Korean J. Math. **27** (2019), No. 4, pp. 831–841
https://doi.org/10.11568/kjm.2019.27.4.831

OPTIMAL RADIOCOLORING OF TREES

Xiaoling Zhang

Abstract. A Radiocoloring (RC) of a graph G is a function f from the vertex set $V(G)$ to the set of all non-negative integers (labels) such that $|f(u) - f(v)| \geq 2$ if $d(u, v) = 1$ and $|f(u) - f(v)| \geq 1$ if $d(u, v) = 2$. The number of discrete labels and the range of labels used are called order and span, respectively. In this paper, we concentrate on the minimum order span Radiocoloring Problem (RCP) of trees. The optimization version of the minimum order span RCP that tries to find, from all minimum order assignments, one that uses the minimum span. We provide attainable lower and upper bounds for trees. Moreover, a complete characterization of caterpillars (as a subclass of trees) with the minimum order span is given.

1. Introduction

The Frequency Assignment Problem (FAP) [1] in radio networks is a well-studied, interesting problem, aiming at assigning frequencies to transmitters exploiting frequency reuse while keeping signal interference to acceptable levels. The FAP is, in many cases, studied as a graph coloring problem, in which the vertices represent transmitters, the edges represent interference between two transmitters and the colors represent the frequencies. A *Radiocoloring* (RC) of a graph G is a function f from

---

Received November 31, 2018. Revised Ocober 1, 2019. Accepted November 26, 2019.

2010 Mathematics Subject Classification: 05C15.

Key words and phrases: radiocoloring; order; span; tree; caterpillar.

This research is supported by NSFC No. 11601265 and High-level Talent Innovation and Entrepreneurship Project of Quanzhou City, China, No. 2017Z033.

the vertex set $V(G)$ to the set of all non-negative integers (labels) such that $|f(u) - f(v)| \geq 2$ if $d(u,v) = 1$ and $|f(u) - f(v)| \geq 1$ if $d(u,v) = 2$, where $d(u,v)$ denotes the distance between u and v. The number of discrete labels and the range of labels used are called *order* and *span*, respectively.

Real networks reserve bandwidth (range of frequencies) rather than distinct frequencies. In this case, an assignment seeks to use as small range of frequencies as possible [2–5]. For more details, one may refer to the surveys [6,7]. It is sometimes desirable to use as few distinct frequencies of a given bandwidth (span) as possible, since the unused frequencies are available for other use. However, there are cases where the primary objective is to minimize the number of frequencies used and the span is a secondary objective, since we do not want to reserve unnecessary large span. These optimization versions of the Radiocoloring Problem (RCP) are the main objects of study in this work and are defined as follows.

Definition 1 (Minimum order RCP). The optimization version of the RCP that tries to minimize the order. The optimal order is called X_{order}.

Definition 2 (Minimum span RCP). The optimization version of the RCP that tries to minimize the span. The optimal span is called X_{span}.

Definition 3 (Minimum span order RCP). The optimization version of the RCP that tries to find from all minimum span assignments, one that uses as few labels as possible. The order of such an assignment is called X'_{order}.

Definition 4 (Minimum order span RCP). The optimization version of the RCP that tries to find, from all minimum order assignments, one that uses a minimum span. The span of such an assignment is called X'_{span}.

It is easy to see that $X_{order} \leq X'_{order}$ and $X_{span} \leq X'_{span}$. Also, it holds that $X_{order} \leq X_{span} + 1$. Another variation of FAP is related to the square of a graph, which is defined as follows: the square G^2 of a graph G is given by $V(G^2) = V(G)$ and $uv \in E(G^2)$ if and only if $d(u,v) \leq 2$. The problem is to color the square of a graph G, so that no two neighbor vertices (in G^2) get the same color. The objective is to use the minimum number of colors, denoted $\chi(G^2)$ and called *chromatic number* of the square of the graph. [8, 9] first observed that for any graph G, $X_{order}(G)$ is the same as the chromatic number of G^2, i.e. $X_{order}(G) = \chi(G^2)$. However, notice that, the set of colors used in the computed assignments of the two problems are different. The colors of the distance one vertices in the RCP should be at frequency distance two instead of one in the coloring of the G^2. However, from a valid coloring of G^2 we can always reach a valid RC of G by doubling the assigned color of each vertex. Observe that $\chi(G^2) \leq X_{span}(G) + 1 \leq 2\chi(G^2) - 1$.

It has been proved that the problem of the minimum span RCP is NP-complete, even for graphs of diameter 2 [10]. In [11], Lin proved that the problem of coloring the square of a general graph (i.e. the minimum order RCP) is NP-complete. It was also shown that the minimum span order RCP is NP-complete for planar graphs in [8]. To our knowledge, the minimum order span RCP has not been investigated before.

