ON THE GROWTH OF SOLUTIONS OF SOME NON-LINEAR COMPLEX DIFFERENTIAL EQUATIONS

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Abstract. In this paper we study the growth of solutions of some non-linear complex differential equations in connection to Brück conjecture using the theory of complex differential equation.

1. Introduction and main results

In this paper, by meromorphic function we shall always mean a meromorphic function in the complex plane. We adopt the standard notations in the Nevanlinna Theory of meromorphic functions as explained in [4, 6, 10, 11]. It will be convenient to let $E$ denote any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence.

For any non-constant meromorphic function $f(z)$, we denote by $S(r, f)$ any quantity satisfying $S(r, f) = o(T(r, f))$ as $r \to \infty, r \notin E$, where $T(r, f)$ is the Nevanlinna characteristic function of $f$. A meromorphic function $a(z)$ is said to be small with respect to $f(z)$ if $T(r, a) = S(r, f)$. We denote by $S(f)$ the collection of all small functions with respect to $f$. Clearly $C \cup \{\infty\} \in S(f)$ and $S(f)$ is a field over the set of complex

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numbers.

For any two non-constant meromorphic functions \( f \) and \( g \), and \( a \in S(f) \cap S(g) \), we say that \( f \) and \( g \) share a IM(CM) provided that \( f - a \) and \( g - a \) have the same zeros ignoring(counting) multiplicities.

Let \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) be an entire function. We define by \( \mu(r, f) = \max\{|a_n|r^n : n = 0, 1, 2, \ldots\} \) the maximum term of \( f \) and by \( \nu(r, f) = \max\{m : \mu(r, f) = |a_m|r^m\} \) the central index of \( f \). In this paper we also need the following definition:

**Definition 1.1.** Let \( f(z) \) be a non-constant entire function. Then the order \( \sigma(f) \), the lower order \( \mu(f) \) and the hyper-order \( \sigma_2(f) \) of \( f(z) \) are defined as follows:

\[
\sigma(f) = \limsup_{r \to +\infty} \frac{\log T(r, f)}{\log r} = \limsup_{r \to +\infty} \frac{\log \log M(r, f)}{\log r}
\]

\[
\mu(f) = \liminf_{r \to +\infty} \frac{\log T(r, f)}{\log r} = \liminf_{r \to +\infty} \frac{\log \log M(r, f)}{\log r}
\]

\[
\sigma_2(f) = \limsup_{r \to +\infty} \frac{\log \log T(r, f)}{\log r} = \limsup_{r \to +\infty} \frac{\log \log \log M(r, f)}{\log r},
\]

where and in the sequel

\[
M(r, f) = \max_{|z|=r} |f(z)|.
\]

In 1976, Rubel and Yang [9] proved that if a non-constant entire function \( f \) and its derivative \( f' \) share two distinct finite complex numbers CM, then \( f \equiv f' \). What will be the relation between \( f \) and \( f' \), if an entire function \( f \) and its derivative \( f' \) share one finite complex number CM?

In 1996 Brück [1] made the following conjecture:

**Conjecture 1.1.** Let \( f \) be a non-constant entire function satisfying \( \sigma_2(f) < \infty \), where \( \sigma_2(f) \) is not a positive integer. If \( f \) and \( f' \) share one finite complex number \( a \) CM, then

\[
\frac{f' - a}{f - a} = c,
\]

for some finite complex number \( c \neq 0 \).
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In the same paper, Brück showed that the conjecture is true when \(a = 0\). He also proved that the conjecture is true for \(a \neq 0\) provided that \(f\) satisfies the additional assumption \(N(r,0; f') = S(r,f)\) and in this case the order restriction on \(f\) can be omitted.

Gundersen and Yang [3] proved that the conjecture is true for functions of finite order.

**Theorem 1.1.** Let \(f\) be a non-constant entire function of finite order. If \(f\) and \(f'\) share one finite complex number \(a\) CM, then

\[
\frac{f' - a}{f - a} = c,
\]

for some finite complex number \(c \neq 0\).

In 2009, Chang and Zhu [2] proved that Theorem 1.1 remains valid when the complex number \(a\) is replaced by a function.

