}\{J_0(2\sqrt{k\xi}) : \xi -> s\},$$

or

$$\sqrt{\frac{2}{\pi}} \int_0^{+\infty} \phi(t)\; \{\frac{1}{s} \sin \frac{t}{s}\}dt = \frac{e^{-\frac{k}{s}}}{s}.$$

For the sake of simplicity, let us take

$$\frac{1}{s} = \omega.$$

Then we get

$$\sqrt{\frac{2}{\pi}} \int_0^{+\infty} \phi(t)\; \sin \omega t\, dt = e^{-k\omega},$$

the left hand side by definition is the Fourier sine transform of certain functions, so that

$$\mathcal{F}_s\{\phi(t); t -> \omega\} = e^{-k\omega}.$$

Using the inversion formula for the Fourier sine transform to obtain

$$\phi(t) = \sqrt{\frac{2}{\pi}} \int_0^{+\infty} e^{-k\omega} \sin(t\omega)d\omega,$$

or

$$\phi(t) = \sqrt{\frac{2}{\pi}} \frac{t}{t^2 + k^2}.$$

Corollary 2.1. *We have the following integral representation for the Bessel's function of order zero.*

$$J_0(2\sqrt{k\xi}) = \frac{2}{\pi}\int_0^{+\infty} \frac{t}{t^2+k^2}\, bei(2\sqrt{\xi t})dt.$$

Let us solve the following partial differential equation with non - constant coefficients by means of exponential operators method.

Example 2.2. Let us solve the following initial value problem

$$u_t = \alpha\lambda t^{\lambda-1}u - \frac{\beta}{x}u_x, \qquad \alpha,\beta > 0, \lambda > 1, \qquad u(x,0) = \psi(x).$$

Solution. In order to solve the above PDE, we can rewrite the above PDE in the following form

$$\frac{\partial u}{\partial t} = (\alpha\lambda t^{\lambda-1} - \frac{\beta}{x}\frac{\partial}{\partial x})u,$$

by solving the first order PDE with respect to t, we get the formal solution as follows

$$u(x,t) = e^{\alpha t^\lambda} e^{-\frac{\beta t}{x}\frac{d}{dx}}\psi(x),$$

the right hand side of the above equation can be treated using third part of the Lemma 2.1.

$$u(x,t) = e^{\alpha t^\lambda}\psi(\sqrt{x^2-\beta^2 t^2}).$$

Note. It is easy to verify that

$$u(x,0) = \psi(x).$$

3. Stieltjes transform

Definition 3.1. The Stieltjes transform is defined as follows [7]

$$\mathcal{S}\{f(t),s\} = \int_0^\infty \frac{f(t)dt}{t+s}.$$

It is well known that the *second iterate* of the Laplace transform is the Stieltjes transform, that is

$$\mathcal{L}^2\{f(t);s\} = \mathcal{L}\{\mathcal{L}\{f(t);u\},s\} = \mathcal{S}\{f(t),s\} = F(s).$$

The complex inversion formula for the Stieltjes transform is defined as follows

$$\begin{aligned} f(t) =&\mathcal{S}^{-1}\{F(s), t\} = \mathcal{L}^{-1}\{\mathcal{L}^{-1}\{F(p); s\}, t\} \\ =&\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}\left(\frac{1}{2\pi i}\int_{c'-i\infty}^{c'+i\infty} F(p)e^{ps}dp\right) ds. \end{aligned}$$

EXAMPLE 3.1. The following identity holds true

$$\mathcal{L}\{Z_\rho^\nu(\xi)\} = \frac{1}{s}\mathcal{S}\{u^\nu e^{-u^\rho} : \frac{1}{s}\}.$$

SOLUTION.

$$\mathcal{L}\{Z_\rho^\nu(\xi)\} = \int_0^\infty \frac{e^{-u^\rho}}{u^{1-\nu}}(\frac{u}{su+1})du = \int_0^\infty (\frac{u^\nu e^{-u^\rho}}{su+1})du = \frac{1}{s}\mathcal{S}\{u^\nu e^{-u^\rho} : \frac{1}{s}\}.$$

LEMMA 3.1. *Let us assume that* $\mathcal{L}f(t) = F(s)$, $\mathcal{L}[g(t)] = G(s)$ *and* $\mathcal{S}f(t) = H(r)$, *then we have the following integral relation*

$$\int_0^\infty F(s)G(s)ds = \int_0^\infty g(r)H(r)dr.$$

Provided that all integrals involved converge absolutely.

