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GENERALIZED CONDITIONAL INTEGRAL TRANSFORMS, CONDITIONAL CONVOLUTIONS AND FIRST VARIATIONS

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Abstract. We study various relationships that exist among generalized conditional integral transform, generalized conditional convolution and generalized first variation for a class of functionals defined on $K[0,T]$, the space of complex-valued continuous functions on $[0,T]$ which vanish at zero.

1. Definitions and preliminaries

Let $C_0[0,T]$ denote one-parameter Wiener space; that is, the space of all $\mathbb{R}$-valued continuous functions $x(t)$ on $[0,T]$ with $x(0)=0$. Let $\mathcal{M}$ denote the class of all Wiener measurable subsets of $C_0[0,T]$ and let m denote Wiener measure. $(C_0[0,T],\mathcal{M},m)$ is a complete measure space and we denote the Wiener integral of a Wiener integrable functional F by

$$E_x[F(x)] = \int_{C_0[0,T]} F(x)\, m(dx). \tag{1.1}$$

Throughout this paper our starting point is the generalized Wiener integral

$$E_x[F(Z_h(x,\cdot))] = \int_{C_0[0,T]} F(Z_h(x,\cdot))\, m(dx), \tag{1.2}$$

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where Z_h is the Gaussian process

$$Z_h(x,t)=\int_0^t h(u)dx(u) \tag{1.3}$$

with $h(\neq 0)$ is in $L_2[0,T]$, and $\int_0^t h(u)dx(u)$ denotes the Paley-Wiener-Zygmund stochastic integral. Of course if $h(t)\equiv 1$ on $[0,T]$, then $Z_h(x,t)=x(t)$ and so the generalized Wiener integral (1.2) reduces to the Wiener integral (1.1). We will simply refer to the integral (1.2) as a Wiener integral.

The Gaussian process Z_h has mean zero and covariance function $E_x[Z_h(x,s)Z_h(x,t)]=a(\min\{s,t\})$ where $a(t)=\int_0^t h^2(u)du$. In addition $Z_h(x,t)$ is stochastically continuous in t on $[0,T]$.

Let $K=K[0,T]$ be the space of all $\mathbb{C}$-valued continuous functions defined on $[0,T]$ which vanish at $t=0$ and let α and β be nonzero complex numbers. In [2], Cameron and Martin defined a Fourier-Wiener transform of functionals defined on $K[0,T]$. In [3], Cameron and Storvick defined a Fourier-Feynman transform of functionals defined on $C_0[0,T]$. In a unifying paper [14], Lee defined an integral transform $\mathcal{F}_{\alpha,\beta}$ of analytic functionals on an abstract Wiener space. For certain values of the parameters α and β and for certain classes of functionals, the Fourier-Wiener transform, the Fourier-Feynman transform and the Gauss transform are special cases of the integral transform $\mathcal{F}_{\alpha,\beta}$.

In [21], Yeh studied conditional Wiener integrals of functionals defined on $C_0[0,T]$. In [8], Chung and Skoug introduced the concept of a conditional Feynman integral, while in [15], Park and Skoug introduced the concept of a conditional Fourier-Feynman transform and a conditional convolution for functionals defined on $C_0[0,T]$.

In this paper we establish various relationships that exist among generalized conditional integral transform, generalized conditional convolution and generalized first variation for a class of functionals defined on $K[0,T]$.

We finish this section by stating definitions of integral transform $\mathcal{F}_{\alpha,\beta}$, convolution $(F*G)_\alpha$ and first variation δF for functionals defined on K. The main focus of [11] was to establish various relationships holding among $\mathcal{F}_{\alpha,\beta}F$, $\mathcal{F}_{\alpha,\beta}G$, $(F*G)_\alpha$, δF and δG.

Definition 1.1. Let F be a functional defined on K. Then the integral transform $\mathcal{F}_{\alpha,\beta}F$ of F is defined by

$$\mathcal{F}_{\alpha,\beta}F(y)\equiv E_x[F(\alpha x+\beta y)],\quad y\in K \tag{1.4}$$

if it exists [5, 11, 13, 15, 19].

Definition 1.2. Let F and G be functionals defined on K. Then the convolution $(F * G)_\alpha$ of F and G is defined by

$$(F * G)_\alpha(y) \equiv E_x\Big[F\Big(\frac{y+\alpha x}{\sqrt{2}}\Big)G\Big(\frac{y-\alpha x}{\sqrt{2}}\Big)\Big], \quad y \in K \tag{1.5}$$

if it exists [5, 10, 11, 19, 20, 22].

Definition 1.3. Let F be a functional defined on K and let $w \in K$. Then the first variation δF of F is defined by

$$\delta F(y|w) \equiv \frac{\partial}{\partial t}F(y+tw)|_{t=0}, \quad y \in K \tag{1.6}$$

if it exists [1, 4, 11, 18].

2. Generalized conditional integral transforms and generalized conditional convolution

Let $X : C_0[0,T] \to \mathbb{R}$ be a Wiener measurable functional and let $F : C_0[0,T] \to \mathbb{C}$ be a Wiener integrable functional. Then for $\eta \in \mathbb{R}$, $E[F\|X](\eta)$ denotes the conditional Wiener integral of F given X [6, 8, 16, 21]. In [16], Park and Skoug gave a formula for expressing conditional Wiener integrals in terms of ordinary(i.e., non-conditional) Wiener integrals; namely that for $X(x) = x(T)$,

$$E_x[F(x)\|X(x)](\eta) = E_x[F(x(\cdot) - \frac{\cdot}{T}x(T) + \frac{\cdot}{T}\eta)]. \tag{2.1}$$

Similarly for the condition $X_h(x) = Z_h(x,T)$, we can get the formula for expressing conditional Wiener integrals,

$$E_x[F(Z_h(x,\cdot)\|X_h(x)](\eta) = E_x[F(Z_{T,\eta}^{\{h,a\}}(x,\cdot))],$$

where $Z_{T,\eta}^{\{h,a\}}(x,\cdot) = Z_h(x,\cdot) - \frac{a(\cdot)}{a(T)}Z_h(x,T) + \frac{a(\cdot)}{a(T)}\eta$.