In this paper, we concentrate on the minimum order span RCP on trees. The optimization version of the minimum order span RCP that tries to find, from all minimum order assignments, one that uses the minimum span. In Section 2, we provide attainable lower and upper bounds for trees. Moreover, a complete characterization of caterpillars (as a subclass of trees) with the minimum order span is given in Section 3.

2. The minimum order span of tree

In this section, we study the minimum order span of trees by presenting attainable lower and upper bounds.

A vertex v is called *k-vertex* if $d(v) = k$, where $d(v)$ is the degree of v in G. Let $\Delta(G)$ denote the maximum degree of G. When the context is clear, we use Δ instead of $\Delta(G)$. For a vertex v in G, let $N(v) = \{w|vw \in E(G)\}$ and $N[v] = N(v) \cup \{v\}$. For integers i and j with $i \leq j$, we denote $[i, j]$ as the set $\{i, i+1, \cdots, j-1, j\}$. The following lemma is easy to verify.

LEMMA 2.1.

1) [2] $\Delta + 1 \leq X_{span}(T) \leq \Delta + 2$, *where T is a tree.*

2) [12] $X_{order}(T) = \chi(T^2) = \Delta + 1$, *where T is a tree with at least one edge.*

3) $\Delta + 1 \leq X_{order}(G) \leq |V(G)|$.

4) $X_{span}(G) \leq X'_{span}(G)$, *and $X'_{span}(G) = X_{span}(G)$ if the diameter of G is at most two.*

Theorem 2.1. *Let T be a tree. Then $\Delta + 1 \leq X'_{span}(T) \leq 2\Delta$. Furthermore, both the lower bound and upper bound are attainable.*

Proof. Firstly, $X_{order}(T) = \Delta + 1$ and $X'_{span}(T) \geq X_{span}(T) \geq \Delta + 1$ by Lemma 2.1. Now, we obtain the upper bound by a first-fit (greedy) labeling. First, order $V(T)$ so that $V(T) = \{v_1, v_2, \cdots, v_n\}$, where, for all $i > 1$, v_i is attached just once to $\{v_1, v_2, \cdots, v_{i-1}\}$. We describe a radiocoloring of T: label v_1 as 0; then successively label $v_2, v_3, \cdots, v_n$, by the lowest available element of $\{0, 2, 4, \cdots, 2\Delta\}$. It can be seen that the labeling, defined above, is a radiocoloring of T with order $\Delta + 1$ and span 2Δ. Thus $X'_{span}(T) \leq 2\Delta$.

Finally, both values can occur. The value $\Delta+1$ holds for the star $K_{1,\Delta}$. On the other hand, if T is a complete Δ-ary tree of height at least 2, where it means all non-leaf vertices are Δ-vertices. Then $X'_{span}(T) = 2\Delta$. This implies the conclusion. □

The following theorem provides a sufficient condition for $X'_{span}(T) = 2\Delta$.

Theorem 2.2. *Let T be a tree. If T contains a Δ-vertex v_0 satisfying that all the vertices in $N(v_0)$ are Δ-vertices. Then $X'_{span}(T) = 2\Delta$.*

Proof. Let f be a radiocoloring of T with order $\Delta + 1$ and $f(V(T)) = \{a_1, a_2, \cdots, a_{\Delta+1}\}$. Let $N(v_0) = \{v_1, v_2, \cdots, v_\Delta\}$. Then $f(N[v_i]) = \{a_1, a_2, \cdots, a_{\Delta+1}\}$ for each $i \in [0, \Delta]$, since v_i is a Δ-vertex. Without loss of generality, $f(v_i) = a_{i+1}$ for $i = 0, 1, \cdots, \Delta$. Then $|a_1 - a_j| \geq 2$ for all $j \neq 1$ and $|a_2 - a_j| \geq 2$ for all $j \neq 2$. Generally, we have $|a_i - a_j| \geq 2$ for all $i \neq j$. Thus, $\max_{v \in V(T)} f(v) \geq 2\Delta$, which implies $X'_{span}(T) \geq 2\Delta$. On the other hand, according to Theorem 2.1, we have $X'_{span}(T) = 2\Delta$. □

Lemma 2.2. *Let T be a tree and f be a radiocoloring of T with order $\Delta + 1$ and span $\Delta + 1$. Then we have $f(u) = 0$ for each Δ-vertex u and $f(v) \in \{0, 2, \Delta + 1\}$ for each $(\Delta - 1)$-vertex v; or $f(u) = \Delta + 1$ for each Δ-vertex u and $f(v) \in \{0, \Delta - 1, \Delta + 1\}$ for each $(\Delta - 1)$-vertex v.*