**Theorem 1.2.** Let \(f\) be a non-constant entire function of finite order and \(a = a(z) (\neq 0)\) be an entire function such that \(\sigma(a) < \sigma(f) < \infty\). If \(f\) and \(f'\) share a CM, then

\[
\frac{f' - a}{f - a} = c,
\]

for some finite complex number \(c \neq 0\).

In 2016, Li and Yi [8] investigated the Brück conjecture and proved that Theorem 1.2 remains true when \(f'\) is replaced by a linear differential polynomial of \(f\), namely \(L(f) = f^{(k)} + a_{k-1}f^{(k-1)} + \ldots + a_1f' + a_0f\), where \(k\) is a positive integer and \(a_{k-1}, ..., a_0\) are complex constants. They proved the following result:

**Theorem 1.3.** Let \(f\) be a non-constant entire function such that \(\sigma(f) < \infty\), and let \(a (\neq 0)\) be an entire function such that \(\sigma(a) < \sigma(f)\). If \(f - a\) and \(L(f) - a\) share 0 CM, where \(L(f)\) is defined as above, then \(\sigma(f) = 1\) and one of the following two cases will occur:

(i) \(L(f) - a = c(f - a)\), where \(c\) is some non-zero constant.

(ii) \(f\) is a solution of the equation \(L(f) - a = (f - a)e^{p_1z + p_0}\) such that \(\sigma(f) = \mu(f) = 1\), where not all \(a_0, a_1, ..., a_{k-1}\) are zeros, \(p_1 \neq 0\) and \(p_0\) are complex numbers.
Question 1.1. It is an interesting question to investigate that what will happen if we replace the linear differential polynomial by a non-linear differential polynomial in Theorem 1.3.

In this connection we need the following definition:

Let \( n_{0j}, n_{1j}, n_{2j}, \ldots, n_{kj} \) are non-negative integers. The expression
\[
M_j[f] = f^{n_{0j}} (f^{(1)})^{n_{1j}} (f^{(2)})^{n_{2j}} \ldots (f^{(k)})^{n_{kj}},
\]
is called a differential monomial generated by \( f \) of degree \( d(M_j) = \sum_{i=0}^{k} n_{ij} \) and weight \( \Gamma_{M_j} = \sum_{i=0}^{k} (i+1)n_{ij} \). The sum
\[
P[f] = \sum_{j=1}^{l} a_j M_j[f],
\]
is called a differential polynomial generated by \( f \) of degree \( \overline{d}(P) = \max \{d(M_j) : 1 \leq j \leq l \} \) and weight \( \Gamma_P = \max \{\Gamma_{M_j} : 1 \leq j \leq l \} \), where \( a_j \) is complex constant for \( j = 1, 2, \ldots, l \). The numbers \( d_P = \min \{d(M_j) : 1 \leq j \leq l \} \) and \( k \) (the highest order of the derivative of \( f \) in \( P[f] \)) are called respectively the lower degree and the order of \( P[f] \). \( P[f] \) is said to be homogeneous differential polynomial of degree \( d \) if \( \overline{d}_P = \frac{d}{d_P} = d \). \( P[f] \) is called a linear differential polynomial generated by \( f \) if \( \overline{d}_P = 1 \). Otherwise, \( P[f] \) is called non-linear differential polynomial. We denote by \( Q_j = \Gamma_{M_j} - d(M_j) = \sum_{i=1}^{k} i.n_{ij} \) for \( 1 \leq j \leq l \).

In this paper we prove the following theorems which improve and generalizes Theorems 1.1, 1.2 and 1.3.

Theorem 1.4. Let \( f \) be a non-constant entire function with \( \sigma(f) < \infty \) and let \( a(\not\equiv 0) \) be entire function such that \( \sigma(a) < \sigma(f) \). If \( f^d(z) - a(z) \) and \( P[f] - a(z) \) share \( 0 \) CM, where \( P[f] = M[f] + \sum_{j=1}^{l} a_j M_j[f] \) is a differential polynomial of \( f \) of degree \( d \), and \( M[f] \) is a differential monomial of \( f \) of degree \( d \). Then \( \sigma(f) = 1 \) and one of the following two cases will occur:
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(i) $f$ is a solution of the equation $P[f] - a(z) = c(f^d - a(z))$, where $c$ is some non-zero constant.

