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β-FUZZY FILTERS IN MS-ALGEBRAS

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Abstract. In this paper, we introduce the concept of β-fuzzy filters in MS-algebras and β-fuzzy filters are characterized in terms of boosters. It is proved that the lattice of β-fuzzy filters is isomorphic to the fuzzy ideal lattice of boosters.

1. Introduction

Blyth and Varlet [5] introduced the class **MS** of all MS-algebras which is a common abstraction of de Morgan algebras and Stone algebras. Blyth and Varlet [6] characterized the subvarieties of **MS**. The class **MS** contains the well-known classes for examples Boolean algebras, de Morgan algebras, Kleene algebras and Stone algebras. And also Rao [11] introduced the concepts of boosters and β-filters of MS-algebras.

On the other hand, fuzzy set theory was introduced by Zadeh [15] which is a generalization of classical set theory. Next Rosenfeld [10] applied it to group theory and developed the theory of fuzzy subgroups. Also, many scholars have worked on fuzzy lattice theory. They introduced the concepts of fuzzy sublattices, fuzzy ideals, fuzzy prime ideals, in a lattice and gave some interesting results (see [1, 2, 8, 9, 12, 14]).

Recently, Alaba and Alemayehu [3] introduced the notion of closure fuzzy ideals of MS-algebras. And also Alaba, Taye and Alemayehu [4] introduced the concept of δ-fuzzy ideals in MS-algebras. In this paper,

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the notions of β-fuzzy filters are introduced in MS-algebras. It is proved that the lattice of β-fuzzy filters is isomorphic to the fuzzy ideals of lattice of boosters and it is proved that the class of all β-fuzzy filters forms a complete distributive lattice. It also observed that the minimal elements of the poset of all prime fuzzy filters of an MS-algebra are β-fuzzy filters, and every proper β-fuzzy filters of L is the intersection of all prime β-fuzzy filters containing it.

2. Preliminaries

In this section, we recall some definitions and results which will be used in this paper. For ordinary crisp theory of β- filters of MS-algebras, we refer to [11].

Definition 2.1. [5] An MS-algebra is an algebra $(L, \vee, \wedge, ^\circ, 0, 1)$ of type $(2, 2, 1, 0, 0)$ such that $(L, \vee, \wedge, 0, 1)$ is a bounded distributive lattice and $a \to a^\circ$ is a unary operation satisfies: $a \leq a^{\circ\circ}$, $(a \wedge b)^\circ = a^\circ \vee b^\circ$ and $1^\circ = 0$.

Lemma 2.2. [5] *For any two elements a, b of an MS-algebra, we have the following:*

(1) $0^\circ = 1$,
(2) $a \leq b \Rightarrow b^\circ \leq a^\circ$,
(3) $a^{\circ\circ\circ} = a^\circ$,
(4) $(a \vee b)^\circ = a^\circ \wedge b^\circ$,
(5) $(a \vee b)^{\circ\circ} = a^{\circ\circ} \vee b^{\circ\circ}$,
(6) $(a \wedge b)^{\circ\circ} = a^{\circ\circ} \wedge b^{\circ\circ}$.

Definition 2.3. [11] Let L be an MS-algebra. Then for any $a \in L$, define the booster of a as follows: $(a)^+ = \{x \in L : x \vee a^\circ = 1\}$. Note that $(0)^+ = L$ and $(1)^+ = \{1\}$.

Let us denote the set of all boosters of an MS-algebra L by $\mathcal{B}_0(L)$. Then we have the following:

Theorem 2.4. [11] *For an MS-algebra L, the set $\mathcal{B}_0(L)$ of all boosters is a complete distributive lattice on its own. Note that for any boosters $(a)^+, (b)^+$ of $\mathcal{B}_0(L)$, define the operations $\cap$ and $\sqcup$ as $(a)^+ \cap (b)^+ = (a \vee b)^+$ and $(a)^+ \sqcup (b)^+ = (a \wedge b)^+$. Note that $(a \vee b)^+$ and $(a \wedge b)^+$ are infimum and supremum for both $(a)^+$ and $(b)^+$ in $\mathcal{B}_0(L)$ respectively.*

Definition 2.5. [11] (1) For any filter F of L, define an operator β as $\beta(F) = \{(x)^+ : x \in F\}$.

(2) For any ideal I of $\mathcal{B}_0(L)$, define an operator $\overleftarrow{\beta}$ as $\overleftarrow{\beta}(I) = \{x \in L : (x)^+ \in I\}$.

Definition 2.6. [7] A proper filter F of L is a prime filter if $A \cap B \subseteq F$ implies $A \subseteq F$ or $B \subseteq F$ for any fuzzy filters of A and B of L.

We recall that for any nonempty set S, the characteristic function of S,

$$\chi_S(x) = \begin{cases} 1 & if\ x \in S, \\ 0 & if\ x \notin S. \end{cases}$$

Definition 2.7. [15] Let μ be a fuzzy subset of S and let $\alpha \in [0, 1]$. Then the set

$$\mu_\alpha = \{x \in L : \alpha \leq \mu(x)\}$$

is called a level subset of μ.

A fuzzy subset μ of L is proper if it is a non constant function. A fuzzy subset μ such that $\mu(x) = 0$ for all $x \in L$ is an improper fuzzy subset.

Definition 2.8. [10] Let μ and θ be fuzzy subsets of a set L. Define the fuzzy subsets $\mu \cup \theta$ and $\mu \cap \theta$ of L as follows: for each $x \in L$, $(\mu \cup \theta)(x) = \mu(x) \vee \theta(x)$ and $(\mu \cap \theta)(x) = \mu(x) \wedge \theta(x)$. Then $\mu \cup \theta$ and $\mu \cap \theta$ are called the union and intersection of μ and θ respectively.

We define the binary operations $''\vee''$ and $''\wedge''$ on all fuzzy subsets of a lattice L as: $(\mu \vee \theta)(x) = \sup\{\mu(a) \wedge \theta(b) : a, b \in L, a \vee b = x\}$ and $(\mu \wedge \theta)(x) = \sup\{\mu(a) \wedge \theta(b) : a, b \in L, a \wedge b = x\}$.

If μ and θ are fuzzy ideals of L, then $\mu \wedge \theta = \mu \cap \theta$, and $\mu \vee \theta$ is a fuzzy ideal generated by $\mu \cup \theta$.

Definition 2.9. [12] A fuzzy subset μ of a bounded lattice L is said to be a fuzzy ideal of L, if for all $x, y \in L$,

1. $\mu(0) = 1$,
2. $\mu(x \vee y) \geq \mu(x) \wedge \mu(y)$
3. $\mu(x \wedge y) \geq \mu(x) \vee \mu(y)$ $for\ all\ x, y \in L$.

In [12], Swamy and Raju observed that, a fuzzy subset μ of a bounded lattice L is a fuzzy ideal of L if and only if $\mu(0) = 1$ and $\mu(x \vee y) =$

$\mu(x) \wedge \mu(y)$ $for\ all\ x, y \in L$.

Definition 2.10. [12] A fuzzy subset μ of a bounded lattice L is said to be a fuzzy filter of L, if for all $x, y \in L$,

1. $\mu(1) = 1$,
2. $\mu(x \vee y) \geq \mu(x) \wedge \mu(y)$
3. $\mu(x \wedge y) \geq \mu(x) \vee \mu(y)$ $for\ all\ x, y \in L$.