Proof. Notice that $f(u) \in \{0, \Delta + 1\}$ for each Δ-vertex u. Now if $f(u_0) = 0$ for some Δ-vertex u_0, then $f(V(T)) = [0, \Delta + 1]/\{1\}$ since the order of f is $\Delta + 1$. Suppose there is some Δ-vertex u_1 such that $f(u_1) = \Delta + 1$. This implies $f(N(u_1)) = [0, \Delta - 1]$. But this contradicts to $f(V(T)) = [0, \Delta + 1]/\{1\}$. So $f(u) = 0$ for each Δ-vertex u. In the case, we have $f(v) \in \{0, 2, \Delta + 1\}$ for each $(\Delta - 1)$-vertex v. Similarly, if $f(u_0) = \Delta + 1$ for some Δ-vertex u_0, then $f(u) = \Delta + 1$ for each Δ-vertex u and $f(v) \in \{0, \Delta - 1, \Delta + 1\}$ for each $(\Delta - 1)$-vertex v. □

THEOREM 2.3. *Let T be a tree. If T contains one of the following configurations, then $X'_{span}(T) \geq \Delta + 2$.*

(C1) *There exist two Δ-vertices u, v such that $d(u, v) \leq 2$.*

(C2) *There exist a Δ-vertex v_0 and three $(\Delta - 1)$-vertices v_1, v_2, v_3 such that $d(v_i, v_j) \leq 2$ for any $i \neq j$.*

(C3) *There exist four $(\Delta - 1)$-vertices v_1, v_2, v_3, v_4 such that $d(v_i, v_j) \leq 2$ for any $i \neq j$.*

(C4) *Let v_0 be a Δ-vertex. If there exist at least $\Delta - 3$ vertices in $N(v_0)$ such that each of them is adjacent to two $(\Delta - 1)$-vertices.*

(C5) *Let $P = v_1 v_2 \cdots v_n$ be a path of T satisfying that $d(v_i) = \Delta$ or $\Delta - 1$ for all $i \in [1, n]$. There exist two consecutive Δ-vertices v_i, v_j ($i < j$) in P such that $d(v_i, v_j) \neq 0 (\mathrm{mod}\ 3)$.*

(C6) *Let u_0, u_1, u_2 be three Δ-vertices and v_0, v_1, v_2 be three $(\Delta - 1)$-vertices such that $d(u_i, v_i) \leq 2$ for all $i \in \{0, 1, 2\}$ and $d(v_i, v_j) \leq 2$ for any $i \neq j$.*

Proof. Suppose that T admits a radiocoloring f with order $\Delta + 1$ and span $\Delta + 1$. Without loss of generality, let $f(u) = 0$ for each Δ-vertex u and $f(v) \in \{0, 2, \Delta + 1\}$ for each $(\Delta - 1)$-vertex v by Lemma 2.2.

(C1) Notice that $f(u) = f(v) = 0$. It is a contradiction to $d(u, v) \leq 2$.

(C2) In the case, $f(v_0) = 0$ and $\{f(v_1), f(v_2), f(v_3)\} = \{0, 2, \Delta + 1\}$, again a contradiction to $d(v_i, v_j) \leq 2$ for any $i \neq j$.

(C3) Note that $f(v_i) \in \{0, 2, \Delta + 1\}$ for all $i \in [1, 4]$. But this is impossible, since $d(v_i, v_j) \leq 2$ for any $i \neq j$.

(C4) Let $N(v_0) = \{v_1, v_2, \cdots, v_\Delta\}$. Then $f(v_0) = 0$ and $f(N(v_0)) = [2, \Delta + 1]$. If v_i is adjacent to two $(\Delta - 1)$-vertices, then the two $(\Delta - 1)$-vertices must be labeled by 2 and $\Delta + 1$, respectively. This implies $f(v_i) \notin \{2, 3, \Delta, \Delta + 1\}$. Therefore, there are at most $\Delta - 4$ vertices in $N(v_0)$ such that each of them is adjacent to two $(\Delta - 1)$-vertices, which is a contradiction to the assumption.

(C5) In this case, $f(v_i), f(v_{i+1}), \cdots, f(v_j) = \overline{0, 2, (\Delta + 1)}, \cdots, \underline{0, 2, (\Delta + 1)}, 0$ or $\underline{0, (\Delta + 1), 2}, \cdots, \underline{0, (\Delta + 1), 2}, 0$. This implies $d(v_i, v_j) \equiv 0 (\mathrm{mod}\ 3)$, a contradiction.