In this paper we will always condition by

$$X_h(x) = Z_h(x,T), \tag{2.2}$$

where Z_h is given by (1.3).

Definition 2.1. For $F : K[0,T] \to \mathbb{C}$ we define generalized conditional integral transform, $\mathcal{F}_{\alpha,\beta}(F\|X_h)(y,\eta)$ of F given X_h by the formula

$$\mathcal{F}_{\alpha,\beta}(F\|X_h)(y,\eta) = E_x[F(\alpha Z_h(x,\cdot) + \beta y)\|X_h(x) = \eta] \tag{2.3}$$

for $y \in K$ and $\eta \in \mathbb{R}$ if it exists.

DEFINITION 2.2. For functionals F and G defined on K, we define generalized conditional convolution, $((F * G)_\alpha \| X_h)(y,\eta)$, of $(F * G)_\alpha$ given X_h by the formula

(2.4)
$$((F{*}G)_\alpha \| X_h)(y,\eta) = E_x[F\Big(\frac{y+\alpha Z_h(x,\cdot)}{\sqrt{2}}\Big)G\Big(\frac{y-\alpha Z_h(x,\cdot)}{\sqrt{2}}\Big)||X_h(x)=\eta]$$

for $y \in K$ and $\eta \in \mathbb{R}$ if it exists.

Next we give a definition of generalized first variation $\delta_{h_1,h_2}F$ of a functional F on K.

DEFINITION 2.3. Let F be a functional defined on K and let $w \in K$ and $h_1, h_2 \in L_2[0,T]$. Then the generalized first variation $\delta_{h_1,h_2}F$ of F is defined by

$$\delta_{h_1,h_2}F(y|w) = \frac{\partial}{\partial r}F(Z_{h_1}(y,\cdot) + rZ_{h_2}(w,\cdot))|_{r=0} \tag{2.5}$$

for $y \in K$ if it exists.

REMARK 2.4. (i)When $h \equiv 1$ on $[0,T]$, the generalized conditional integral transform and generalized conditional convolution are reduced to conditional integral transform and conditional convolution, respectively, which are defined and studied in [12].

(ii) When $h_1 = h_2 \equiv 1$ on $[0,T]$, our definition of the generalized first variation is reduced to the first variation studied in $[1,4,11,12,18]$.

(iii) Using the formula for expressing conditional Wiener integral with conditioning function $X_h(x) = Z_h(x,T)$ in [17], we have

$$\mathcal{F}_{\alpha,\beta}(F\|X_h)(y,\eta) = E_x[F(\alpha Z_{T,\eta}^{\{h,a\}}(x,\cdot) + \beta y(\cdot))] \tag{2.6}$$

and

(2.7)
$$\begin{aligned}&((F*G)_\alpha\|X_h)(y,\eta)\\ &= E_x\Big[F\Big(\frac{1}{\sqrt{2}}(y(\cdot)+\alpha Z_{T,\eta}^{\{h,a\}}(x,\cdot))\Big)G\Big(\frac{1}{\sqrt{2}}(y(\cdot)-\alpha Z_{T,\eta}^{\{h,a\}}(x,\cdot))\Big)\Big]\end{aligned}$$

where in (2.6) and (2.7) the existence of either side implies the other side and their equality.

Under rather mild conditions on F and G, our first theorem shows that the generalized conditional integral transform of the generalized conditional convolution is the product of generalized conditional integral transforms.

THEOREM 2.5. *Let α and β be nonzero complex numbers. Assume that for $F : K \to \mathbb{C}$ and $G : K \to \mathbb{C}$, $\mathcal{F}_{\alpha,\beta}(((F*G)_\alpha\|X_h)(\cdot,\eta_1)\|X_h)$, $\mathcal{F}_{\alpha,\beta}(F\|X_h)$ and $\mathcal{F}_{\alpha,\beta}(G\|X_h)$ all exist for a.e.$\eta_1 \in \mathbb{R}$. Then*

$$\begin{aligned}&\mathcal{F}_{\alpha,\beta}(((F*G)_\alpha\|X_h)(\cdot,\eta_1)\|X_h)(y,\eta_2)\\ &= \mathcal{F}_{\alpha,\beta}(F\|X_h)\Big(\frac{y}{\sqrt{2}},\frac{\eta_2+\eta_1}{\sqrt{2}}\Big)\mathcal{F}_{\alpha,\beta}(G\|X_h)\Big(\frac{y}{\sqrt{2}},\frac{\eta_2-\eta_1}{\sqrt{2}}\Big)\end{aligned} \tag{2.8}$$

for all $y \in K$ and a.e.$\eta_2 \in \mathbb{R}$.

Proof. From equations (2.3) through (2.7) we have the following;

$$\begin{aligned}R \equiv&\mathcal{F}_{\alpha,\beta}(((F*G)_\alpha\|X_h)(\cdot,\eta_1)\|X_h)(y,\eta_2)\\ =&E_x[E_w[F(\frac{\beta y(\cdot)}{\sqrt{2}}+\frac{\alpha}{\sqrt{2}}Z^{\{h,a\}}_{T,\eta_2+\eta_1}(x+w,\cdot))\\ &\cdot G(\frac{\beta y(\cdot)}{\sqrt{2}}+\frac{\alpha}{\sqrt{2}}Z^{\{h,a\}}_{T,\eta_2-\eta_1}(x-w,\cdot))]].\end{aligned}$$

Since $Z_h(x+w,\cdot)-\frac{a(\cdot)}{a(T)}Z_h(x+w,T)$ and $Z_h(x-w,\cdot)-\frac{a(\cdot)}{a(T)}Z_h(x-w,T)$ are independent processes as can be seen by checking their covariance function, we can see that

$$\begin{aligned}R =&E_x[E_w[F(\frac{\beta y(\cdot)}{\sqrt{2}}+\frac{\alpha}{\sqrt{2}}Z^{\{h,a\}}_{T,\eta_2+\eta_1}(x+w,\cdot))]]\\ &\cdot E_x[E_w[G(\frac{\beta y(\cdot)}{\sqrt{2}}+\frac{\alpha}{\sqrt{2}}Z^{\{h,a\}}_{T,\eta_2-\eta_1}(x-w,\cdot))]].\end{aligned}$$