In [12] a fuzzy subset μ of a bounded lattice L is a fuzzy filter of L if and only if $\mu(1) = 1$ and $\mu(x \vee y) = \mu(x) \wedge \mu(y)$ $for\ all\ x, y \in L$.

Theorem 2.11. [12] *Let μ be a fuzzy subset of L. Then μ is a fuzzy ideal of L if and only if, for any $\alpha \in [0, 1]$, μ_α is an ideal of L.*

Definition 2.12. [2] (1) A proper fuzzy ideal μ of L is called a prime fuzzy ideal if for any two fuzzy ideals η, ν of L, $\eta \cap \nu \subseteq \mu$ implies $\eta \subseteq \mu$ or $\nu \subseteq \mu$.

(2) A proper fuzzy filter μ of L is called a prime fuzzy filter if for any two fuzzy filters η, ν of L, $\eta \cap \nu \subseteq \mu$ implies $\eta \subseteq \mu$ or $\nu \subseteq \mu$.

Theorem 2.13. [13] *For any $\alpha \in [0, 1)$, the fuzzy subset P_α^1 of L given by $P_\alpha^1(x) = \begin{cases} 1 & if\ x \in P, \\ \alpha & if\ \ x \notin P \end{cases}$ $for\ all\ x \in L$ is a prime fuzzy filter if and only if P is a prime filter of L.*

Throughout the next sections L stands for an MS-algebra unless otherwise mentioned.

3. β-Fuzzy Filters in MS-algebras

In this section, the concept of β-fuzzy filters is introduced in MS-algebras and basic properties of β-fuzzy filters are observed.

Definition 3.1. A fuzzy subset μ of $\mathcal{B}_0(L)$ is called a fuzzy ideal of $\mathcal{B}_0(L)$ if $\mu((1)^+) = 1$ and $\mu((a)^+ \sqcup (b)^+) \geq \mu((a)^+) \wedge \mu((b)^+))$ and $\mu((a)^+ \cap (b)^+) \geq \mu((a)^+) \vee \mu((b)^+))$,$\forall\ (a)^+, (b)^+ \in \mathcal{B}_0(L)$.

Example 3.2. Consider the MS-algebra L described as the following Hasse diagram 1

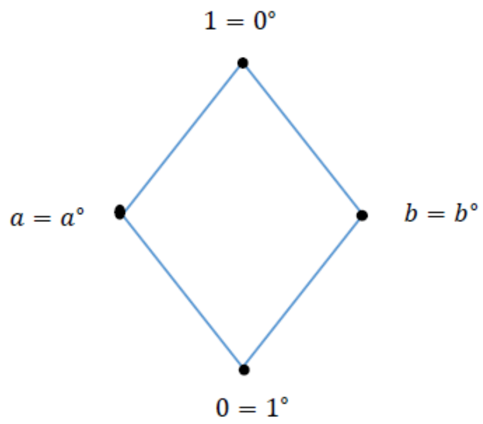


diagram 1

Clearly $\mathcal{B}_0(L) = \{(0)^+, (a)^+, (b)^+, (1)^+\}$. Define a fuzzy subset μ of $\mathcal{B}_0(L)$ as $\mu((0)^+) = 0.5 = \mu((a)^+)$, $\mu((b)^+) = 0.8$ and $\mu((1)^+) = 1$. It is easily verified that μ is a fuzzy ideal of $\mathcal{B}_0(L)$.

Now we introduce two operators β and $\overleftarrow{\beta}$ in the following:

DEFINITION 3.3. Let L be an MS-algebra.

(1) For any fuzzy filter θ of L and for any x in L, define an operator β as follows: $\beta(\theta)((x)^+) = \sup\{\theta(y) : (x)^+ = (y)^+, y \in L\}$.

(2) For any fuzzy ideal μ of $\mathcal{B}_0(L)$ and for any x in L, define an operator $\overleftarrow{\beta}$ as follows: $\overleftarrow{\beta}(\mu)(x) = \mu((x)^+)$.

LEMMA 3.4. *For any MS-algebra L, we have the following:*

(1) For any fuzzy filter θ of L, $\beta(\theta)$ is a fuzzy ideal of $\mathcal{B}_0(L)$,

(2) For any fuzzy ideal μ of $\mathcal{B}_0(L)$, $\overleftarrow{\beta}(\mu)$ is a fuzzy filter of L,

(3) β and $\overleftarrow{\beta}$ are isotones.

Proof. (1) Let θ be a fuzzy filter of L. Then clearly $\beta(\theta)((1)^+) = 1$. For any $(a)^+, (b)^+$ in $\mathcal{B}_0(L)$,

$$\begin{aligned}
&\beta(\theta)((a)^+) \wedge \beta(\theta)((b)^+)\\
&= \sup\{\theta(x) : (x)^+ = (a)^+\} \wedge \sup\{\theta(y) : (y)^+ = (b)^+\}\\
&= \sup\{\theta(x) \wedge \theta(y) : (x)^+ = (a)^+, (y)^+ = (b)^+\}\\
&\leq \sup\{\theta(x \wedge y) : (x \wedge y)^+ = (a \wedge b)^+\}\\
&= \beta(\theta)((a \wedge b)^+) = \beta(\theta)((a)^+ \sqcup (b)^+),
\end{aligned}$$

$$\begin{aligned}
&\beta(\theta)((a)^+) \vee \beta(\theta)((b)^+) \\
&\quad = \sup\{\theta(x) : (x)^+ = (a)^+\} \vee \sup\{\theta(y) : (y)^+ = (b)^+\} \\
&\quad = \sup\{\theta(x) \vee \theta(y) : (x)^+ = (a)^+, (y)^+ = (b)^+\} \\
&\quad \leq \sup\{\theta(x \vee y) : (x \vee y)^+ = (a \vee b)^+\} \\
&\quad = \beta(\theta)((a \vee b)^+) = \beta(\theta)((a)^+ \cap (b)^+)
\end{aligned}$$

Therefore $\beta(\theta)$ is a fuzzy ideal of $\mathcal{B}_0(L)$.

(2) For any fuzzy ideal μ of $\mathcal{B}_0(L)$, $\overleftarrow{\beta}(\mu)(1) = \mu((1)^+) = 1$. For any a, b in L, $\overleftarrow{\beta}(\mu)(a \wedge b) = \mu((a \wedge b)^+) = \mu((a)^+ \sqcup (b)^+) \geq \mu((a)^+) \wedge \mu((b)^+) = \overleftarrow{\beta}(\mu)(a) \wedge \overleftarrow{\beta}(\mu)(b)$ and $\overleftarrow{\beta}(\mu)(a \vee b) = \mu((a \vee b)^+) = \mu((a)^+ \cap (b)^+) \geq \mu((a)^+) \vee \mu((b)^+) = \overleftarrow{\beta}(\mu)(a) \vee \overleftarrow{\beta}(\mu)(b)$.

(3) Suppose that μ and θ are fuzzy filters of L such that $\mu \subseteq \theta$. $\beta(\mu)((x)^+) = \sup\{\mu(y) : (y)^+ = (x)^+\} \leq \sup\{\theta(y) : (y)^+ = (x)^+\} = \beta(\theta)((x)^+)$. Therefore β is an isotone. Similarly, we get that $\overleftarrow{\beta}$ is an isotone. □

THEOREM 3.5. *The map $\mu \to \overleftarrow{\beta}\beta(\mu)$ is a closure operator on the lattice of fuzzy filters of L. i.e., for any fuzzy filters μ and θ of L,*

(1) $\mu \subseteq \overleftarrow{\beta}\beta(\mu)$,
(2) $\mu \subseteq \theta \Rightarrow \overleftarrow{\beta}\beta(\mu) \subseteq \overleftarrow{\beta}\beta(\theta)$,
(3) $\overleftarrow{\beta}\beta\{\overleftarrow{\beta}\beta(\mu)\} = \overleftarrow{\beta}\beta(\mu)$.