(C6) Note that $f(v_i) \in \{2, \Delta+1\}$ since $f(u_i) = 0$ and $d(u_i, v_i) \leq 2$ for all $i \in \{0,1,2\}$. But $d(v_i, v_j) \leq 2$ for any $i \neq j$, a contradiction. □

3. The minimum order span of caterpillars

In the previous section we obtain attainable lower and upper bounds for trees. When we restrict to the subclass of trees, such as caterpillars, a complete characterization will be given in this section.

A tree T is called a *caterpillar* if the removal of all vertices of degree 1 results in a path, called the *spine*. If $\Delta = 2$, then T is a path.

THEOREM 3.1. [2] *Let P_n be a path on n vertices. Then*

$$X_{span}(P_n) = \begin{cases} 2, & \text{if } n = 2, \\ 3, & \text{if } n = 3,4, \\ 4, & \text{if } n \geq 5. \end{cases}$$

THEOREM 3.2. *Let $P_n = v_1 v_2 \cdots v_n$ be a path on n vertices. Then*

$$X'_{span}(P_n) = \begin{cases} 2, & \text{if } n = 2, \\ 3, & \text{if } n = 3, \\ 4, & \text{if } n \geq 4. \end{cases}$$

Proof. Clearly, $X_{order}(P_2) = 2$ and $X_{order}(P_n) = 3$ if $n \geq 3$ by Lemma 2.1. For $n = 2, 3$, $X'_{span}(P_2) = X_{span}(P_2) = 2$, $X'_{span}(P_3) = X_{span}(P_3) = 3$ using the Lemma 2.1 and the fact that the diameters of P_2 and P_3 are at most 2.

For $n = 4$, suppose for contradiction that there is a radiocoloring f with order 3 and span 3. If $f(v_2) = 1$, then $\{f(v_1), f(v_3)\} = \{3, 4\}$ and $f(v_4) = 0$ or 2, a contradiction to $X_{order}(P_4) = 3$. If $f(v_2) = 2$, then $\{f(v_1), f(v_3)\} = \{0, 4\}$, a contradiction to the fact that the span of f is 3. The discussion for $f(v_3)$ is the same as for $f(v_2)$. So $\{f(v_2), f(v_3)\} = \{0, 3\}$. But now $\{f(v_1), f(v_4)\} = \{1, 2\}$, again a contradiction to $X_{order}(P_4) = 3$. Hence $X'_{span}(P_4) \geq 4$. For $n \geq 5$, $X'_{span}(P_n) \geq X_{span}(P_n) = 4$ in view of Lemma 2.1 and Theorem 3.1.

For $n \geq 4$, we define a radiocoloring with order 3 and span 4 as follows:

$$f(v_i) = \begin{cases} 0, & \text{if } i \equiv 0 \pmod 3, \\ 2, & \text{if } i \equiv 1 \pmod 3, \\ 4, & \text{if } i \equiv 2 \pmod 3. \end{cases}$$

Therefore, $X'_{span}(P_n) \leq 4$ for $n \geq 4$. □

We are now ready to consider $\Delta \geq 3$. Let T be a caterpillar with the spine $v_1 v_2 \cdots v_n$. Let $N\{v_k\} = N(v_k) \backslash \{v_{k-1}, v_{k+1}\}$, which denotes the leaf neighbors of v_k.

THEOREM 3.3. *Let T be a caterpillar with the spine $v_1 v_2 \cdots v_n$. Then $\Delta + 1 \leq X'_{span}(T) \leq \Delta + 3$.*

Proof. Firstly, $X'_{span}(T) \geq \Delta + 1$ follows from Theorem 2.1.

Now we prove that $X'_{span}(T) \leq \Delta + 3$ by giving a radiocoloring f of T with order $\Delta + 1$ and span $\Delta + 3$. Firstly, the vertices on the spine are labeled by the sequence $0, 2, 4$ repeatedly. Next, $f(N\{v_i\}) \subseteq [6, \Delta + 3]$ for each $i \in [1, n]$. It is straightforward to check that f is a radiocoloring of T with order $\Delta + 1$ and span $\Delta + 3$. Hence $X'_{span}(T) \leq \Delta + 3$. □

The following two theorems give a characterization of the lower and upper bounds, respectively.