Also the processes $\frac{Z_h(x+w,\cdot)}{\sqrt{2}}$ and $\frac{Z_h(x-w,\cdot)}{\sqrt{2}}$ are each equivalent to the process $Z_h(x,\cdot)$, and the equation (2.6) give us the following result;

$$\begin{aligned}R =&E_x[F(\frac{\beta y(\cdot)}{\sqrt{2}}+\alpha Z^{\{h,a\}}_{T,\frac{\eta_2+\eta_1}{\sqrt{2}}}(x,\cdot))]E_x[G(\frac{\beta y(\cdot)}{\sqrt{2}}+\alpha Z^{\{h,a\}}_{T,\frac{\eta_2-\eta_1}{\sqrt{2}}}(x,\cdot))]\\ =&\mathcal{F}_{\alpha,\beta}(F\|X_h)(\frac{y}{\sqrt{2}},\frac{\eta_2+\eta_1}{\sqrt{2}})\mathcal{F}_{\alpha,\beta}(G\|X_h)(\frac{y}{\sqrt{2}},\frac{\eta_2-\eta_1}{\sqrt{2}})\end{aligned}$$

which completes the proof. □

Next we describe the class of functionals that we work with in this paper. Let $\{\theta_1, \theta_2, \ldots\}$ be a complete orthonormal set of $\mathbb{R}$-valued functions in $L_2[0,T]$. Furthermore assume that each θ_j is of bounded variation on $[0,T]$. Then for each $y \in K$ and $j \in \{1, 2, \ldots\}$, the Riemann-Stieltjes integral $\langle \theta_j, y\rangle \equiv \int_0^T \theta_j(t)\, dy(t)$ exists. Furthermore

$$|\langle \theta_j, y\rangle| = |\theta_j(T)y(T) - \int_0^T y(t)\, d\theta_j(t)| \le M_j \|y\|_\infty \tag{2.9}$$

with

$$M_j = |\theta_j(T)| + \mathrm{Var}(\theta_j, [0,T]). \tag{2.10}$$

For $0 \le \sigma < 1$, let E_σ be the space of all functionals $F : K \to \mathbb{C}$ of the form

$$F(y) = f(\langle \theta_1, y\rangle, \ldots, \langle \theta_n, y\rangle) = f(\langle \vec{\theta}, y\rangle) \tag{2.11}$$

for some positive integer n, where $f(\lambda_1, \ldots, \lambda_n) = f(\vec{\lambda})$ is an entire function of n complex variables $\lambda_1, \ldots, \lambda_n$ of exponential type; that is to say

$$|f(\vec{\lambda})| \le A_F \exp\{B_F \sum_{j=1}^{n} |\lambda_j|^{1+\sigma}\} \tag{2.12}$$

for some positive constants A_F and B_F.

In [12], the current authors and Skoug showed that for all F and G in E_σ, $\mathcal{F}_{\alpha,\beta}(F\|X)$ and $((F*G)_\alpha\|X)$ exist and belong to E_σ for all nonzero complex numbers α and β and the condition by $X(x) = x(T)$ while $\delta F(y|w)$ exists and belongs to E_σ for all y and w in K.

For F of the form (2.11), as we will see below in Theorem 2.7, when we evaluate the generalized conditional integral transform of F given X_h, we encounter the Riemann-Stieltjes integrals $\langle \theta_j(\cdot), Z_{T,\eta}^{\{h,a\}}(x, \cdot)\rangle$. Letting

$$b_j = \frac{1}{a(T)} \int_0^T \theta_j(t)\, da(t), \tag{2.13}$$

we see that

$$\langle \theta_j(\cdot), Z_{T,\eta}^{\{h,a\}}(x, \cdot)\rangle = \langle \theta_j(\cdot) - b_j, Z_h(x, \cdot)\rangle + \eta b_j \tag{2.14}$$

for $x \in C_0[0,T]$.

Take a $n\times n$ matrix $C=(c_{j,k})$ and an orthonormal set $\{\phi_1,\ldots,\phi_n\}$ on $[0,T]$ satisfying

$$\vec{\theta}-\vec{b}=\vec{\phi}C. \tag{2.15}$$

For details on the matrix C and the orthonormal set $\{\phi_1,\ldots,\phi_n\}$, see Section 2 of [12].

The following lemma [7], which follows quite easily from the definition of the Paley-Wiener-Zygmund stochastic integral, (2.14) and (2.15), plays a key role in the proof of Theorem 2.7. In view of the following lemma, throughout this paper we require h to be in $L_\infty[0,T]$ with $\{\phi_1 h,\ldots,\phi_n h\}$ be an orthogonal set in $L_2[0,T]$ rather than simply in $L_2[0,T]$.

Lemma 2.6. *For each $\phi\in L_2[0,T]$ and each $h\in L_\infty[0,T]$,*

$$\int_0^T \phi(t)dZ_h(x,t)=\int_0^T \phi(t)h(t)dx(t) \tag{2.16}$$

for s-a.e. $x\in C_0[0,T]$, that is, $\langle\phi,Z_h(x,\cdot)\rangle=\langle\phi h,x\rangle$.