Proof. (1) For any $x \in L$, $\overleftarrow{\beta}\beta(\mu)(x) = \sup\{\mu(y) : (x)^+ = (y)^+\} \geq \mu(x)$.

(2) It is obvious, since β and $\overleftarrow{\beta}$ are isotones.

(3) For any $x \in L$,

$$\begin{aligned}
\overleftarrow{\beta}\beta\{\overleftarrow{\beta}\beta(\mu)\}(x) &= \beta\{\overleftarrow{\beta}\beta(\mu)\}((x)^+) \\
&= \sup\{\overleftarrow{\beta}\beta(\mu)(y) : (y)^+ = (x)^+, y \in L\} \\
&= \sup\{\beta(\mu)((y)^+) : (y)^+ = (x)^+, y \in L\} \\
&= \beta(\mu)((x)^+) = \overleftarrow{\beta}\beta(\mu)(x)
\end{aligned}$$

□

THEOREM 3.6. *For any MS-algebra L, β is a homomorphism of the lattice of fuzzy filters of L into the lattice of fuzzy ideals of $\mathcal{B}_0(L)$.*

Proof. Let $\mathcal{FF}(L)$ be the set of all fuzzy filters of L and $\mathcal{FIB}_0(L)$ be the set of all fuzzy ideals in $\mathcal{B}_0(L)$. For any $\mu, \theta \in \mathcal{FF}(L)$, $\mu \cap \theta \subseteq \mu$ and $\mu \cap \theta \subseteq \theta$. This implies $\beta(\mu \cap \theta) \subseteq \beta(\mu)$ and $\beta(\mu \cap \theta) \subseteq \beta(\theta)$. We get $\beta(\mu \cap \theta) \subseteq \beta(\theta) \cap \beta(\mu)$.
Also

$$\begin{aligned}
&(\beta(\mu) \cap \beta(\theta))((x)^+) \\
&= \beta(\mu)((x)^+) \wedge \beta(\theta)((x)^+) \\
&= \sup\{\mu(a) : (a)^+ = (x)^+\} \wedge \sup\{\theta(b) : (b)^+ = (x)^+\} \\
&\leq \sup\{\mu(a \vee b) : (a \vee b)^+ = (x)^+\} \wedge \sup\{\theta(a \vee b) : (a \vee b)^+ = (x)^+\} \\
&= \sup\{\mu(a \vee b) \wedge \theta(a \vee b) : (a \vee b)^+ = (x)^+\} \\
&= \sup\{(\mu \cap \theta)(a \vee b) : (a \vee b)^+ = (x)^+\} \\
&= \beta(\mu \cap \theta)((x)^+).
\end{aligned}$$

Thus $\beta(\mu \cap \theta) = \beta(\mu) \cap \beta(\theta)$.
Since $\mu \subseteq \mu \vee \theta$ and $\mu \subseteq \mu \vee \theta$, $\beta(\mu) \subseteq \beta(\mu \vee \theta)$ and $\beta(\mu) \subseteq \beta(\mu \vee \theta)$. This gives $\beta(\mu) \sqcup \beta(\mu) \subseteq \beta(\mu \vee \theta)$. Again

$$\begin{aligned}
&(\beta(\mu \vee \theta))((x)^+) \\
&= \sup\{(\mu \vee \theta)(a) : (a)^+ = (x)^+\} \\
&= \sup\{\sup\{\mu(a_1) \wedge \theta(a_2) : a = a_1 \wedge a_2\} : (a)^+ = (x)^+\} \\
&\leq \sup\{\sup\{\mu(b_1) \wedge \theta(b_2) : (b_1)^+ = (a_1)^+, (b_2)^+ = (a_2)^+\} : \\
&\quad (a_1 \wedge a_2)^+ = (x)^+\} \\
&= \sup\{\sup\{\mu(b_1) : (b_1)^+ = (a_1)^+\} \wedge \sup\{\theta(b_2) : (b_2)^+ = (a_2)^+\} : \\
&\quad (a_1)^+ \sqcup (a_2)^+ = (x)^+\} \\
&= \sup\{\beta(\mu)((a_1)^+) \wedge \beta(\theta)((a_2)^+) : (a_1)^+ \sqcup (a_2)^+ = (x)^+\} \\
&= (\beta(\mu) \sqcup \beta(\theta))((x)^+)
\end{aligned}$$

This implies $\beta(\mu \vee \theta) \subseteq \beta(\mu) \sqcup \beta(\mu)$. Therefore $\beta(\mu \vee \theta) = \beta(\mu) \sqcup \beta(\theta)$ and clearly $\chi_{\{1\}}, \chi_L$ are the smallest and the largest fuzzy filters of L respectively and also $\beta(\chi_{\{1\}}), \beta(\chi_L)$ are smallest and greatest fuzzy ideals of $\mathcal{B}_0(L)$ respectively. Therefore β is a homomorphism from $\mathcal{FF}(L)$ into $\mathcal{FIB}_0(L)$. □

COROLLARY 3.7. *For any two fuzzy filters μ and θ of an MS-algebra L, we have $\overleftarrow{\beta}\beta(\mu \cap \theta) = \overleftarrow{\beta}\beta(\mu) \cap \overleftarrow{\beta}\beta(\theta)$.*

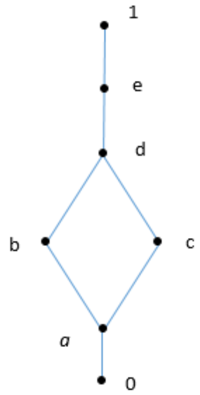
Proof. By Theorem 3.6 , $\beta(\mu\cap\theta) = \beta(\mu)\cap\beta(\theta)$. Now $\overleftarrow{\beta}\beta(\mu\cap\theta)(y) = \beta(\mu\cap\theta)((y)^+) = \beta(\mu)((y)^+) \wedge \beta(\theta)((y)^+) = \overleftarrow{\beta}\beta(\mu)((y)) \wedge \overleftarrow{\beta}\beta(\theta)((y))$. Therefore $\overleftarrow{\beta}\beta(\mu\cap\theta) = \overleftarrow{\beta}\beta(\mu) \cap \overleftarrow{\beta}\beta(\theta)$. □

Now we introduce the notion of β-fuzzy filters in MS-algebras.

Definition 3.8. A fuzzy filter μ of L is called a β-fuzzy filter if $\overleftarrow{\beta}\beta(\mu) = \mu$.

Example 3.9. Consider the MS-algebra L given in diagram 1, define a fuzzy subset μ of L as $\mu(a) = \mu(0) = 0.5$ and $\mu(1) = \mu(b) = 1$. Then μ is a β-fuzzy filter of L.

Example 3.10. Let us consider the MS-algebra L described in the diagram 2



	0	a	b	c	d	e	1
∘	1	e	e	e	e	e	0

diagram 2

The fuzzy subsets μ and θ of L defined as $\mu(a) = \mu(b) = 0.3, \mu(c) = \mu(d) = 0.6$, $\mu(e) = \mu(1) = 1$ and $\mu(0) = 0$, and $\theta(a) = \theta(b) = \theta(c) = \theta(d) = \theta(e) = \theta(1) = 1$ and $\theta(0) = 0.6$. Then it can be easily verified that μ is a fuzzy filter of L but not a β-fuzzy filter of L, and θ is a β-fuzzy filter of L.