Proof. It is not difficult to verify. □

NOTE.The above Lemma has an interesting application as below

EXAMPLE 3.2. Let us take $f(t) = J_0(2\sqrt{\alpha t})$, $g(t) = (\frac{t}{\beta})^{\frac{\nu}{2}}I_\nu(2\sqrt{\beta t})$, then we have $\mathcal{L}(J_0(2\sqrt{\alpha t})) = F(s) = \frac{e^{-\frac{\alpha}{s}}}{s}$, $\mathcal{L}((\frac{t}{\beta})^{\frac{\nu}{2}}I_\nu(2\sqrt{\beta t})) = G(s) = \frac{e^{\frac{\beta}{s}}}{s^{\nu+1}}$ and
$H(r) = \mathcal{S}f(t) = K_0(2\sqrt{\alpha r})$, by setting the above information in the Lemma 3.1., we arrive at

$$\int_0^\infty (\frac{e^{-\frac{\alpha}{s}}}{s})(\frac{e^{\frac{\beta}{s}}}{s^{\nu+1}})ds = \int_0^\infty (\frac{r}{\beta})^{\frac{\nu}{2}}I_\nu(2\sqrt{\beta r})K_0(2\sqrt{\alpha r})dr.$$

After evaluating the first integral, we obtain

$$\int_0^\infty (\frac{r}{\beta})^{\frac{\nu}{2}}I_\nu(2\sqrt{\beta r})K_0(2\sqrt{\alpha r})dr = \frac{\Gamma(\nu+1)}{(\alpha-\beta)^{\nu+1}}, \qquad ,\alpha > \beta \geq 0.$$

4. Main Results

In [8]the authors established explicit solutions of Cauchy type problems for fractional diffusion - wave partial differential equations involving the Riemann - Liouville fractional derivatives of order $\alpha > 0$. They also considered fractional differential equations involving the partial Caputo fractional derivative with respect to time, t and the Laplacian with respect to x, with order $n-1 \leq \alpha < n$. They applied the joint Laplace-Fourier integral transforms to construct analytic solutions of Cauchy type and Cauchy problems for fractional diffusion - wave and evolution equations.

Solution to Time Fractional Linearized KdV via The Joint Laplace- Fourier Transform.

The KdV equations are attracting many researchers, and a great deal of works has already been done in some of these equations. In this section, we will implement the joint Laplace- Fourier transforms to construct an exact solution for a variety of the KdV equation with the time fractional derivative in the Caputo sense.
To the best of our knowledge, this kind of Kdv equation has not been studied in any detail.

Problem 4.1. Solving the following time fractional non-homogeneous linearized KdV, is not yet considered

$$\begin{aligned}&{}^cD_t^{\frac{1}{2}}u + \alpha u + \beta u_x + \gamma u_{xxx} = \phi(x)\\ &u(x,0) = f(x).\end{aligned}$$

Solution. By taking the joint Laplace - Fourier transform of equation and using boundary condition, we get the following transformed equation

$$\hat{\bar{U}}(w,s) = \frac{1}{\sqrt{s}}\frac{F(w)}{\sqrt{s} + (iw\beta - i\gamma w^3 - \alpha)} + \frac{1}{s}\frac{\Phi(w)}{\sqrt{s} + (iw\beta - i\gamma w^3 - \alpha)}.$$

For the sake of simplicity, let us assume that $\tau = iw\beta - i\gamma w^3 - \alpha$, and using the inverse Laplace transform of the transformed equation to obtain

$$\hat{U}(w,t) = \mathcal{L}^{-1}\{\frac{1}{\sqrt{s}}\frac{F(w)}{\sqrt{s}+\tau} + \frac{\Phi(w)}{s(\sqrt{s}+\tau)}; s-> t\},$$

or

$$\hat{U}(w,t) = (\frac{1}{\sqrt{\pi t}})*\frac{F(w)}{2t\sqrt{\pi t}}\int_0^{+\infty} e^{-(\frac{\xi^2}{4t}+\tau\xi)}d\xi + h(t)*\frac{\Phi(w)}{2t\sqrt{\pi t}}\int_0^{+\infty} e^{-(\frac{\xi^2}{4t}+\tau\xi)}d\xi.$$