Theorem 2.7. *Let $F\in E_\sigma$ be given by (2.11), $h\in L_\infty[0,T]$ with $\|\phi_j h\|_2^2>0$ for $j=1,2,\cdots n$, and let X_h be given by (2.2). Then the generalized conditional integral transform $\mathcal{F}_{\alpha,\beta}(F\|X_h)(y,\eta)$ exists, belongs to E_σ and is given by the formula*

$$\mathcal{F}_{\alpha,\beta}(F\|X_h)(y,\eta)=K_h(\eta;\langle\vec{\theta},y\rangle) \tag{2.17}$$

for all $y\in K$ and a.e.$\eta\in\mathbb{R}$, where

(2.18)

$$K_h(\eta;\vec{\lambda})$$
$$=((2\pi)^n\prod_{j=1}^n\|\phi_j h\|_2^2)^{-\frac12}\int_{\mathbb{R}^n} f(\alpha\vec{u}C+\alpha\eta\vec{b}+\beta\vec{\lambda})\exp\Big\{-\frac12\sum_{j=1}^n\frac{u_j^2}{\|\ \phi_j h\|_2^2}\Big\}\,d\vec{u}.$$

Proof. For each $y\in K$ and $a.e.\eta\in\mathbb{R}$,

$$\mathcal{F}_{\alpha,\beta}(F\|X_h)(y,\eta)=E_x[f(\alpha\langle\vec{\theta},Z_{T,\eta}^{\{h,a\}}(x,\cdot)\rangle+\beta\langle\vec{\theta},y\rangle)].$$

Using (2.14) and (2.15), we have

$$\begin{aligned}\mathcal{F}_{\alpha,\beta}(F\|X_h)(y,\eta)&=E_x[f(\alpha\langle\vec{\theta}-\vec{b},Z_h(x,\cdot)\rangle+\alpha\eta\vec{b}+\beta\langle\vec{\theta},y\rangle)]\\&=E_x[f(\alpha\langle\vec{\phi},Z_h(x,\cdot)\rangle C+\alpha\eta\vec{b}+\beta\langle\vec{\theta},y\rangle)].\end{aligned} \tag{2.19}$$

By Lemma 2.6 and a well-known Wiener integration theorem, we see that the last expression is equal to $K_h(\eta;\langle\vec{\theta},y\rangle)$, where $K_h(\eta;\cdot)$ is given by (2.18). By [9, Theorem 3.15] $K_h(\eta;\vec{\lambda})$ is an entire function. Moreover by the inequality (2.12) we have

$$\begin{aligned}
&|K_h(\eta;\vec{\lambda})|\\
\le&((2\pi)^n\prod_{j=1}^n\|\phi_j h\|_2^2)^{-1/2}A_F\exp\Big\{B_F(3|\beta|)^{1+\sigma}\sum_{j=1}^n|\lambda_j|^{1+\sigma}\Big\}\\
&\cdot\int_{\mathbb{R}^n}\exp\Big\{B_F(3|\alpha|)^{1+\sigma}\sum_{j=1}^n\Big(|(\vec{u}C)_j|^{1+\sigma}+|\eta b_j|^{1+\sigma}\Big)-\frac{1}{2}\sum_{j=1}^n\frac{u_j^2}{\|\phi_j h\|_2^2}\Big\}\,d\vec{u}\\
=&A_{\mathcal{F}_{\alpha,\beta;h}F}\exp\Big\{B_{\mathcal{F}_{\alpha,\beta;h}F}\sum_{j=1}^n|\lambda_j|^{1+\sigma}\Big\},
\end{aligned}$$

where $B_{\mathcal{F}_{\alpha,\beta;h}F}=B_F(3|\beta|)^{1+\sigma}$, and

$$\begin{aligned}
A_{\mathcal{F}_{\alpha,\beta;h}F}=&A_F((2\pi)^n\prod_{j=1}^n\|\phi_j h\|_2^2)^{-1/2}\int_{\mathbb{R}^n}\exp\Big\{B_F(3|\alpha|)^{1+\sigma}\\
&\sum_{j=1}^n\Big(|(\vec{u}C)_j|^{1+\sigma}+|\eta b_j|^{1+\sigma}\Big)-\frac{1}{2}\sum_{j=1}^n\frac{u_j^2}{\|\phi_j h\|_2^2}\Big\}\,d\vec{u}<\infty.
\end{aligned}$$

Hence $\mathcal{F}_{\alpha,\beta}(F\|X_h)(y,\eta)\in E_\sigma$ as a function of y. □

REMARK 2.8. For any $F\in E_\sigma$ and $G\in E_\sigma$ we can always express F by equation (2.11) and G by

$$G(y)=g(\langle\theta_1,y\rangle,\cdots,\langle\theta_n,y\rangle) \tag{2.20}$$

using the same positive integer n.

In our next theorem we show that the generalized conditional convolution of functionals from E_σ is an element of E_σ.

THEOREM 2.9. *Let $F,G\in E_\sigma$ be given by (2.11) and (2.20) with corresponding entire functions f and g, respectively. And let X_h be given by (2.2) with $h\in L_\infty[0,T]$ and $\|\phi_j h\|_2^2>0$ for $j=1,\cdots,n$. Then the generalized conditional convolution $((F*G)_\alpha\|X_h)(y,\eta)$ exists for all $y\in K$ and a.e.$\eta\in\mathbb{R}$, belongs to E_σ, and is given by the formula*

$$((F*G)_\alpha\|X_h)(y,\eta)=L_h(\eta;\langle\vec{\theta},y\rangle) \tag{2.21}$$

where

(2.22)

$$L_h(\eta;\vec\lambda) = ((2\pi)^n\prod_{j=1}^n\|\phi_jh\|_2^2)^{-1/2}\int_{\mathbb{R}^n} f\Big(\frac{\vec\lambda+\alpha\vec uC+\alpha\eta\vec b}{\sqrt2}\Big)g\Big(\frac{\vec\lambda-\alpha\vec uC-\alpha\eta\vec b}{\sqrt2}\Big) \exp\Big\{-\frac12\sum_{j=1}^n\frac{u_j^2}{\|\phi_jh\|_2^2}\Big\}\,d\vec u.$$