In the following Theorem, we characterize β-fuzzy filters in terms of its level subsets and characteristic functions.

Theorem 3.11. *For proper fuzzy subset μ of L, μ is a β-fuzzy filter if and only if $\mu_\alpha, \forall\alpha \in [0,1]$, is a β-filter of L.*

Proof. Suppose that μ is a β-fuzzy filter of L. Then $(\overleftarrow{\beta}\beta(\mu))_\alpha = \mu_\alpha$. To prove each level subset of μ is a β-filter of L, it is enough to show $\overleftarrow{\beta}\beta(\mu_\alpha) = \mu_\alpha$. Clearly $\mu_\alpha \subseteq \overleftarrow{\beta}\beta(\mu_\alpha)$. Next, let $x \in \overleftarrow{\beta}\beta(\mu_\alpha)$. Then $(x)^+ \in \beta(\mu_\alpha)$. This implies there exists $y \in \mu_\alpha$ such that $(x)^+ = (y)^+$, and so $\mu(y) \geq \alpha$ such that $(x)^+ = (y)^+$. This gives $\beta(\mu)((x)^+) = \sup\{\mu(y) : (x)^+ = (y)^+\} \geq \alpha$ and so $\overleftarrow{\beta}\beta(\mu)(x) \geq \alpha$. we have $x \in (\overleftarrow{\beta}\beta(\mu))_\alpha$. Thus $\overleftarrow{\beta}\beta(\mu_\alpha) \subseteq \mu_\alpha$. Therefore $\overleftarrow{\beta}\beta(\mu_\alpha) = \mu_\alpha$.
Conversely, it is clear that $\mu \subseteq \overleftarrow{\beta}\beta(\mu)$. Next, let $\alpha = \overleftarrow{\beta}\beta(\mu)(x) = \sup\{\mu(y) : (y)^+ = (x)^+\}$. Then for each $\epsilon > 0$, there is $a \in L, (a)^+ = (x)^+$ such that $\mu(a) > \alpha - \epsilon$. Since ϵ is arbitrary then $\mu(a) \geq \alpha$ such that $(a)^+ = (x)^+$. This implies $a \in \mu_\alpha$. Thus $x \in \overleftarrow{\beta}\beta(\mu_\alpha) = \mu_\alpha$. Hence $\mu(x) \geq \alpha = \overleftarrow{\beta}\beta(\mu)(x)$. Therefore $\mu = \overleftarrow{\beta}\beta(\mu)$. □

Corollary 3.12. *For a nonempty subset F of L, F is a β-filter if and only if χ_F is a β-fuzzy filter of L.*

In the following theorem, the class of all β-fuzzy filters of an MS-algebra can be characterized in terms of boosters.

Theorem 3.13. *A fuzzy filter μ of an MS-algebra L is a β-fuzzy filter if and only if for all $x, y \in L, (x)^+ = (y)^+$ implies $\mu(x) = \mu(y)$.*

Proof. Suppose that $\mu(x) = \overleftarrow{\beta}\beta(\mu)(x), \forall x \in L$ and $x, y \in L$ such that $(x)^+ = (y)^+$. This implies $\mu(x) = \overleftarrow{\beta}\beta(\mu)((x) = \beta(\mu)((x)^+) = \beta(\mu)((y)^+) = \overleftarrow{\beta}\beta(\mu)(y) = \mu(y)$.
Conversely, suppose that $\forall x, y \in L, (x)^+ = (y)^+$ implies $\mu(x) = \mu(y)$. Now $\overleftarrow{\beta}\beta(\mu)(x) = \sup\{\mu(y) : (y)^+ = (x)^+\} = \mu(x)$. Therefore $\overleftarrow{\beta}\beta(\mu) = \mu$. □

Theorem 3.14. *Let $\{\mu_i : i \in \Omega\}$ be a family of β-fuzzy filters in an MS-algebra L. Then $\cap_{i\in\Omega}\mu_i$ is a β-fuzzy filter of L.*

Corollary 3.15. *Let L be an MS-algebra. Then the set $FF_\beta(L)$ of all β-fuzzy filters of L is a complete distributive lattice with relation $\subseteq$. The sup and inf of any subfamily $\{\mu_i | i \in \Omega\}$ of β-fuzzy filters are $\overleftarrow{\beta}\beta(\bigvee \mu_i)$ and $\cap_{i\in\Omega}\mu_i$ respectively, where $\bigvee \mu_i$ is their supremum in the lattice of fuzzy filters of L.*

Lemma 3.16. *For any fuzzy ideal μ of $\mathcal{B}_0(L)$, $\beta\overleftarrow{\beta}(\mu) = \mu$.*

Proof. Let $(x)^+ \in \mathcal{B}_0(L)$. Now $\beta\overleftarrow{\beta}(\mu)((x)^+) = \sup\{\overleftarrow{\beta}(\mu)(y) : (y)^+ = (x)^+\} = \sup\{\mu((y)^+) : (y)^+ = (x)^+\} = \mu((x)^+)$. Therefore $\beta\overleftarrow{\beta}(\mu) = \mu$. □

Using Corollary 3.15 and Lemma 3.16, we prove that the lattice of β-fuzzy filters of L is isomorphic to the lattice of fuzzy ideals of $\mathcal{B}_0(L)$.

THEOREM 3.17. *Let L be an MS-algebra. Then there is an isomorphism of the lattice of β-fuzzy filters of L onto the lattice of fuzzy ideals of $\mathcal{B}_0(L)$.*

Proof. Let $\mathcal{FF}_\beta(L)$ be the set of all β-fuzzy filters of L, $\mathcal{FIB}_0(L)$ be the set of all fuzzy ideals of $\mathcal{B}_0(L)$ and $f : \mathcal{FF}_\beta(L) \to \mathcal{FIB}_0(L)$ be a mapping defined by $f(\mu) = \beta(\mu)$, for any $\mu \in FF_\beta(L)$. Then clearly f is one-to-one. Let μ be any fuzzy ideal of $\mathcal{B}_0(L)$. Then $\overleftarrow{\beta}(\mu)$ is a fuzzy filter of L. By Lemma 3.16, $\overleftarrow{\beta}\beta(\overleftarrow{\beta}(\mu)) = \overleftarrow{\beta}(\beta\overleftarrow{\beta}(\mu)) = \overleftarrow{\beta}(\mu)$. Thus $\overleftarrow{\beta}(\mu)$ is a β-fuzzy filter of L. Now $f(\overleftarrow{\beta}(\mu)) = \beta(\overleftarrow{\beta}(\mu)) = \mu$. This gives f is onto. Let μ, θ be any two β-fuzzy filters of L. Then clearly $f(\mu\cap\theta) = \beta(\mu\cap\theta) = \beta(\mu)\cap\beta(\theta)$. Again $f(\overleftarrow{\beta}\beta(\mu\vee\theta)) = \beta(\overleftarrow{\beta}\beta(\mu\vee\theta)) = \beta(\mu\vee\theta) = \beta(\mu)\sqcup\beta(\theta)$. Therefore f is an isomorphism of the lattice of β-fuzzy filters of L onto the lattice of fuzzy ideals of $\mathcal{B}_0(L)$. □

4. Prime β-Fuzzy Filters and Maximal β-Fuzzy Filters of MS-algebras

In this section, we study prime β-fuzzy filters and maximal β-fuzzy filters of MS-algebras and discussed about some properties of them.