(ii) $f$ is a solution of the equation $P[f] - a(z) = (f^d - a(z))e^{p_1z + p_0}$ such that $\sigma(f) = \mu(f) = 1$, where not all $a_1, a_2, \ldots, a_l$ are zeros, $p_1 \neq 0$ and $p_0$ are complex numbers.

Proceeding as in the proof of Theorem 1.4 of this paper, we can prove the following theorem.

**Theorem 1.5.** Let $f$ be a non-constant entire function such that $\sigma(f) < \infty$ and let $a(\neq 0)$ and $\beta$ be entire functions such that $\max\{\sigma(a), \sigma(\beta)\} < \sigma(f)$. If $f^d(z) - a(z)$ and $P[f] + \beta(z) - a(z)$ share $0$ CM, where $P[f] = M[f] + \sum_{j=1}^{l} a_j M_j[f]$ is a differential polynomial of $f$ of degree $d$, and $M[f]$ is a differential monomial of $f$ of degree $d$. Then $\sigma(f) = 1$ and one of the following two cases will occur:

(i) $f$ is a solution of the equation $P[f] + \beta(z) - a(z) = c(f^d(z) - a(z))$, where $c$ is some non-zero constant.

(ii) $f$ is a solution of the equation $P[f] + \beta(z) - a(z) = (f^d(z) - a(z))e^{p_1z + p_0}$ such that $\sigma(f) = \mu(f) = 1$, where not all $a_1, a_2, \ldots, a_l$ are zeros, $p_1 \neq 0$ and $p_0$ are complex numbers.

From Theorem 1.4 we get the following corollary:

**Corollary 1.1.** Let $f$ be a non-constant entire function such that $\sigma(f) < \infty$ and let $a(\neq 0)$ be entire function such that $\sigma(a) < \sigma(f)$. If $f^d(z) - a(z)$ and $M[f] - a(z)$ share $0$ CM, where $M[f]$ is a differential monomial of $f$ of degree $d$. Then $\sigma(f) = 1$ and $f$ is a solution of the equation $M[f] - a(z) = c(f^d - a(z))$, where $c$ is some non-zero constant.

From Theorem 1.5 we get the following corollary:

**Corollary 1.2.** Let $f$ be a non-constant entire function such that $\sigma(f) < \infty$ and let $a(\neq 0)$ and $\beta$ be entire functions such that $\max\{\sigma(a), \sigma(\beta)\} < \sigma(f)$. If $f^d(z) - a(z)$ and $M[f] + \beta(z) - a(z)$ share $0$ CM, where $M[f]$ is a differential monomial of $f$ of degree $d$. Then $\sigma(f) = 1$ and $f$ is a solution of the equation $M[f] + \beta(z) - a(z) = c(f^d(z) - a(z))$, where $c$
is some non-zero constant.

The following is the supportive example of (i) of Theorem 1.4.

**Example 1.1.** Let \( f(z) = 1 - e^z \) and \( P[f] = f'f + f \). Then \( \sigma(f) = 1 \) and \( P[f] - a(z) = c(f^2(z) - a(z)) \), where \( c = 1 \) and \( a(z) = z + 1 \).

The following is the supportive example of (ii) of Theorem 1.4.

**Example 1.2.** Let \( f(z) = 1 + e^z \) and \( P[f] = f^2 - (f'')^2 - f' + 2 \). Then \( P[f] - 1 \) and \( f^2 - 1 \) share \( 0 \) \( \text{CM} \), \( \sigma(f) = 1 \) and \( P[f] - 1 = (f^2 - 1)e^{-z} \).

**Example 1.3.** Let \( f(z) = a(z) = e^z \) and \( P[f] = f^2 - f^2 + 2f - 1 \). Then \( f^2 - a \) and \( P[f] - a \) share \( 0 \) \( \text{CM} \) and \( \sigma(f) = \sigma(a) = 1 \) but \( P[f] - a = e^{-z}(f^2 - a) \). This example shows that the condition \( \{ \sigma(a) < \sigma(f) \} \) in (i) of Theorem 1.4 is the best possible.