THEOREM 3.4. *Let T be a caterpillar with the spine $v_1 v_2 \cdots v_n$. Then $X'_{span}(T) = \Delta + 3$ if and only if T contains three consecutive Δ-vertices.*

Proof. Let f be a radiocoloring of T with order $\Delta + 1$. If T contains three consecutive Δ-vertices, denoted by v_i, v_{i+1}, v_{i+2}. Then $f(v_i)$ is at least two apart from each label except itself. An similar case for $f(v_{i+1})$ and $f(v_{i+2})$. Therefore, we conclude that $X'_{span}(T) \geq \Delta + 3$. On the other hand, by Theorem 3.3, we have $X'_{span}(T) = \Delta + 3$.

If T does not contain three consecutive Δ-vertices, then we construct a radiocoloring f of T as follows: label each v_i by 0, 2 or $\Delta + 2$ such that each Δ-vertex is labeled by 0 or 2. Furthermore, $f(N\{v_i\}) \subseteq [4, \Delta + 1]$ if $f(v_i) = 0$ or 2; $f(N\{v_i\}) \subseteq [4, \Delta]$ if $f(v_i) = \Delta + 2$. It is easy to show that the labeling f, constructed above, is a radiocoloring of T with order $\Delta + 1$ and span $\Delta + 2$. Thus, $X'_{span}(T) \leq \Delta + 2$. This completes the proof of Theorem 3.4. □

THEOREM 3.5. *Let T be a caterpillar with the spine $v_1 v_2 \cdots v_n$. Then $X'_{span}(T) = \Delta + 1$ if and only if T has no structures as follows:*

(I) *There exist two consecutive Δ-vertices v_i, v_j such that $d(v_i, v_j) \neq 0 (\text{mod } 3)$ and $d(v_k) = \Delta - 1$ for all $k \in [i + 1, j - 1]$;*

(II) *There exist two consecutive* Δ*-vertices* v_i, v_j *such that* $d(v_i, v_j) \equiv 2(\text{mod } 3)$ *and* $d(v_k) < \Delta - 1$ *for some* $k \in \{i+1, i+4, \cdots, j-4, j-1\}$, *and* $d(v_l) = \Delta - 1$ *for all* $l \in [i+1, j-1]$ *except* k.

Proof. We first prove the sufficiency. If (I) holds, then $X'_{span}(T) \geq \Delta + 2$ by the (C5) of Theorem 2.3. So it is enough to show (II) derives $X'_{span}(T) \geq \Delta + 2$. For the contrary, suppose f is a radiocoloring of T with order $\Delta + 1$ and span $\Delta + 1$. Without loss of generality, let $f(v_i) = f(v_j) = 0$. If $d(v_{i+1}) < \Delta - 1$ and $d(v_l) = \Delta - 1$ for all $l \in [i+2, j-1]$, then $f(v_{j-3t}) = 0$ for each $t \in [0, \frac{j-i-2}{3}]$. Thus $f(v_{i+2}) = 0$ since $d(v_i, v_j) \equiv 2(\text{mod } 3)$. But this is impossible, since $f(v_i) = 0$ and $d(v_i, v_{i+2}) = 2$. Similar arguments can be made for $k \in \{i+4, i+7, \cdots, j-4, j-1\}$. Therefore, $X'_{span}(T) \geq \Delta + 2$.

Suppose that T has no structures (I)-(II). We now prove the necessity by constructing a radiocoloring f of T with order $\Delta + 1$ and span $\Delta + 1$. Let $v_{i_1}, v_{i_2}, \cdots, v_{i_k}$ be all Δ-vertices, where $i_1 < i_2 < \cdots < i_k$.

First, label $v_{i_1}, v_{i_1-1}, \cdots, v_2, v_1$ by the sequence $0, 2, (\Delta+1)$ or $0, (\Delta+1), 2$ repeatedly. Then, for $k \in [1, i_1]$, $f(N\{v_k\}) \subseteq [3, \Delta]$ if $f(v_k) = 0$; $f(N\{v_k\}) \subseteq [4, \Delta]$ if $f(v_k) = 2$; $f(N\{v_k\}) \subseteq [3, \Delta - 1]$ if $f(v_k) = \Delta + 1$.

Next, for any two consecutive Δ-vertices v_{i_j} and $v_{i_{j+1}}$, we treat the following four cases.

Case 1. $d(v_{i_j}, v_{i_{j+1}}) \equiv 0(\text{mod } 3)$ and $d(v_k) < \Delta$ for all $k \in [i_j+1, i_{j+1}-1]$.