DEFINITION 4.1. A proper β-fuzzy filter μ of L is called a prime β-fuzzy filter if for any fuzzy filters θ and ν such that $\theta\cap\nu \subseteq \mu$, we have $\theta \subseteq \mu$ or $\nu \subseteq \mu$.

LEMMA 4.2. *A proper β-filter P of L is a prime β-filter of L , $\alpha \in [0,1)$ if and only if*

$$P^1_\alpha(z) = \begin{cases} 1 & if\ z \in P \\ \alpha & otherwise \end{cases}$$

is a prime β-fuzzy filter of L.

Proof. Suppose that P is a proper β-filter of L and $\alpha \in [0,1)$. It is easily verified P^1_α is a proper fuzzy filter of L. Now, we prove that P^1_α is a prime fuzzy filter of L. Let θ and λ be fuzzy filters of L such that $\theta \nsubseteq P^1_\alpha$ and $\lambda \nsubseteq P^1_\alpha$. Then there exist $x, y \in L$ such that $\theta(x) > P^1_\alpha(x)$ and $\lambda(y) > P^1_\alpha(y)$. This implies $x \notin P$ and $y \notin P$, and so $x \vee y \notin P$ and $P^1_\alpha(x \vee y) = \alpha$. It follows that $\theta(x) \wedge \lambda(y) > \alpha$. Since θ and λ are isotones, we have $(\theta \cap \lambda)(x \vee y) = \theta(x \vee y) \wedge \lambda(x \vee y) \geq \theta(x) \wedge \lambda(x) > \alpha = P^1_\alpha(x \vee y)$. This implies $\theta \cap \lambda \nsubseteq P^1_\alpha$. Thus P^1_α is a prime fuzzy filter of L. Next, we prove that P^1_α is a prime β-fuzzy filter of L. Since P is a prime β-filter of L and $\alpha \in [0,1)$, for any $x, y \in L$ such that $(x)^+ = (y)^+$,
If $P^1_\alpha(x) = 1$, then $x \in P$. This implies $y \in P$ and $P^1_\alpha(y) = 1$.
If $P^1_\alpha(x) = \alpha$, then $x \notin P$. This implies $y \notin P$ and $P^1_\alpha(y) = \alpha$. Hence P^1_α is a prime β-fuzzy filter of L.
Conversely, suppose that P^1_α is a prime fuzzy filter of L. If I and J are any filters of L such that $I \cap J \subseteq P$, then $(I \cap J)^1_\alpha = I^1_\alpha \cap J^1_\alpha \subseteq P^1_\alpha$. This implies $I^1_\alpha \subseteq P^1_\alpha$ or $J^1_\alpha \subseteq P^1_\alpha$, so that $I \subseteq P$ or $J \subseteq P$. Therefore P is a prime filter of L. Now, suppose that P^1_α is a prime β-fuzzy filter of L and for any $x, y \in L$ such that $(x)^+ = (y)^+$. Let $x \in P$. Then $1 = P^1_\alpha(x) = P^1_\alpha(y)$. This implies $y \in P$. Hence P is a prime β-filter of L. $\square$

EXAMPLE 4.3. In diagram 2, $A = \{1\}$, $B = \{1, e\}$, $C = \{1, e, d\}$, $D = \{1, e, d, b\}$, $E = \{1, e, d, c, \}$, $F = \{1, a, b, c, d, e\}$ are filters of L and all except for C are prime filters of L and also F is a prime β-filter of L.

In addition, it is easily verified that A^1_α,B^1_α,D^1_α, E^1_α and F^1_α are prime fuzzy filters of L, for any $\alpha \in [0,1)$.
F^1_α is a prime β-fuzzy filter of L, and A^1_α, B^1_α, D^1_α and E^1_α are not prime β-fuzzy filters of L .

COROLLARY 4.4. *A proper filter P is a prime β-filter of L if and only if χ_P is a prime β-fuzzy filter of L.*

Proof. Suppose that P is a prime β-filter of L. First we prove that χ_P is a prime fuzzy filter of L. For any μ and λ be any fuzzy filters of L such that $\theta \cap \lambda \subseteq \chi_P$. Suppose that $\lambda \nsubseteq \chi_P$ and $\lambda \nsubseteq \chi_P$. This implies there exist $x, y \in L$ such that $\lambda(x) > \chi_P(x)$ and $\theta(y) > \chi_P(y)$. This implies $x \notin P$ and $y \notin P$. Since P is a prime filter, $x \vee y \notin P$. Thus $\chi_P(x \vee y) = 0$.
Now $(\lambda \cap \theta)(x \vee y) = \lambda(x \vee y) \wedge \theta(x \vee y) \geq \lambda(x) \wedge \theta(y) > \chi_P(x) \wedge \chi_P(y) = 0 = \chi_P(x \vee y)$. This implies $\theta \cap \lambda \nsubseteq \chi_P$, which is a contradiction. Thus

χ_P is a prime filter of L. Next we prove that χ_P is a prime β-fuzzy filter. For all $x, y \in L$ such that $(x)^+ = (y)^+$.
If $\chi_P(x) = 1$, then $x \in P$. This implies $y \in P$. Thus $\chi_P(y) = 1$.
If $\chi_P(x) = 0$, then $x \notin P$. This implies $y \notin P$. Thus $\chi_P(y) = 0$.
Hence χ_P is a prime β-fuzzy filter of L.
Conversely, suppose that χ_P is a prime β-filter of L. First we prove that P is a prime filter of L. Let I and J be any filters of L such that $I \cap J \subseteq P$.

$$\begin{aligned}
&\Rightarrow \quad \chi_{I\cap J} \subseteq \chi_P \\
&\Rightarrow \quad \chi_I \subseteq \chi_P \ or \ \chi_J \subseteq \chi_P \\
&\Rightarrow \quad I \subseteq P \ or \ J \subseteq P
\end{aligned}$$

This implies P is a prime filter.
Next, we prove that P is a prime β-filter of L. Suppose for all $x, y \in L$ such that $(x)^+ = (y)^+$. Let $x \in P$. Then $\chi_P(x) = 1 = \chi_P(y)$. Thus $y \in P$. Hence P is a prime β-fuzzy filter of L. □

THEOREM 4.5. *A proper fuzzy filter μ of L is a prime β-fuzzy filter if and only if $Img(\mu) = \{1, \alpha\}$, where $\alpha \in [0, 1)$ and the set $\mu_* = \{x \in L : \mu(x) = 1\}$ is a prime β-filter of L.*