**Theorem 1.6.** In Theorem 1.4 if we replace the condition \( \{ \sigma(a) < \sigma(f) \} \) by \( \{ \sigma(a) < \mu(f) \} \) and all other conditions remains the same, then also the conclusion of the theorem is true.

**Theorem 1.7.** In Theorem 1.5 if we replace the condition \( \{ \max\{\sigma(a), \sigma(\beta)\} < \sigma(f) \} \) by \( \{ \max\{\sigma(a), \sigma(\beta)\} < \mu(f) \} \) and all other conditions remains the same, then also the conclusion of the theorem is true.

From Theorem 1.6 we get the following corollary:

**Corollary 1.3.** Let \( f \) be a non-constant entire function such that \( \sigma(f) < \infty \) and let \( a \neq 0 \) be entire function such that \( \sigma(a) < \mu(f) \). If \( f^d(z) - a(z) \) and \( M[f] - a(z) \) share \( 0 \) \( \text{CM} \), where \( M[f] \) is a differential monomial of \( f \) of degree \( d \). Then \( \sigma(f) = 1 \) and \( f \) is a solution of the equation \( M[f] - a(z) = c(f^d - a(z)) \), where \( c \) is some non-zero constant.

From Theorem 1.7 we get the following corollary:

**Corollary 1.4.** Let \( f \) be a non-constant entire function such that \( \sigma(f) < \infty \) and let \( a \neq 0 \) and \( \beta \) be entire functions such that \( \max\{\sigma(a), \sigma(\beta)\} < \mu(f) \). If \( f^d(z) - a(z) \) and \( M[f] + \beta(z) - a(z) \) share \( 0 \) \( \text{CM} \), where \( M[f] \) is a differential monomial of \( f \) of degree \( d \). Then \( \sigma(f) = 1 \) and \( f \) is a solution of the equation \( M[f] + \beta(z) - a(z) = c(f^d - a(z)) \), where \( c \) is some non-zero constant.
2. Preparatory Lemmas

In this section we state some lemmas needed in the sequel.

**Lemma 2.1.** [6] Let \( f(z) \) be a transcendental entire function, \( \nu(r, f) \) be the central index of \( f(z) \). Then there exists a set \( E \subset (1, +\infty) \) with finite logarithmic measure such that for some point \( z \) satisfying \( |z| = r \not\in [0, 1] \cup E \) and \( |f(z)| = M(r, f) \), we get
\[
\begin{align*}
\frac{f^{(j)}(z)}{f(z)} &= \left\{ \frac{\nu(r, f)}{z} \right\}^j (1 + o(1)), \quad j \in \mathbb{N}.
\end{align*}
\]

**Lemma 2.2.** [5] Let \( f(z) \) be an entire function of finite order \( \sigma(f) = \sigma < +\infty \) and let \( \nu(r, f) \) be the central index of \( f \). Then
\[
\begin{align*}
\sigma(f) &= \limsup_{r \to +\infty} \frac{\log \nu(r, f)}{\log r} \\
\mu(f) &= \liminf_{r \to +\infty} \frac{\log \nu(r, f)}{\log r}.
\end{align*}
\]

And if \( f \) is a transcendental entire function of hyper order \( \sigma_2(f) \), then
\[
\limsup_{r \to +\infty} \frac{\log \log \nu(r, f)}{\log r} = \sigma_2(f).
\]

**Lemma 2.3.** [7] Let \( f(z) \) be a transcendental entire function and let \( E \subset [1, +\infty) \) be a set having finite logarithmic measure. Then there exists \( \{ z_n = r_n e^{i\theta_n} \} \) such that \( |f(z_n)| = M(r_n, f), \theta_n \in [0, 2\pi) \), \( \lim_{n \to +\infty} \theta_n = \theta_0 \in [0, 2\pi) \), \( r_n \not\in E \) and if \( 0 < \sigma(f) < +\infty \), then for any given \( \varepsilon > 0 \) and sufficiently large \( r_n \),
\[
r_n^{\sigma(f) - \varepsilon} < \nu(r_n, f) < r_n^{\sigma(f) + \varepsilon}.
\]

**Lemma 2.4.** ([6], Corollary 2.3.4) Let \( f \) be a transcendental meromorphic function and \( k \) be a positive integer. Then \( m(r, f^{(k)}/f) = S(r, f) \), outside of a possible exceptional set \( E \) of finite linear measure, and if \( f \) is of finite order of growth, then \( m(r, f^{(k)}/f) = O(\log r) \).