Firstly, label $v_{i_j}, v_{i_j+1}, \cdots, v_{i_{j+1}}$ in the following sequence: $\underline{0, 2, (\Delta+1)}, \cdots, \underline{0, 2, (\Delta+1)}, 0$ or $\underline{0, (\Delta+1), 2}, \cdots, \underline{0, (\Delta+1), 2}, 0$. Secondly, for $k \in [i_j + 1, i_{j+1} - 1]$, $f(N\{v_k\}) \subseteq [3, \Delta]$ if $f(v_k) = 0$; $f(N\{v_k\}) \subseteq [4, \Delta]$ if $f(v_k) = 2$; $f(N\{v_k\}) \subseteq [3, \Delta - 1]$ if $f(v_k) = \Delta + 1$, as illustrated in Figure 1 (a).

Case 2. $d(v_{i_j}, v_{i_{j+1}}) \equiv 1(\text{mod } 3)$ and $d(v_k) < \Delta - 1$ for some $k \in [i_j + 1, i_{j+1} - 1]$.

Start label $v_{i_j}, v_{i_j+1}, \cdots, v_{i_{j+1}}$ by the sequence: $\underline{0, 2, (\Delta+1)}, \cdots, \underline{0, 2, (\Delta+1)}$, 0 or $\underline{0, (\Delta+1), 2}, \cdots, \underline{0, (\Delta+1), 2}, 0$ by inserting 4 such that v_k is labeled by 4. Next, for $t \notin \{k-1, k, k+1\}$, $f(N\{v_t\}) \subseteq [3, \Delta]$ if $f(v_t) = 0$; $f(N\{v_t\}) \subseteq [4, \Delta]$ if $f(v_t) = 2$; $f(N\{v_t\}) \subseteq [3, \Delta - 1]$ if $f(v_t) = \Delta + 1$. Finally, for $t \in \{k-1, k, k+1\}$, we label the leaf neighbours of v_t in the way of X, as illustrated in Figure 2. Consequently, we use the labelling way shown in Figure 1 (b).