Proof. The converse part of this theorem follows from Lemma 4.2. Suppose that μ is a prime β-fuzzy filter. Clearly $1 \in Im(\mu)$ and since μ is proper, there is $x \in L$ such that $\mu(x) < 1$. We prove that $\mu(x) = \mu(y)$ for all $x, y \in L - \mu_*$. Suppose that $\mu(x) \neq \mu(y)$ for some $x, y \in L - \mu_*$. Without loss of generality we can assume that $\mu(y) < \mu(x) < 1$. Define fuzzy subsets θ and λ as follows:

$$\theta(z) = \begin{cases} 1 & if\ z \in [x) \\ 0 & otherwise. \end{cases}$$

and

$$\lambda(z) = \begin{cases} 1 & if\ z \in \mu_* \\ \mu(x) & otherwise. \end{cases}$$

for all $z \in L$. Then it can be easily verified that both θ and λ are fuzzy filters of L. Let $z \in L$. If $z \in \mu_*$, then $(\theta \cap \lambda)(z) \leq 1 = \mu(z)$. If $z \in [x) - \mu_*$, then $z = x \vee z$, and we have $(\theta \cap \lambda)(z) = \theta(z) \wedge \lambda(z) = 1 \wedge \mu(x) = \mu(x) \leq \mu(z)$.
Also if $z \notin [x)$, then $\theta(z) = 0$, so that $(\theta \cap \lambda)(z) = 0 \leq \mu(z)$. Therefore $\theta \cap \lambda \subseteq \mu$. But we have $\theta(x) = 1 > \mu(x)$ and $\lambda(y) = \mu(x) > \mu(y)$. This

implies $\lambda \nsubseteq \mu$ and $\theta \nsubseteq \lambda$, which is a contradiction. Thus $\mu(x) = \mu(y)$ for all $x, y \in L - \mu_*$ and hence $Im(\mu) = \{1, \alpha\}$ for some $\alpha \in [0, 1)$. Let $P = \{x \in L : \mu(x) = 1\}$. Since μ is proper, we get that P is a proper filter of L. Let $\alpha \neq 1$. Then

$$\mu(z) = \begin{cases} 1 & if\ z \in P \\ \alpha & if\ z \notin P. \end{cases}$$

Hence by Lemma 4.2, $P = \mu_*$. □

Now we introduce maximal β-filters of an MS-algebra L.

Definition 4.6. A proper fuzzy filter μ of an MS-algebra L is called a maximal fuzzy filter of L if $Im\mu = \{1, \alpha\}$, where $\alpha \in [0, 1)$ and the level filter $\mu_* = \{x \in L : \mu(x) = 1\}$ is a maximal filter.

Definition 4.7. A proper fuzzy filter μ of an MS-algebra L is called a maximal β-fuzzy filter of L if $Im\mu = \{1, \alpha\}$, where $\alpha \in [0, 1)$ and the level filter μ_* is a maximal β-filter.

Example 4.8. In Example 3.9, $\mu_\star = \{1, b\}$, $Im\mu = \{1, \alpha\}$ and μ_* is a maximal β-filter of L. Hence μ is a maximal β-fuzzy filter of L.

Theorem 4.9. *Every maximal fuzzy filter of an MS-algebra is a β-fuzzy filter.*

Proof. Suppose that μ is a maximal fuzzy filter of L. Then μ_* is a maximal filter and $Im\mu = \{1, \alpha\}$. This implies μ_* is maximal and $\mu_\alpha = L$. Since every maximal filter is a β-filter of L. This implies the level subsets of L is β filters of L. Hence μ is a β-fuzzy filter of L. □

Corollary 4.10. *Every maximal β-fuzzy filter of L is a maximal fuzzy filter.*

Corollary 4.11. *Every maximal β-fuzzy filter of L is a prime β-fuzzy filter.*

Theorem 4.12. *Let L be an MS-algebra. If μ is minimal in the class of all prime fuzzy filters containing a given β-fuzzy filter, then μ is a β-fuzzy filter.*

Proof. Suppose μ is minimal in the class of all prime fuzzy filters containing a β-fuzzy filter θ of L. Since μ is a prime fuzzy filter of L, there exists a prime filter P of L such

$$\mu(z) = \begin{cases} 1 & if\ z \in P \\ \alpha & otherwise. \end{cases}$$

for some $\alpha \in [0, 1)$. Suppose that μ is not a β-fuzzy filter of L. Then there exist $x, y \in L$, $(x)^+ = (y)^+$ such that $\mu(x) \neq \mu(y)$. Without loss of generality $\mu(x) = 1$ and $\mu(y) = \alpha$. Consider a fuzzy ideal ϕ of L defined by

$$\phi(z) = \begin{cases} 1 & if\ z \in (L - P) \vee (x \vee y] \\ \alpha & otherwise. \end{cases}$$

Then $\theta \cap \phi \leq \alpha$. Otherwise there exists $a \in L$ such that $\phi(a) = 1$. This implies $a \in (L - P) \vee (x \vee y]$.

$$\implies \quad a = r \vee s\ for\ some\ r \in (L - P)\ and\ s \in (x \vee y]$$

$$\implies \quad a = r \vee s = r \vee (s \wedge ((x \vee y)) = (r \vee s) \wedge (s \vee x \vee y) \leq s \vee x \vee y$$

Since θ is a β-fuzzy filter of L, $\alpha < \theta(r \vee s) \leq \theta(r \vee x \vee y) \leq \mu(r \vee x \vee y)$. Also $x^+ = y^+$ implies $(r \vee x \vee y)^+ = (r \vee y)^+$. This implies $\theta(r \vee x \vee y) = \theta(r \vee y) \leq \mu(r \vee y) = 1$. Since μ is a prime filter, $\mu(r) = 1$ or $\mu(y) = 1$, which is a contradiction. Thus $\theta \cap \phi \leq \alpha$. This implies there exists a prime fuzzy filter η such that $\eta \cap \phi \leq \alpha$ and $\theta \subseteq \eta$. Clearly $x \vee y \in (L - P) \vee (x \vee y]$. This implies $\phi(x \vee y) = 1$ and $\phi \cap \eta \leq \alpha$. Hence $\eta(x \vee y) \leq \alpha < \mu(x \vee y) = 1$. This implies $\mu \nsubseteq \eta$. Therefore μ is not minimal in the class of all prime fuzzy filters containing a given β-fuzzy filter, which is a contradiction. Therefore μ is a β-fuzzy filter. □

Corollary 4.13. *Let L be an MS-algebra. Then prime β-fuzzy filters of L are one to one correspondence with the prime fuzzy ideals of $\mathcal{B}_0(L)$.*

Proof. From Theorem 3.17 we have seen that β-fuzzy filters of L are one to one correspondence with the fuzzy ideals of $\mathcal{B}_0(L)$. Now we prove that if μ is a prime β-fuzzy filter then $\beta(\mu)$ is a prime fuzzy ideal of $\mathcal{B}_0(L)$ and vice versa. Let μ be a prime β-fuzzy filter of L. Then $\beta(\mu)$ is a fuzzy ideal of $\mathcal{B}_0(L)$. Let θ and ν be any ideals of $\mathcal{B}_0(L)$. Then there exist a β-fuzzy filter of L, ϕ and ψ such that $\theta = \beta(\phi)$ and $\nu = \beta(\psi)$. Assume $\beta(\phi) \cap \beta(\psi) \subseteq \beta(\mu)$. Then $\beta(\phi \cap \psi) \subseteq \beta(\mu)$ and so $\phi \cap \psi \subseteq \mu$. Since μ is a prime β-filter of L, then $\phi \subseteq \mu$ or $\psi \subseteq \mu$. This gives $\beta(\phi) \subseteq \beta(\mu)$ or $\beta(\psi) \subseteq \beta(\mu)$.

Let μ be a prime ideal of $\mathcal{B}_0(L)$. Then there exists a β-fuzzy filter of η

of L such that $\mu = \beta(\eta)$. Let ϕ, ψ be any fuzzy filters of L such that $\phi \cap \psi \subseteq \eta$. Then $\beta(\phi \cap \psi) = \beta(\phi) \cap \beta(\psi) \subseteq \beta(\eta)$. Since $\beta(\eta)$ is a prime ideal of L, then $\beta(\phi) \subseteq \beta(\eta)$ or $\beta(\psi) \subseteq \beta(\eta)$ and so $\phi \subseteq \eta$ or $\psi \subseteq \eta$. This implies η is a prime β-fuzzy filter of L. Thus prime β-fuzzy filters of L are one to one correspondence with the prime fuzzy ideals of $\mathcal{B}_0(L)$. □

In the following Theorem we prove the existence of prime β-fuzzy filters in MS-algebra.