**Lemma 2.5.** [8] Let \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) be an entire function, let \( \mu(r, f) \) be the maximum term of \( f \), and let \( \nu(r, f) \) be the central index.
Then for $0 < r < R$ we have

$$M(r, f) < \mu(r, f) \left\{ \nu(R, f) + \frac{R}{R - r} \right\}.$$ 

**Lemma 2.6.** ([6], Lemma 1.1.2) Let $g : (0, +\infty) \to R$, $h : (0, +\infty) \to R$ be monotone increasing functions such that $g(r) \leq h(r)$ outside of an exceptional set $F$ of finite logarithmic measure. Then for any $\alpha > 1$, there exists $r_0 > 0$ such that $g(r) \leq h(r^\alpha)$ for all $r > r_0$.

### 3. Proof of Main Theorems

In this section we present the proof of the main result of the paper.

Proof of Theorem 1.4:

Since $f^d - a$ and $P[f] - a$ share $0$ $CM$, we get

$$P[f] - a = e^\phi,$$

where $\phi$ is an entire function. Again from $\sigma(a) < \sigma(f)$, we have $\sigma(f) > 0$, which implies that $f$ is a transcendental entire function.

Now, we consider the following two cases:

**Case I:**

$$\liminf_{r \to \infty} \frac{\log \nu(r, f)}{\log r} > 1.$$  

Then from (3.2) and Lemma 2.2, we get

$$\mu(f) = \liminf_{r \to \infty} \frac{\log \nu(r, f)}{\log r} > 1.$$  

Since $f$ is a transcendental entire function, we have

$$M(r, f) \to \infty \text{ as } r \to \infty.$$  

Again since $f$ is a transcendental entire function, by Lemma 2.1 there exist subset $F_j \subset (1, \infty)$ ($1 \leq j \leq n$) with finite logarithmic measure.
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such that for some point \( z_r = re^{i\theta(r)} \), \( \theta(r) \in [0, 2\pi) \) satisfying \( |z_r| = r \notin F_j \) and \( M(r, f) = |f(z_r)| \), we have

\[
\frac{f^{(j)}(z_r)}{f(z_r)} = \left( \frac{\nu(r, f)}{z_r} \right)^j \{1 + o(1)\} \quad (1 \leq j \leq n), \quad \text{as} \quad r \notin \bigcup_{j=1}^n F_j \quad \text{and} \quad r \to \infty.
\]

By Definition 1.1, Lemma 2.6, Definition 1.1.1 and Theorem 1.1.3 from [12] and the assumption \( \sigma(a) < \sigma(f) \), there exists an infinite sequence of points \( z_{r_n} = r_ne^{i\theta(r_n)} \) satisfying \( M(r_n, f) = |f(z_{r_n})| \), where \( r_n \in I \setminus \bigcup_{j=1}^n F_j \), \( I \subseteq \mathbb{R}^+ \) is a subset with logarithmic measure \( \int_I \frac{dt}{t} = \infty \) such that

\[
\lim_{r_n \to \infty} \log \log M(r_n, f) / \log r_n = \sigma(f)
\]

and

\[
\lim_{r_n \to \infty} M(r_n, a) = 0.
\]

Since

\[
P[f] - a = \frac{P[f] - a}{f^d - a} \cdot 1 - \frac{a}{f^d},
\]

using (3.2),(3.4)-(3.7) in (3.8) we get

\[
\frac{P[f](z_{r_n}) - a(z_{r_n})}{f^d(z_{r_n}) - a(z_{r_n})} = R \left( \frac{\nu(r_n, f)}{z_{r_n}} \right)^Q \{1 + o(1)\}, \quad \text{as} \quad r_n \to \infty,
\]

where \( Q = \max \{ \Gamma_M - d(M) : M \text{ is a monomial in } P[f] \} \) and \( R \) is a complex number.