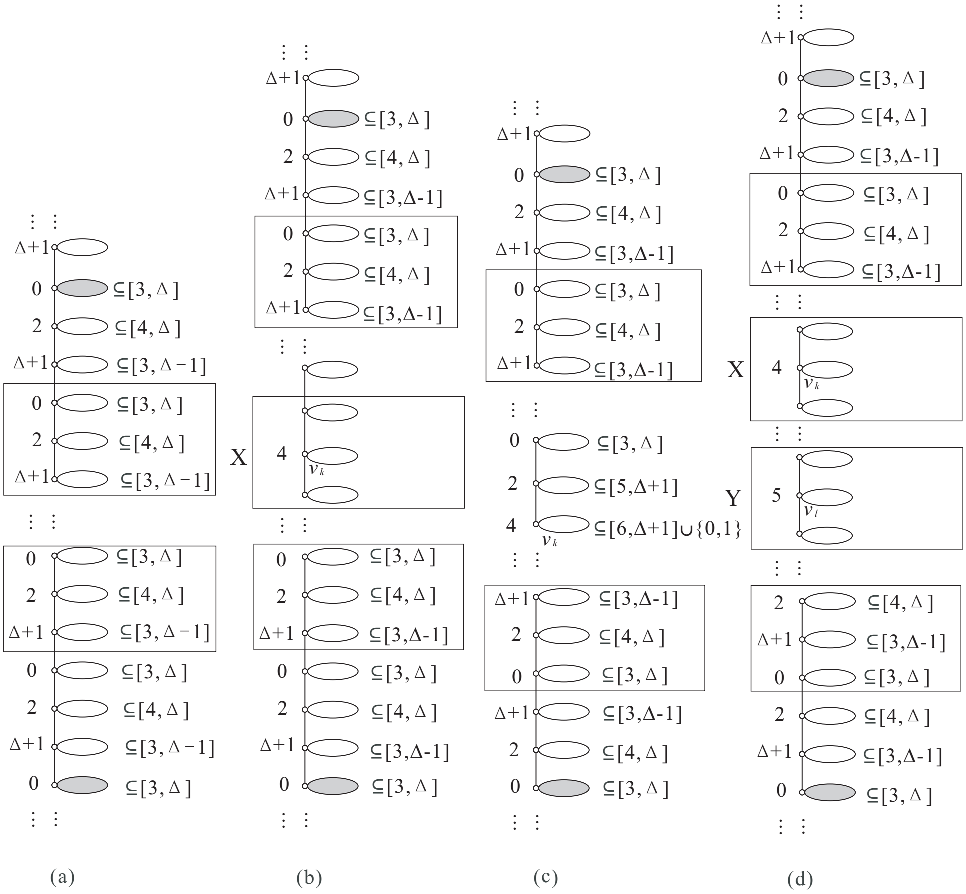

FIGURE 1. The radiocoloring for the case (a) $d(v_{i_j}, v_{i_{j+1}}) \equiv 0 (\mathrm{mod}\ 3)$; (b) $d(v_{i_j}, v_{i_{j+1}}) \equiv 1 (\mathrm{mod}\ 3)$ and $d(v_k) < \Delta - 1$ for some $k \in [i_j + 1, i_{j+1} - 1]$; (c) $d(v_{i_j}, v_{i_{j+1}}) \equiv 2 (\mathrm{mod}\ 3)$ and $d(v_k) < \Delta - 1$ for some $k \notin \{i_j + 1, i_j + 4, \cdots, i_{j+1} - 1\}$; (d) $d(v_{i_j}, v_{i_{j+1}}) \equiv 2 (\mathrm{mod}\ 3)$ and $d(v_k) < \Delta - 1, d(v_l) < \Delta - 1$ for some $k, l \in \{i_j + 1, i_j + 4, \cdots, i_{j+1} - 1\}$.

Case 3. $d(v_{i_j}, v_{i_{j+1}}) \equiv 2 (\mathrm{mod}\ 3)$ and $d(v_k) < \Delta - 1$ for some $k \notin \{i_j + 1, i_j + 4, \cdots, i_{j+1} - 1\}$.

We first label $v_{i_j}, v_{i_j+1}, \cdots, v_{i_{j+1}}$ by the sequence: $\underline{0, 2, (\Delta + 1)}, \cdots, \underline{0, 2, (\Delta + 1)}, 0, 2, 4, \underline{(\Delta + 1), 2, 0}, \cdots, \underline{(\Delta + 1), 2, 0}$ or $0, \overline{(\Delta + 1), 2}, \cdots, \underline{0, (\Delta + 1), 2}, 0, (\Delta + 1), 4, \underline{2, (\Delta + 1), 0}, \cdots, \underline{2, (\Delta + 1), 0}$ such that v_k is

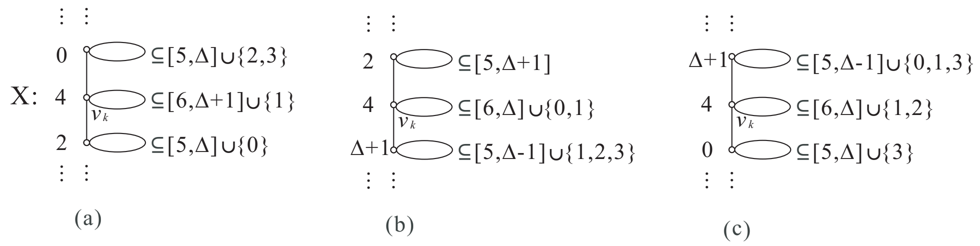

FIGURE 2. A radiocoloring of $\{v_{k-1}, v_k, v_{k+1}\}$ and their leaf neighbours with order $\Delta + 1$ and span $\Delta + 1$.

labeled by 4. Secondly, for $t \notin \{k-1, k, k+1\}$, $f(N\{v_t\}) \subseteq [3, \Delta]$ if $f(v_t) = 0$; $f(N\{v_t\}) \subseteq [4, \Delta]$ if $f(v_t) = 2$; $f(N\{v_t\}) \subseteq [3, \Delta - 1]$ if $f(v_t) = \Delta + 1$. Finally, $f(N\{v_{k-1}\}) \subseteq [5, \Delta + 1]$ if $f(v_{k-1}) = 2$; $f(N\{v_k\}) \subseteq [6, \Delta + 1] \cup \{0, 1\}$; $f(N\{v_{k+1}\}) \subseteq [5, \Delta - 1] \cup \{0, 4\}$ if $f(v_{k+1}) = \Delta + 1$, which is depicted in in Figure 1 (c).

Case 4. $d(v_{i_j}, v_{i_{j+1}}) \equiv 2 (\mathrm{mod}\ 3)$ and $d(v_k) < \Delta - 1, d(v_l) < \Delta - 1$ for some $k, l \in \{i_j + 1, i_j + 4, \cdots, i_{j+1} - 1\}$.

Initially, we label $v_{i_j}, v_{i_j+1}, \cdots, v_{i_{j+1}}$ by the sequence: $\underline{0, 2, (\Delta + 1)}, \cdots, \underline{0, 2, (\Delta + 1)}, 0$ or $\underline{0, (\Delta + 1), 2}, \cdots, \underline{0, (\Delta + 1), 2}, 0$ by inserting 4, 5 such that v_k, v_l are labeled by 4 and 5 respectively. Next, for $t \notin \{k-1, k, k+1, l-1, l, l+1\}$, $f(N\{v_t\}) \subseteq [3, \Delta]$ if $f(v_t) = 0$; $f(N\{v_t\}) \subseteq [4, \Delta]$ if $f(v_t) = 2$; $f(N\{v_t\}) \subseteq [3, \Delta - 1]$ if $f(v_t) = \Delta + 1$. Lastly, for $t \in \{k - 1, k, k + 1\}$, we label the leaf neighbours of v_t in the way of X, as shown in Figure 2. For $t \in \{l - 1, l, l + 1\}$, we label the leaf neighbours of v_t in the way of Y, as shown in Figure 3. Namely, we use the labelling way shown in Figure 1 (d).