THEOREM 4.14. *Let $\alpha \in [0, 1)$, μ be a β-fuzzy filter and σ be a fuzzy ideal of an MS-algebra L such that $\mu \cap \sigma \leq \alpha$. Then there exists a prime β-fuzzy filter η such that $\mu \subseteq \eta$ and $\eta \cap \sigma \leq \alpha$.*

Proof. Put $\xi = \{\theta \in \mathcal{FF}_\beta(L) : \mu \subseteq \theta,\ \theta \cap \sigma \leq \alpha\}$. Clearly $\mu \in \xi$, $\xi \neq \emptyset$, and $(\xi, \subseteq)$ is a poset. Let $Q = \{\mu_i : i \in \Omega\}$ be a chain in ξ. We prove that $\cup_{i\in\Omega}\mu_i \in \xi$. Clearly $(\cup_{i\in\Omega}\mu_i)(1) = 1$. For any $x, y \in L$,

$$\begin{aligned}
(\cup_{i\in\Omega}\mu_i)(x) \wedge (\cup_{i\in\Omega}\mu_i)(y) &= \sup\{\mu_i(x) : i \in \Omega\} \wedge \sup\{\mu_j(y) : j \in \Omega\} \\
&= \sup\{\mu_i(x) \wedge \mu_j(y) : i, j \in \Omega\} \\
&\leq \sup\{(\mu_i \cup \mu_j)(x) \wedge (\mu_i \cup \mu_j)(y) : i, j \in \Omega\}
\end{aligned}$$

Since Q is a chain, $\mu_i \subseteq \mu_j$ or $\mu_j \subseteq \mu_i$. Without loss of generality, assume $\mu_j \subseteq \mu_i$. This implies $\mu_i \cup \mu_j = \mu_i$. This shows,

$$\begin{aligned}
(\cup_{i\in\Omega}\mu_i)(x) \wedge (\cup_{i\in\Omega}\mu_i)(y) &\leq \sup\{\mu_i(x) \wedge \mu_i(y),\ i \in \Omega\} \\
&= \sup\{\mu_i(x \wedge y),\ i \in \Omega\} \\
&= (\cup_{i\in\Omega}\mu_i)(x \wedge y)
\end{aligned}$$

Again $(\cup_{i\in\Omega}\mu_i)(x) = \sup\{\mu_i(x) : i \in \Omega\} \leq \sup\{\mu_i(x \vee y) : i \in \Omega\} = (\cup_{i\in\Omega}\mu_i)(x \vee y)$. Similarly $(\cup_{i\in\Omega}\mu_i)(y) \leq (\cup_{i\in\Omega}\mu_i)(x \vee y)$. This implies $(\cup_{i\in\Omega}\mu_i)(x) \vee (\cup_{i\in\Omega}\mu_i)(y) \leq (\cup_{i\in\Omega}\mu_i)(x \vee y)$. Hence $\cup_{i\in\Omega}\mu_i$ is a fuzzy filter of L. Now prove that $(\cup_{i\in\Omega}\mu_i)$ is a β-fuzzy filter.

$$\begin{aligned}
\overleftarrow{\beta}\,\beta(\cup_{i\in\Omega}\mu_i)(x) &= \sup\{(\cup_{i\in\Omega}\mu_i)(a) : (x)^+ = (a)^+,\ a \in L\} \\
&= \sup\{\sup\{(\mu_i)(a) :,\ i \in \Omega\} : (x)^+ = (a)^+,\ a \in L\} \\
&= \sup\{\sup\{(\mu_i)(a) : (x)^+ = (a)^+,\ a \in L\},\ i \in \Omega\} \\
&= \sup\{\overleftarrow{\beta}\,\beta(\mu_i)(x), i \in \Omega\} = \sup\{\mu_i(x), i \in \Omega\} \\
&= (\cup_{i\in\Omega}\mu_i)(x)
\end{aligned}$$

Thus $\cup_{i\in\Omega}\mu_i$ is a β-fuzzy filter of L. Since $\mu_i \cap \sigma \leq \alpha$ for each $i \in \Omega$,

$$\begin{aligned}
((\cup_{i\in\Omega}\mu_i) \cap \sigma)(x) &= (\cup_{i\in\Omega}\mu_i)(x) \wedge \sigma(x) \\
&= \sup\{\mu_i(x),\ i \in \Omega\} \wedge \sigma(x) \\
&= \sup\{\mu_i(x) \wedge \sigma(x),\ i \in \Omega\} \\
&= \sup\{(\mu_i \wedge \sigma)(x),\ i \in \Omega\} \leq \alpha
\end{aligned}$$

Thus $(\cup_{i\in\Omega}\mu_i) \cap \sigma) \leq \alpha$. Hence $\cup_{i\in\Omega}\mu_i \in \xi$. By applying Zorn's Lemma, we get a maximal element, say δ, i.e, δ is a β-fuzzy filter of L such that $\mu \subseteq \delta$ and $\delta \cap \theta \leq \alpha$. Next we show that δ is a prime β-fuzzy filter of L. Assume that δ is not a prime β-fuzzy filter. Let $\lambda_1,\ \lambda_2 \in FF(L)$, and $\lambda_1 \cap \lambda_2 \subseteq \delta$ such that$\lambda_1 \nsubseteq \delta$ and $\lambda_2 \nsubseteq \delta$. If we put $\delta_1 = \overleftarrow{\beta}\beta(\lambda_1 \vee \delta)$ and $\delta_2 = \overleftarrow{\beta}\beta(\lambda_2 \vee \delta)$, then both δ_1, δ_2 are β-fuzzy filters of L properly containing δ. Since δ is a maximal in ξ, we get $\delta_1, \delta_2 \notin \xi$. This indicates $\delta_1 \cap \theta \nleq \alpha$ and $\delta_2 \cap \theta \nleq \alpha$. This implies there exist $x, y \in L$ such that $(\delta_1 \cap \sigma)(x) > \alpha$ and $(\delta_2 \cap \sigma)(y) > \alpha$. We have $(\delta_1 \cap \sigma)(x \vee y) \wedge (\delta_2 \cap \sigma)(x \vee y) \geq (\delta_1 \cap \sigma)(x) \wedge (\delta_2 \cap \sigma)(y) > \alpha$, which implies

$$\begin{aligned}
&(\delta_1 \cap \sigma)(x \vee y) \wedge (\delta_2 \cap \sigma)(x \vee y) \\
&= ((\delta_1 \cap \theta) \cap (\delta_2 \cap \sigma))(x \vee y) \\
&= ((\delta_2 \cap \delta_2) \cap \sigma)(x \vee y) \\
&= ((\overleftarrow{\beta}\beta(\lambda_1 \vee \delta) \cap \overleftarrow{\beta}\beta(\lambda_2 \vee \delta)) \cap \sigma)(x \vee y) \\
&= (\overleftarrow{\beta}\beta((\lambda_1 \cap \lambda_2) \vee \delta) \cap \sigma)(x \vee y) \\
&= (\overleftarrow{\beta}\beta(\delta) \cap \sigma)(x \vee y) \ \ \textit{becouse } \lambda_1 \subseteq \delta \ \ \textit{and} \ \ \lambda_2 \subseteq \delta \\
&= (\delta \cap \theta)(x \vee y) \\
&> \alpha
\end{aligned}$$