From (3.9), we have

\[
\log \left| \frac{P[f](z_{r_n}) - a(z_{r_n})}{f^d(z_{r_n}) - a(z_{r_n})} \right| = Q \{ \log \nu(r_n, f) - \log r_n \} + o(1), \quad \text{as} \quad r_n \to \infty.
\]
From (3.1), Lemma 2.4 and the condition \(\sigma(a) < \sigma(f) < \infty\), we get
\[
T(r, e^\phi) \leq 2T(r, f) + O(\log r)
\]
\[
\Rightarrow \log T(r, e^\phi) \leq \log T(r, f) + O(\log \log r)
\]
\[
\Rightarrow \frac{\log T(r, e^\phi)}{\log r} \leq \frac{\log T(r, f)}{\log r} + O(1)
\]
(3.11)
\[
\Rightarrow \sigma(e^\phi) \leq \sigma(f) < \infty \text{ as } r \to \infty,
\]
which implies that \(\phi\) is a polynomial.

Let
(3.12)
\[
\phi = p_m z^m + p_{m-1} z^{m-1} + \ldots + p_1 z + p_0,
\]
where \(p_0, p_1, \ldots, p_{m-1}, p_m\) are complex constants with \(p_m \neq 0\).

It follows from (3.12) that
\[
\lim_{|z| \to \infty} \frac{\phi(z)}{p_m z^m} = 1 \quad \text{and} \quad |\phi(z)/p_m z^m| > \frac{1}{\epsilon} \quad \text{as} \quad |z| > r_0, \quad \text{when} \quad r_0 \text{ is a sufficiently large positive number.}
\]

From this and (3.1), we get
(3.13)
\[
m \log |z| + \log |p_m| - 1 \leq \log |\phi(z)| \leq \log \log e^\phi = \left| \log \log \frac{P[f] - a}{P[f] - a} \right| \quad \text{as} \quad |z| \to \infty.
\]

From (3.9), (3.13), Lemma 2.2 and the condition \(\sigma(f) < \infty\), we get
\[
m \log |z_r^n| + \log |p_m| - 1 \leq \left| \log \log \frac{P[f](z_{r_n}) - a(z_{r_n})}{f^d(z_{r_n}) - a(z_{r_n})} \right|
\]
\[
= \left| \log \log \frac{P[f](z_{r_n}) - a(z_{r_n})}{f^d(z_{r_n}) - a(z_{r_n})} \right| + i \arg \left( \log \frac{P[f](z_{r_n}) - a(z_{r_n})}{f^d(z_{r_n}) - a(z_{r_n})} \right)
\]
\[
\leq \left| \log \log \frac{P[f](z_{r_n}) - a(z_{r_n})}{f^d(z_{r_n}) - a(z_{r_n})} \right| + 2\pi
\]
\[
\leq \log \log \nu(r_n, f) + \log \log r_n + O(1)
\]
\[
\leq 2 \log \log r_n + O(1), \quad \text{as} \quad r_n \to \infty
\]
(3.14)
\[
\Rightarrow m \log |z_{r_n}| + \log |p_m| - 1 \leq 2 \log \log r_n + O(1), \quad \text{as} \quad r_n \to \infty
\]
which is impossible. Thus \(\phi\) is a constant and so (3.9) becomes
(3.15)
\[
\left( \frac{\nu(r_n, f)}{z_{r_n}} \right)^Q \{1 + o(1)\} = c \quad \text{as} \quad r_n \to \infty,
\]
where $c$ is some non-zero constant.