Proof. For each $y\in K$ and $a.e.\eta\in\mathbb{R}$,

$$\begin{aligned} L \equiv&((F*G)_\alpha\|X_h)(y,\eta)\\ =&E_x\Big[f\Big(\frac{1}{\sqrt2}(\langle\vec\theta,y\rangle+\alpha\langle\vec\theta,Z_h(x,\cdot)-\frac{a(\cdot)}{a(T)}Z_h(x,T)+\frac{a(\cdot)}{a(T)}\eta\rangle)\Big)\\ &g\Big(\frac{1}{\sqrt2}(\langle\vec\theta,y\rangle-\alpha\langle\vec\theta,Z_h(x,\cdot)-\frac{a(\cdot)}{a(T)}Z_h(x,T)+\frac{a(\cdot)}{a(T)}\eta\rangle)\Big)\Big].\end{aligned}$$

By (2.14), (2.15) and a well-known Wiener integration theorem, we see that the last expression is equal to $L_h(\eta;\langle\vec\theta,y\rangle)$ where $L_h(\eta;\vec\lambda)$ is given by (2.22). By [9, Theorem 3.15], $L_h(\eta;\vec\lambda)$ is an entire function and

$$\begin{aligned}&|L_h(\eta;\vec\lambda)|\\ &\le((2\pi)^n\prod_{j=1}^n\|\phi_jh\|_2^2)^{-\frac12}A_FA_G\exp\Big\{(B_F+B_G)\Big(\frac{3}{\sqrt2}\Big)^{1+\sigma}\sum_{j=1}^n|\lambda_j|^{1+\sigma}\Big\}\\ &\cdot\int_{\mathbb{R}^n}\exp\Big\{(B_F+B_G)\Big(\frac{3|\alpha|}{\sqrt2}\Big)^{1+\sigma}\sum_{j=1}^n\Big(|(\vec uC)_j|^{1+\sigma}+|\eta b_j|^{1+\sigma}-\frac12\frac{u_j^2}{\|\phi_jh\|_2^2}\Big)\Big\}\,d\vec u\\ &=A_{(F*G)_\alpha;h}\exp\Big\{B_{(F*G)_\alpha;h}\sum_{j=1}^n|\lambda_j|^{1+\sigma}\Big\},\end{aligned}$$

where

$$B_{(F*G)_\alpha;h}=(B_F+B_G)\Big(\frac{3}{\sqrt2}\Big)^{1+\sigma}$$

and

$$A_{(F*G)_\alpha;h} = A_F A_G\Big((2\pi)^n \prod_{j=1}^n \|\phi_j h\|_2^2\Big)^{-\frac{1}{2}} \int_{\mathbb{R}^n} \exp\Big\{(B_F+B_G)\Big(\frac{3|\alpha|}{\sqrt{2}}\Big)^{1+\sigma} \sum_{j=1}^n\Big(|(\vec{u}C)_j|^{1+\sigma}+|\eta b_j|^{1+\sigma}\Big) - \frac{1}{2}\sum_{j=1}^n \frac{u_j^2}{\|\phi_j h\|_2^2}\Big\}\, d\vec{u} < \infty.$$

Hence $((F*G)_\alpha\|X_h)(y,\eta) \in E_\sigma$ as a function of y. □

REMARK 2.10. Unlike the convolution product, the generalized conditional convolution product is not commutative because $((F*G)_\alpha\|X_h)(y,\eta) = ((G*F)_\alpha\|X_h)(y,-\eta)$. However the usual additive distribution properties hold for the generalized conditional convolution product.

As in [12, Theorem 2.6], we can show that the generalized first variation $\delta_{h_1,h_2}F(y|w)$ of functionals F in E_σ is an element of E_σ, both as a function of y for fixed w and as a function of w for fixed y.

THEOREM 2.11. *Let $F \in E_\sigma$ be given by (2.11) and let h_1 and h_2 be in $L_\infty[0,T]$. Then for all y and w in K,*

$$\begin{aligned} \delta_{h_1,h_2}F(y|w) &= \sum_{j=1}^n \langle\theta_j, Z_{h_2}(w,\cdot)\rangle f_j(\langle\vec{\theta}, Z_{h_1}(y,\cdot)\rangle) \\ &= \sum_{j=1}^n \langle\theta_j, Z_{h_2}(w,\cdot)\rangle F_j(Z_{h_1}(y,\cdot)) \end{aligned} \tag{2.23}$$

where $f_j(\vec{\lambda}) = \frac{\partial}{\partial\lambda_j} f(\lambda_1,\dots,\lambda_n)$ and $F_j(\cdot) = f_j(\langle\vec{\theta},\cdot\rangle)$. In addition, as a function of y, $\delta_{h_1,h_2}F(y|w)$ is an element of E_σ with $B_{\delta_{h_1,h_2}F(\cdot|w)} = 2^{1+\sigma}B_F$ and with

$$A_{\delta_{h_1,h_2}F(\cdot|w)} = A_F \exp\{2^{1+\sigma}B_F\}(\|w\|_\infty \sum_{j=1}^n N_j)$$

where $|\langle\theta_j h_2, w\rangle| \le N_j\|w\|_\infty$, and $N_j = |\theta_j(T)h_2(T)| + Var(\theta_j h_2,[0,T])$. Furthermore, as a function of w, $\delta_{h_1,h_2}F(y|w)$ is an element of E_σ with $B_{\delta_{h_1,h_2}F(y|\cdot)} = 1$ and with

$$A_{\delta_{h_1,h_2}F(y|\cdot)} = ne^{-\frac{1}{2}}A_F \exp\Big\{2^{1+\sigma}B_F\Big(1+\sum_{j=1}^n(\|h_1\|_\infty\|y\|_\infty M_j)^{1+\sigma}\Big)\Big\}.$$

REMARK 2.12. Note that in view of theorems 2.7, 2.9, and 2.11 above, all of the functionals that arise in Section 3 below are automatically elements of E_σ.

3. Various relationships involving the concepts

In this section we establish the various relationships involving the three concepts of generalized conditional integral transform, generalized conditional convolution and generalized first variation for functionals belonging to E_σ. These various relationships, as well as alternative expressions for some of them, are given by formula (2.8) above, formulas (3.4) through (3.6), (3.8) and (3.10) through (3.12) below. It is interesting to note that the left hand side of each of these formulas involve exactly two of the three concepts, while each right hand side involves at most one of these concepts.