At this stage, inverting the Fourier transform to get

$$u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{(-ixw)}\{(\frac{1}{\sqrt{\pi t}}) * \frac{F(w)}{2t\sqrt{\pi t}}\int_0^{+\infty} e^{-(\frac{\xi^2}{4t}+\tau\xi)}d\xi\}dw +$$

$$+\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{(-ixw)}\{h(t) * \frac{\Phi(w)}{2t\sqrt{\pi t}}\int_0^{+\infty} e^{-(\frac{\xi^2}{4t}+\tau\xi)}d\xi\}dw.$$

By setting $\tau = iw\beta - i\gamma w^3 - \alpha$, and changing the order of integration, we have

$$u(x,t) = (\frac{1}{\sqrt{\pi t}})*\frac{1}{2t\sqrt{\pi t}}\int_0^{+\infty} e^{-(\frac{\xi^2}{4t})}\{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} F(w)e^{-(ixw-\tau\xi)}\, dw\}d\xi +$$

$$+h(t) * \frac{1}{2t\sqrt{\pi t}}\int_0^{+\infty} e^{-(\frac{\xi^2}{4t})}\{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \Phi(w)e^{-(ixw-\tau\xi)}\, dw\}d\xi,$$

equivalently

$$u(x,t) = (\frac{1}{\sqrt{\pi t}})*\frac{1}{2t\sqrt{\pi t}}\int_0^{+\infty} e^{-(\frac{\xi^2}{4t}+\alpha\xi)}\{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} F(w)e^{-i(x+\xi\beta)w}e^{i(\gamma\xi)w^3}\, dw\}d\xi$$

$$+h(t) * \frac{1}{2t\sqrt{\pi t}}\int_0^{+\infty} e^{-(\frac{\xi^2}{4t}+\alpha\xi)}\{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \Phi(w)e^{-i(x+\xi\beta)w}e^{i(\gamma\xi)w^3}\, dw\}d\xi.$$

The interior integrals can be evaluated by convolution for the Fourier transform as below

$$u(x,t) = (\frac{1}{\sqrt{\pi t}})*\frac{1}{2t\sqrt{\pi t}}\int_0^{+\infty} e^{-(\frac{\xi^2}{4t}+\alpha\xi)}\{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \frac{f(x+\beta\xi-\varphi)}{\sqrt[3]{3\gamma\xi}}Ai(\frac{\varphi+\beta\xi}{\sqrt[3]{3\gamma\xi}})\, d\varphi\}d\xi.$$

$$+h(t)*\frac{1}{2t\sqrt{\pi t}}\int_0^{+\infty} e^{-(\frac{\xi^2}{4t}+\alpha\xi)}\{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \frac{\phi(x+\beta\xi-\varphi)}{\sqrt[3]{3\gamma\xi}}Ai(\frac{\varphi+\beta\xi}{\sqrt[3]{3\gamma\xi}})\, d\varphi\}d\xi.$$

Note: Let us consider the special case

$$\beta = \alpha = 0, \gamma = \frac{1}{3}.$$

We get the simple standard time fractional non homogeneous KdV,

$$\begin{array}{l} {}^cD_t^{\frac{1}{2}}u + \frac{1}{3}u_{xxx} = \phi(x), \\ u(x,0) = f(x), \end{array}$$

with the following solution

$$u(x,t) = (\frac{1}{\sqrt{\pi t}})*\frac{1}{2t\sqrt{\pi t}}\int_0^{+\infty} e^{-(\frac{\xi^2}{4t})}\{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\frac{f(x-\varphi)}{\sqrt[3]{\xi}}Ai(\frac{\varphi}{\sqrt[3]{\xi}})\,d\varphi\}d\xi$$

$$+h(t)*\frac{1}{2t\sqrt{\pi t}}\int_0^{+\infty} e^{-(\frac{\xi^2}{4t})}\{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\frac{\phi(x-\varphi)}{\sqrt[3]{\xi}}Ai(\frac{\varphi}{\sqrt[3]{\xi}})\,d\varphi\}d\xi.$$

5. Conclusion

The paper is devoted to study the Laplace, Stieltjes integral transforms and their applications in evaluating the integrals. The one dimensional Laplace and Fourier transforms provide a powerful method for analyzing linear systems. The main purpose of this work is to develop methods for evaluating some special integrals, and solution to a variety of non homogeneous KdV equation with fractional derivative. We hope that it will also benefit many researchers in the disciplines of engineering, applied mathematics, and mathematical physics.

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