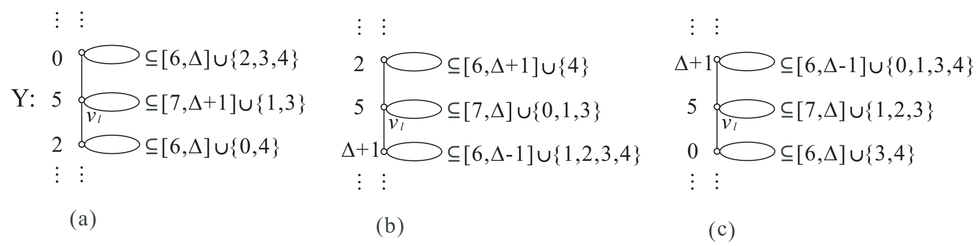

FIGURE 3. A radiocoloring of $\{v_{l-1}, v_l, v_{l+1}\}$ and their leaf neighbours with order $\Delta + 1$ and span $\Delta + 1$.

Finally, we label $v_{i_k}, v_{i_k+1}, \cdots, v_n$ by the sequence $0, 2, (\Delta + 1)$ or $0, (\Delta + 1), 2$ repeatedly. Then, for $k \in [i_k, n]$, $f(N\{v_k\}) \subseteq [3, \Delta]$ if

$f(v_k) = 0$; $f(N\{v_k\}) \subseteq [4, \Delta]$ if $f(v_k) = 2$; $f(N\{v_k\}) \subseteq [3, \Delta - 1]$ if $f(v_k) = \Delta + 1$.

The proof is complete. □

Combining Theorem 3.3, 3.4 and 3.5, we provide a complete characterization of caterpillars with the minimum order span.

References

[1] W.K. Hale, *Frequency assignment: Theory and applications*, Proc. IEEE **68** (1980), 1497–1514.

[2] J.R. Griggs and R.K. Yeh, *Labelling graphs with a condition at distance 2*, SIAM J. Discrete Math. **5** (1992), 586–595.

[3] G.J. Chang and D. Kuo, *The* $L(2,1)$*-labeling problem on graphs*, SIAM J. Discrete Math. **9** (1996), 309–316.

[4] J.P. Georges and D.W. Mauro, *Labeling trees with a condition at distance two*, Discrete Math. **269** (2003), 127–148.

[5] W.F. Wang, *The* $L(2,1)$*-labelling of trees*, Discrete Appl. Math. **154** (2007), 598–603.

[6] R.K. Yeh, *A survey on labeling graphs with a condition at distance two*, Discrete Math. **306** (2006), 1217–1231.

[7] T. Calamoneri, *The* $L(h,k)$*-labelling problem: an updated survey and annotated bibliography*, Comput. J. **54** (2011), 1344–1371.

[8] D.A. Fotakis, S.E. Nikoletseas, V.G. Papadopoulou and P.G. Spirakis, *Radiocoloring in planar graphs: Complexity and approximations*, Theo. Comput. Sci. **340** (2005), 514–538.

[9] D.A. Fotakis, S.E. Nikoletseas, V.G. Papadopoulou and P.G. Spirakis, *NP-Completeness Results and Efficient Approximations for Radiocoloring in Planar Graphs*, In: Proceedings of the 25th International Symposium on Mathematical Foundations of Computer Science, Lecture Notes of Computer Science, vol. 1893, pp. 363-372. Springer (2000).

[10] J. Griggs and D. Liu, *Minimum span channel assignments*, Recent Advances in Radio Channel Assignments, Invited Minisymposium, Discrete Math. (1998).

[11] Y.L. Lin and S. Skiena, *Algorithms for square roots of graphs*, SIAM J. Discrete Math. **8** (1995), 99–118.

[12] K.W. Lih and W.F. Wang, *Coloring the square of an outerplanar graph*, Taiwanese J. Math. **10** (4) (2006), 1015–1023.

Xiaoling Zhang
College of Mathematics and Computer Science
Quanzhou Normal University
Quanzhou 362000, Fujian, P.R. China
E-mail: `xml000999@163.com`