The following lemma plays a key role in establishing several formulas throughout this section.

LEMMA 3.1. *For all $j \in \{1, 2, \dots, n\}$ and $h \in L_\infty[0, T]$ with $\|\phi_j h\|_2^2 > 0$,*

$$E_x[\langle \theta_j - b_j, Z_h(x, \cdot)\rangle] = 0, \tag{3.1}$$

while for all j and l in $\{1, 2, \dots, n\}$,

$$E_x[\langle \theta_j - b_j, Z_h(x, \cdot)\rangle\langle \theta_l - b_l, Z_h(x, \cdot)\rangle] = D_{j,l;h} \tag{3.2}$$

where

$$D_{j,l;h} = \sum_{k=1}^{n} c_{k,j} c_{k,l} \|\phi_k h\|_2^2. \tag{3.3}$$

Proof. By equation (2.15), $\theta_j - b_j = \sum_{k=1}^{n} c_{k,j}\phi_k$ for all $j \in \{1, 2, \dots, n\}$, and hence

$$E_x[\langle \theta_j - b_j, Z_h(x, \cdot)\rangle] = \sum_{k=1}^{n} c_{k,j} E_x[\langle \phi_k, Z_h(x, \cdot)\rangle].$$

But by Lemma 2.6,

$$E_x[\langle \phi_k, Z_h(x, \cdot)\rangle] = (2\pi\|\phi_k h\|_2^2)^{-1/2} \int_{\mathbb{R}} u \exp\Big\{-\frac{1}{2}\frac{u^2}{\|\phi_k h\|_2^2}\Big\} du = 0$$

and so we obtain (3.1).

Similarly for all j and l in $\{1, 2, \cdots, n\}$,

$$E_x[\langle\theta_j - b_j, Z_h(x,\cdot)\rangle\langle\theta_l - b_l, Z_h(x,\cdot)\rangle]$$
$$= \sum_{k=1}^{n}\sum_{m=1}^{n} c_{k,j}c_{m,l}E_x[\langle\phi_k, Z_h(x,\cdot)\rangle\langle\phi_m, Z_h(x,\cdot)\rangle]$$
$$= \sum_{k=1}^{n} c_{k,j}c_{k,l}\|\phi_k h\|_2^2 = D_{j,l;h}$$

as desired, because for $k \neq m$,

$$E_x[\langle\phi_k, Z_h(x,\cdot)\rangle\langle\phi_m, Z_h(x,\cdot)\rangle] = E_x[\langle\phi_k h, x\rangle\langle\phi_m h, x\rangle] = 0$$

and this completes the proof. □

Our first formula (2.8) is useful because it allows us to calculate $\mathcal{F}_{\alpha,\beta}(((F * G)_\alpha\|X_h)(\cdot,\eta_1)\|X_h)(y,\eta_2)$ without ever actually calculating $(F * G)_\alpha$ or $((F * G)_\alpha\|X_h)$.

Theorem 3.2. *Let F and G be as in Theorem 2.9. Then equation (2.8) holds for all $y \in K$ and a.e.$\eta_1, \eta_2 \in \mathbb{R}$.*

Proof. The left hand side of (2.8) exists by Theorem 2.9 and Theorem 2.7, while the right hand side of equation (2.8) exists by Theorem 2.7. The equality in equation (2.8) then follows from Theorem 2.5. □

Our next formula (3.4), giving the conditional convolution of conditional integral transforms, follows from Theorem 2.7, Theorem 2.9 and a well-known Wiener integration formula.

Theorem 3.3. *Let F and G be as in Theorem 2.9. Then for all $y \in K$ and a.e.$\eta_1, \eta_2, \eta_3 \in \mathbb{R}$,*

(3.4)

$$((\mathcal{F}_{\alpha,\beta}(F\|X_h)(\cdot,\eta_1) * \mathcal{F}_{\alpha,\beta}(G\|X_h)(\cdot,\eta_2))_\alpha\|X_h)(y,\eta_3)$$
$$= [(2\pi)^n\prod_{j=1}^{n}\|\phi_j h\|_2^2]^{-\frac{3}{2}}\int_{\mathbb{R}^{3n}} f(\alpha\vec{v}C + \alpha\eta_1\vec{b} + \frac{\beta}{\sqrt{2}}(\langle\vec{\theta}, y\rangle + \alpha\vec{u}C + \alpha\eta_3\vec{b}))$$
$$g(\alpha\vec{w}C + \alpha\eta_2\vec{b} + \frac{\beta}{\sqrt{2}}(\langle\vec{\theta}, y\rangle - \alpha\vec{u}C - \alpha\eta_3\vec{b}))$$
$$\exp\{-\frac{1}{2}\sum_{j=1}^{n}\frac{1}{\|\phi_j h\|_2^2}(u_j^2 + v_j^2 + w_j^2)\}\, d\vec{u}\, d\vec{v}\, d\vec{w}.$$

In our next theorem we obtain a formula for the generalized first variation of the conditional convolution of functionals from E_σ.

THEOREM 3.4. *Let $F \in E_\sigma$ be given by (2.11) and let h, h_1 and h_2 be in $L_\infty[0,T]$. Then for a.e.$\eta \in \mathbb{R}$,*

$$\begin{aligned} &\delta_{h_1,h_2}((F*G)_\alpha \| X_h)(\cdot,\eta)(y|w) \\ &= \sum_{j=1}^{n} \frac{\langle \theta_j, Z_{h_2}(w,\cdot)\rangle}{\sqrt{2}} \Big[((F_j*G)_\alpha \| X_h)(Z_{h_1}(y,\cdot),\eta) \\ &\qquad + ((F*G_j)_\alpha \| X_h)(Z_{h_1}(y,\cdot),\eta)\Big] \end{aligned} \tag{3.5}$$

for all y and w in K.