From (3.15), we get

\[(3.16) \lim_{r_n \to \infty} \frac{\log \nu(r_n, f)}{\log r_n} = 1.\]

By Lemma 2.5, we know that

\[(3.17) M(r_n, f) < \mu(r_n)\{\nu(2r_n, f) + 2\} = |a_{\nu(r_n, f)}| r_n^{\nu(r_n, f)}\{\nu(2r_n, f) + 2\}.\]

Since $|a_j| < M_1$, for all non-negative integer $j$ and some constant $M_1 > 0$, we get from (3.17) that

\[(3.18) \log \log M(r_n, f) \leq \log \nu(r_n, f) + 2 \log \log r_n + C_1,\]

where $C_1 > 0$ is a suitable constant.

From Lemma 2.2 and the condition $\sigma(f) < \infty$, we get

\[(3.19) \log \nu(2r_n, f) < \{1 + o(1)\}(\log r_n + \log 2) \text{ as } r \to \infty.\]

From (3.16), (3.18) and (3.19) we get

\[(3.20) \frac{\log \log M(r_n, f)}{\log r_n} \leq \frac{\log \nu(r_n, f)}{\log r_n},\]

By (3.6), (3.16) and (3.20), we get

\[(3.21) \sigma(f) \leq 1.\]

which is a contradiction by the fact $\mu(f) \leq \sigma(f)$ and (3.3).

**Case II:** Suppose that

\[(3.22) \liminf_{r \to \infty} \frac{\log \nu(r, f)}{\log r} \leq 1.\]

Then from (3.21) and Lemma 2.2, we get

\[(3.23) \mu(f) \leq 1.\]

We consider the following two subcases:

**Subcase I:** Suppose that

\[(3.24) \sigma(f) > 1.\]
By (3.24), Definition 1.1, Lemma 2.6, Definition 1.1.1, Theorem 1.1.3 from [12] and the assumption \( \sigma(a) < \sigma(f) \), there exists an infinite sequence of points \( z_{r_n} = r_n e^{i\theta(r_n)} \) satisfying \( M(r_n, f) = |f(z_{r_n})| \), where \( r_n \in I \setminus \bigcup_{j=1}^n F_j \), \( I \subseteq R^+ \) is a subset with logarithmic measure \( \int_I \frac{dt}{t} = \infty \), such that (3.6) and (3.7) hold. Next proceeding in the same manner as in Case I we get (3.21), which contradicts (3.24).

**Subcase II:** Suppose that

(3.25) \( \sigma(f) \leq 1 \).

We will show that

(3.26) \( \sigma(f) = 1 \).

Suppose that

(3.27) \( \sigma(f) < 1 \).

Then from (3.27) and (3.11), we get \( \sigma(e^\phi) \leq \sigma(f) < 1 \), which implies that \( \phi \) is a constant and so is \( e^\phi \). Thus (3.1) becomes

(3.28) \[
\frac{P[f] - a f}{f^d - a} = c,
\]

where \( c \) is some non-zero constant.

Re-writing (3.28), we get

(3.29) \[
\frac{M[f]}{f^d} + \sum_{j=1}^l a_j \frac{M_j[f]}{f^d} - \frac{a}{f} \frac{1}{f^{d-1}} = c \left( \frac{1}{f} \frac{1}{f^{d-1}} \right).
\]

By Lemma 2.6, Definition 1.1.1, Theorem 1.1.3 from [12] and the assumption \( \sigma(a) < \sigma(f) \), there exists an infinite sequence of points \( z_{r_n} = r_n e^{i\theta(r_n)} \) satisfying \( M(r_n, f) = |f(z_{r_n})| \), where \( r_n \in I \setminus \bigcup_{j=1}^n F_j \), \( I \subseteq R^+ \) is a subset with logarithmic measure \( \int_I \frac{dt}{t} = \infty \), such that (3.6) and (3.7) hold and from 3.29 we have

(3.30) \[
\left( \frac{\nu(r_n, f)}{z_{r_n}} \right)^{\Gamma_M - d(M)} \{1 + o(1)\} + \sum_{j=1}^l a_j \left( \frac{\nu(r_n, f)}{z_{r_n}} \right)^{Q_j} \frac{1}{f(z_{r_n})^{d - d(M_j)}} \{1 + o(1)\} = c
\]

as \( r_n \to \infty \).