This shows $(\delta \cap \sigma)(x \vee y) > \alpha$, which is a contradiction $\delta \cap \sigma \leq \alpha$. This δ is a prime β-fuzzy filter of L. □

Corollary 4.15. *Let μ be a fuzzy β-filter and σ be a fuzzy ideal of an MS-algebra L such that $\mu \cap \sigma = 0$. Then there exists a prime β-fuzzy filter η such that $\mu \subseteq \eta$ and $\eta \cap \sigma = 0$.*

Corollary 4.16. *Let $\alpha \in [0, 1)$, μ be a β-fuzzy filter of an MS-algebra L and $\mu(x) \leq \alpha$. Then there exists a prime β-fuzzy filter θ of L such that $\mu \subseteq \theta$ and $\theta(x) \leq \alpha$.*

Proof. Put $\xi = \{\theta \in \mathcal{FF}_\beta(L) : \mu \subseteq \theta \text{ and } \theta(x) \leq \alpha\}$. Clearly $\mu \in \xi$, $\xi \neq \emptyset$, and $(\xi, \subseteq)$ is a poset. Let $Q = \{\mu_i : i \in \Omega\}$ be a chain in ξ. We prove that $\cup_{i\in\Omega}\mu_i \in \xi$. By Theorem 4.14, $(\cup_{i\in\Omega}\mu_i)$ is a β-fuzzy filter of L. Since $\mu_i \subseteq \theta$ for each $i \in \Omega$ and $\theta(x) \leq \alpha$.

$(\cup_{i\in\Omega}\mu_i)(x) = \sup\{\mu_i(x),\ i \in \Omega\} \leq \theta(x) \leq \alpha$.

Hence $\cup_{i\in\Omega}\mu_i \in \xi$. By applying Zorn's Lemma, we get a maximal element of ξ, say δ, i.e, δ is an β-fuzzy filter of L such that $\mu \subseteq \delta$ and $\delta(x) \leq \alpha$. Next we show that δ is a prime β-fuzzy filter of L. Assume that δ is not a prime β-fuzzy filter. Let $\lambda_1,\ \lambda_2 \in FF(L)$, and $\lambda_1 \cap \lambda_2 \subseteq \delta$ such that $\lambda_1 \nsubseteq \delta$ and $\lambda_2 \nsubseteq \delta$. If we put $\delta_1 = \overleftarrow{\beta}\beta(\lambda_1 \vee \delta)$ and $\delta_2 = \overleftarrow{\beta}\beta(\lambda_2 \vee \delta)$, then both δ_1, δ_2 are β-fuzzy filters of L properly containing δ. Since δ is maximal in ξ, we get $\delta_1, \delta_2 \notin \xi$. This we show that $\delta_1(x) \nleq \alpha$ and $\delta_2(x) \nleq \alpha$. Thus implies $\delta_1(x) > \alpha$ and $\delta_2(x) > \alpha$. We get $(\delta_1(x) \wedge (\delta_2)(x) = (\delta_1 \cap \delta_2)(x) > \alpha$, which implies

$$\begin{aligned}
\delta_1(x) \wedge \delta_2(x) &= (\overleftarrow{\beta}\beta(\lambda_1 \vee \delta) \cap \overleftarrow{\beta}\beta(\lambda_2 \vee \delta))(x) \\
&= (\overleftarrow{\beta}\beta((\lambda_1 \cap \lambda_2) \vee \delta))(x) \\
&= \overleftarrow{\beta}\beta(\delta)(x) \ \textit{becouse}\ \lambda_1 \subseteq \delta \ \textit{and}\ \lambda_2 \subseteq \delta \\
&= \delta(x) \\
&> \alpha
\end{aligned}$$

This shows $\delta(x) > \alpha$, which is a contradiction $\delta(x) \leq \alpha$. Thus δ is a prime β-fuzzy filter of L. □

COROLLARY 4.17. *Let L be an MS-algebra. Then every proper β-fuzzy filters of L is the intersection of all prime β-fuzzy filters containing it.*

Proof. Let μ be a proper β-fuzzy filter of L Put $\eta = \cap\{\theta : \theta \text{ is a prime } \beta\text{-fuzzy filter such that } \mu \subseteq \theta\}$. Now, we prove that $\mu = \eta$. Clearly $\mu \subseteq \eta$. Put $\alpha = \mu(a)$ for some $a \in L$. This implies $\mu \subseteq \mu$ and $\mu(a) \leq \alpha$. Thus by the Corollary 4.16, there exists a prime β-fuzzy filter δ such that $\mu \subseteq \delta$ and $\delta(a) \leq \alpha$. Thus $\eta \subseteq \mu$. Hence $\mu = \eta$. This implies every proper β-fuzzy filters of L is the intersection of all prime β-fuzzy filters containing it. □

References

[1] N. Ajmal, *Fuzzy lattice*, Inform. Sci. **79** (1994), 271–291.

[2] M. Attallah, *Completely fuzzy prime ideals of distributive lattices*, The Journal of Fuzzy Mathematics **8** (1) (2000), 151–156.

[3] B. A. Alaba and T. G. Alemayehu, *Clousure fuzzy ideals of MS-algebras*, Ann. Fuzzy Math. Inform. **16** (2) (2018), 247–260.

[4] B. A. Alaba, M. A. Taye and T. G. Alemayehu, *δ-fuzzy ideals in MS-algebras*, Int. J. Math. and Appl. **6** (2-B) (2018), 273–280.

[5] T. S. Blyth and J. C. Varlet, *On a common abstraction of de Morgan and Stone algebras*, Proc.Roy.Soc.Edinburgh Sect. A (1983), 94A, 301–308.

[6] T. S. Blyth and J. C. Varlet, *Subvarieties of the class of MS-algebras*, Proc.Roy. Soc. Edinburgh Sect. A (1983), 94A, 301–308.

[7] B. A. Davey and H. A. Priestley, *Introduction to Lattices and order*, Second edition Cambrige (2002).

[8] A. Kadji, C. Lele and M. Tonga, *Fuzzy prime and maximal filters of residual lattice*, Soft Comput, Springer-Verlag Berlin Heidelberg (2016) DOI 10.1007/s00500-016-2113-2.

[9] B. B. N. Koguep , C. Nkuimi and C. Lele, *On Fuzzy prime ideals of lattice*, SJPAM **3** (2008), 1–11.

[10] A. Rosenfeld, *Fuzzy groups*, J. Math. Anal. Appl. **35** (1971), 512–517.

[11] M. Sambasiva Rao, *β-Filters of MS-algebras*, Asian-Eur. J. Math. **5** (2) (2012), 1250023(8 pages).

[12] U. M. Swamy and D. V. Raju, *Fuzzy ideals and congruences of lattices*, Fuzzy sets and systems **95** (1998), 249–253.

[13] U. M. Swamy and D. V. Raju, *Irreducibility in algebraic fuzzy systems*, Fuzzy Sets and Systems **41** (1991), 233–241.

[14] Bo. Yuan and W. Wu, *Fuzzy ideals on a distributive lattice*, Fuzzy Sets and Systems **35** (1990), 231–240.

[15] L. A. Zadeh, *Fuzzy sets*, Information and Control **8** (1965), 338–353.

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