Proof. By the definition of the generalized first variation and (2.21) it follows that

$$\begin{aligned} A &\equiv \delta_{h_1,h_2}((F*G)_\alpha \| X_h)(\cdot,\eta)(y|w) \\ &= \frac{\partial}{\partial r}((F*G)_\alpha \| X_h)(Z_{h_1}(y,\cdot) + rZ_{h_2}(w,\cdot),\eta)|_{r=0} \\ &= \frac{\partial}{\partial r} E_x[f(\frac{1}{\sqrt{2}}(\langle \vec{\theta}, Z_{h_1}(y,\cdot) + rZ_{h_2}(w,\cdot)\rangle + \alpha\langle \vec{\phi}, Z_h(x,\cdot)\rangle C + \alpha\eta\vec{b})) \\ &\qquad g(\frac{1}{\sqrt{2}}[\langle \vec{\theta}, Z_{h_1}(y,\cdot) + rZ_{h_2}(w,\cdot)\rangle - \alpha\langle \vec{\phi}, Z_h(x,\cdot)\rangle C - \alpha\eta\vec{b}))]|_{r=0}. \end{aligned}$$

Evaluating partial derivative in the last expression we obtain that

$$\begin{aligned} A = \sum_{j=1}^{n} \frac{\langle \theta_j, Z_{h_2}(w,\cdot)\rangle}{\sqrt{2}} E_x[&f_j(\frac{1}{\sqrt{2}}(\langle \vec{\theta}, Z_{h_1}(y,\cdot)\rangle + \alpha\langle \vec{\phi}, Z_h(x,\cdot)\rangle C + \alpha\eta\vec{b})) \\ &g(\frac{1}{\sqrt{2}}[\langle \vec{\theta}, Z_{h_1}(y,\cdot)\rangle - \alpha\langle \vec{\phi}, Z_h(x,\cdot)\rangle C - \alpha\eta\vec{b})) \\ &+ f(\frac{1}{\sqrt{2}}(\langle \vec{\theta}, Z_{h_1}(y,\cdot)\rangle + \alpha\langle \vec{\phi}, Z_h(x,\cdot)\rangle C + \alpha\eta\vec{b})) \\ &g_j(\frac{1}{\sqrt{2}}[\langle \vec{\theta}, Z_{h_1}(y,\cdot)\rangle - \alpha\langle \vec{\phi}, Z_h(x,\cdot)\rangle C - \alpha\eta\vec{b}))]. \end{aligned}$$

Using (2.21) once more we know that the last expression is equal to the last hand side of (3.5) and this completes the proof. □

In Theorem 3.5 below we obtain a formula for the generalized conditional convolution with respect to the first argument of the variation of the first variation of functionals from E_σ.

THEOREM 3.5. *Let F and G be as in Theorem 2.9 and let h, h_1 and h_2 be in $L_\infty[0,T]$. Then*

$$((\delta_{h_1,h_2}F(\cdot|w) * \delta_{h_1,h_2}G(\cdot|w))_\alpha \| X_h)(y,\eta) = \sum_{j=1}^{n}\sum_{l=1}^{n}\langle\theta_j, Z_{h_2}(w,\cdot)\rangle\langle\theta_l, Z_{h_2}(w,\cdot)\rangle((F_j * G_l)_\alpha \| X_h)(Z_{h_1}(y,\cdot),\eta) \tag{3.6}$$

for all $y, w \in K$ and a.e.$\eta \in \mathbb{R}$.

Proof. Applying the additive distribution properties of the conditional convolution to the expressions given by (2.23) and the corresponding expression for G,

$$\delta_{h_1,h_2}G(y|w) = \sum_{l=1}^{n}\langle\theta_l, Z_{h_2}(w,\cdot)\rangle G_l(Z_{h_1}(y,\cdot)) \tag{3.7}$$

yields equation (3.6) as desired. □

We restrict our attention, in this subsequent, to the functions h_1 and h_2 either of them is constant on $[0,T]$ rather than to be in $L_\infty[0,T]$.

In Theorem 3.6 below we obtain a formula for the generalized conditional convolution product with respect to the second argument of the variation of the first variation of functionals from E_σ.

THEOREM 3.6. *Let F and G be as in Theorem 2.9 and let h, and h_1 be in $L_\infty[0,T]$ and h_2 be a constant function. Then for a.e.$\eta \in \mathbb{R}$,*

$$\begin{aligned}((\delta_{h_1,h_2}F(y|\cdot) * \delta_{h_1,h_2}G(y|\cdot))_\alpha \| X_h)(w,\eta) &= \frac{1}{2}\delta_{h_1,h_2}F(y|w)\delta_{h_1,h_2}G(y|w) \\ &+ \frac{\alpha h_2\eta}{2}\sum_{j=1}^{n}\Big[\delta_{h_1,h_2}G(y|w)b_jF_j(Z_{h_1}(y,\cdot)) - \delta_{h_1,h_2}F(y|w)b_jG_j(Z_{h_1}(y,\cdot))\Big] \\ &- \frac{\alpha^2 {h_2}^2}{2}\sum_{j=1}^{n}\sum_{l=1}^{n}(\eta^2 b_j b_l + D_{j,l:h})F_j(Z_{h_1}(y,\cdot))G_l(Z_{h_1}(y,\cdot))\end{aligned} \tag{3.8}$$

for all y and w in K with $D_{j,l:h}$ given by equation (3.3).