From Lemma 2.3, we get

(3.31) \[
\nu(r_n, f) \leq r_n^{\sigma(f) + \epsilon_0},
\]
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as \( r_n \geq R_0 \), where \( \varepsilon_0 = (1 - \sigma(f))/2 \) and \( R_0 \) is sufficiently large positive number.

From (3.27) and (3.31), we get

\[
\lim_{r_n \to \infty} \left| \frac{\nu(r_n, f)}{z_{r_n}} \right|^Q_j \leq \lim_{r_n \to \infty} \frac{r_n^{\frac{\varepsilon(f)-1}{2}}}{Q_j} = 0 \text{ for } 1 \leq j \leq l
\]

and

\[
\lim_{r_n \to \infty} \left| \frac{\nu(r_n, f)}{z_{r_n}} \right|_{\Gamma M - d(M)} \leq \lim_{r_n \to \infty} r_n^{\frac{\varepsilon(f)-1}{2}(\Gamma M - d(M))} = 0.
\]

From (3.30), (3.32) and (3.33) we get \( c = 0 \), which is a contradiction. Therefore we get

\[
\sigma(f) = 1.
\]

From (3.11) and (3.34) we get \( \sigma(e^\phi) \leq 1 \) and it follows that \( \phi \) is a polynomial of degree \( \deg(\phi) \leq 1 \). If \( \phi \) is a constant, then from (3.1) we get the conclusion \( (i) \) of Theorem 1.2.

Next suppose that \( \phi \) is a polynomial degree \( \deg(\phi) = 1 \). Thus

\[
\phi(z) = p_1 z + p_0,
\]

where \( p_1 \neq 0 \) and \( p_0 \) are complex number.

First of all we prove that \( \mu(f) = 1 \).

From (3.34) it follows that \( \mu(f) \leq 1 \).

Let us suppose that \( \mu(f) < 1 \).

By Definition 1.1 there exists an infinite sequence of positive numbers \( r_n \) such that

\[
\lim_{r_n \to \infty} \frac{\log T(r_n, f)}{\log r_n} = \mu(f).
\]

Again from (3.11), we get

\[
\mu(e^\phi) \leq \lim_{r_n \to \infty} \frac{\log T(r_n, e^\phi)}{\log r_n} \leq \lim_{r_n \to \infty} \frac{\log T(r_n, f)}{\log r_n} = \mu(f) < 1.
\]

\[
\Rightarrow \mu(e^\phi) < 1,
\]

which is a contradiction. Therefore \( \mu(f) = 1 \).
Secondly, we will prove that not all $a_1, a_2, ..., a_l$ are zero. Suppose that $a_j = 0$ for $1 \leq j \leq l$, then we have

$$M[f] - a(z) = (f^d - a(z))e^{p_1 + p_0}. \tag{3.35}$$

From Definition 1.1, Lemma 2.6, Definition 1.1.1, Theorem 1.1.3 from [12] and the assumption $\sigma(a) < \sigma(f)$, there exists an infinite sequence of points $z_{r_n} = r_n e^{i\theta(r_n)}$ satisfying $M(r_n, f) = |f(z_{r_n})|$, where $r_n \in I \setminus \bigcup_{j=1}^n F_j$, $I \subseteq R^+$ is a subset with logarithmic measure $\int_I \frac{dt}{t} = \infty$, such that (3.6) and (3.7) holds.

From (3.6), (3.7) and (3.35), we get

$$\left(\frac{\nu(r_n, f)}{z_{r_n}}\right)^{\Gamma_M - d(M)} \{1 + o(1)\} = e^{p_1 z + p_0} \text{ as } r_n \to \infty. \tag{3.36}$$

From (3.36), we get

$$|p_1| r_n - |p_0| = |p_1| |z_{r_n}| - |p_0| \leq |p_1 z_{r_n} + p_0| \leq \log e^{p_1 z_{r_n} + p_0} + O(1) \leq (\Gamma_M - d(M))|\log \nu(r_n, f)| - \log r_n + O(1) \leq (\Gamma_M - d(M))\{\sigma(f) + 2\} \log r_n + O(1) \text{ as } r_n \to \infty,$$

which is a contradiction, since $p_1 \neq 0$. This completes the proof of (ii) of Theorem 1.4.

References

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