Proof. Using the definition of the conditional convolution product, together with equations (2.23) and (3.7), we obtain that

$$((\delta_{h_1,h_2}F(y|\cdot) * \delta_{h_1,h_2}G(y|\cdot))_\alpha \| X_h)(w,\eta)$$
$$= \frac{1}{2}\sum_{j=1}^{n}\sum_{l=1}^{n} F_j(Z_{h_1}(y,\cdot))G_l(Z_{h_1}(y,\cdot))E_x[\{\langle\theta_j h_2, w\rangle + \alpha h_2\langle\theta_j - b_j, Z_h(x,\cdot)\rangle$$
$$+ \alpha h_2\eta b_j\}\{\langle\theta_l h_2, w\rangle - \alpha h_2\langle\theta_l - b_l, Z_h(x,\cdot)\rangle - \alpha h_2\eta b_l\}].$$

Hence using Lemma 3.1 we see that

(3.9)
$$\begin{aligned}((\delta_{h_1,h_2}&F(y|\cdot) * \delta_{h_1,h_2}G(y|\cdot))_\alpha \| X_h)(w,\eta)\\ &= \frac{1}{2}\sum_{j=1}^{n}\sum_{l=1}^{n} F_j(Z_{h_1}(y,\cdot))G_l(Z_{h_1}(y,\cdot))\{\langle\theta_j h_2, w\rangle\langle\theta_l h_2, w\rangle\\ &\qquad + \alpha h_2\eta(b_j\langle\theta_l h_2, w\rangle - b_l\langle\theta_j h_2, w\rangle) - \alpha^2 h_2^2\eta^2 b_j b_l - \alpha^2 h_2^2 D_{j,l:h}\}.\end{aligned}$$

Finally, using equations (2.23) and (3.7) again, it follows that the right hand side of (3.9) equals the right hand side of (3.8) as desired. □

In Theorem 3.7 below we get a formula for the generalized conditional integral transform with respect to the first argument of the variation while in Theorem 3.8 we get a formula for the generalized conditional integral transform with respect to the second argument of the variation.

THEOREM 3.7. *Let $F \in E_\sigma$ be given by (2.11) and let h be in $L_2[0,T]$, h_2 be in $L_\infty[0,T]$ and h_1 be a constant function. Then for a.e.$\eta \in \mathbb{R}$,*
(3.10)
$$\mathcal{F}_{\alpha,\beta}(\delta_{h_1,h_2}F(\cdot|w)\|X_h)(y,\eta) = \sum_{j=1}^{n}\langle\theta_j, Z_{h_2}(w,\cdot)\rangle\mathcal{F}_{\alpha h_1,\beta h_1}(F_j\|X_h)(y,\eta)$$

and
(3.11)
$$\delta_{h_1,h_2}\mathcal{F}_{\alpha,\beta}(F\|X_h)(\cdot,\eta)(y|w) = \beta\sum_{j=1}^{n}\langle\theta_j, Z_{h_2}(w,\cdot)\rangle\mathcal{F}_{\alpha,\beta h_1}(F_j\|X_h)(y,\eta)$$

for all y and w in K.

Proof. Using the definition of conditional integral transform, equations (2.23) and (3.1), it follows that

$$\begin{aligned}
&\mathcal{F}_{\alpha,\beta}(\delta_{h_1,h_2}F(\cdot|w)\|X_h)(y,\eta)\\
&= E_x[\sum_{j=1}^{n}\langle\theta_j, Z_{h_2}(w,\cdot)\rangle f_j(\alpha\langle h_1\vec{\theta}, Z_{T,\eta}^{\{h,a\}}(x,\cdot)\rangle + \beta\langle h_1\vec{\theta}, y\rangle)]\\
&= \sum_{j=1}^{n}\langle\theta_j, Z_{h_2}(w,\cdot)\rangle E_x[f_j(\langle\vec{\theta}, \alpha h_1 Z_{T,\eta}^{\{h,a\}}(x,\cdot) + \beta h_1 y\rangle)].
\end{aligned}$$

Furthermore, using the definition of generalized first variation and equation (2.19) we obtain that

$$\begin{aligned}
&\delta_{h_1,h_2}\mathcal{F}_{\alpha,\beta}(F\|X_h)(\cdot,\eta)(y|w)\\
&= \beta\sum_{j=1}^{n}\langle\theta_j, Z_{h_2}(w,\cdot)\rangle E_x[f_j(\alpha\langle\vec{\phi}, Z_h(x,\cdot)\rangle C + \alpha\eta\vec{b} + \beta h_1\langle\vec{\theta}, y\rangle)]\\
&= \beta\sum_{j=1}^{n}\langle\theta_j, Z_{h_2}(w,\cdot)\rangle\mathcal{F}_{\alpha,\beta h_1}(F_j\|X_h)(y,\eta)
\end{aligned}$$

as we wished. □

Theorem 3.8. *Let $F \in E_\sigma$ be given by (2.11) and let h, h_1 be in $L_\infty[0,T]$ and h_2 be a constant function. Then for a.e.$\eta \in \mathbb{R}$,*

(3.12)

$$\mathcal{F}_{\alpha,\beta}(\delta_{h_1,h_2}F(y|\cdot)\|X_h)(w,\eta) = \beta\delta_{h_1,h_2}F(y|w) + \alpha\eta h_2\sum_{j=1}^{n} b_j F_j(Z_{h_1}(y,\cdot))$$

for all y and w in K.

Proof. Using the definition of the generalized conditional integral transform, equation (2.23), equation (3.1) and then equation (2.23) again, it follows that

$$
\begin{aligned}
&\mathcal{F}_{\alpha,\beta}(\delta_{h_1,h_2}F(y|\cdot)\|X_h)(w,\eta)\\
&= E_x[\sum_{j=1}^{n}\langle\theta_j h_2, \alpha Z_{T,\eta}^{\{h,a\}}(x,\cdot)+\beta w\rangle f_j(\langle\vec{\theta}, Z_{h_1}(y,\cdot)\rangle)]\\
&= \sum_{j=1}^{n} f_j(\langle\vec{\theta}, Z_{h_1}(y,\cdot)\rangle)h_2 E_x[\beta\langle\theta_j, w\rangle + \alpha\langle\theta_j - b_j, Z_h(x,\cdot)\rangle + \alpha\eta b_j]\\
&= \beta\sum_{j=1}^{n}\langle\theta_j, Z_{h_2}(w,\cdot)\rangle f_j(\langle\vec{\theta}, Z_{h_1}(y,\cdot)\rangle) + \alpha\eta h_2\sum_{j=1}^{n} b_j f_j(\langle\vec{\theta}, Z_{h_1}(y,\cdot)\rangle)
\end{aligned}
$$

as we wished. □

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