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ON THE STABILITY OF THE QUADRATIC-ADDITIVE TYPE FUNCTIONAL EQUATION IN RANDOM NORMED SPACES VIA FIXED POINT METHOD

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Abstract. In this paper, we prove the stability in random normed spaces via fixed point method for the functional equation

$$2f(x+y)+f(x-y)+f(y-x)-f(2x)-f(2y)=0.$$

1. Introduction

In 1940, S. M. Ulam [23] raised a question concerning the stability of homomorphisms: Given a group G_1, a metric group G_2 with the metric $d(\cdot,\cdot)$, and a positive number ε, does there exist a $\delta>0$ such that if a mapping $f: G_1 \to G_2$ satisfies the inequality

$$d(f(xy), f(x)f(y)) < \delta$$

for all $x, y \in G_1$ then there exists a homomorphism $F: G_1 \to G_2$ with

$$d(f(x), F(x)) < \varepsilon$$

for all $x \in G_1$? As mentioned above, when this problem has a solution, we say that the homomorphisms from G_1 to G_2 are stable. In 1941, D. H. Hyers [5] gave a partial solution of Ulam's problem for the case of approximate additive mappings under the assumption that G_1 and G_2 are Banach spaces. Hyers' result was generalized by T. Aoki [1] for additive mappings and Th. M. Rassias [19] for linear mappings

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by considering the stability problem with unbounded Cauchy differences. The paper of Th. M. Rassias has provided a lot of influence in the development of stability problems. The terminology Hyers-Ulam-Rassias stability originated from these historical background. During the last decades, the stability problems of functional equations have been extensively investigated by a number of mathematicians, see [2]-[4], [6]-[15].

Recall, almost all subsequent proofs in this very active area have used Hyers' method, called *a direct method.* Namely, the function F, which is the solution of a functional equation, is explicitly constructed, starting from the given function f, by the formulae $F(x) = \lim_{n\to\infty} \frac{1}{2^n} f(2^n x)$ or $F(x) = \lim_{n\to\infty} 2^n f(\frac{x}{2^n})$. In 2003, V. Radu [18] observed that the existence of the solution F of a functional equation and the estimation of the difference with the given function f can be obtained from the fixed point alternative. In 2008, D. Mihet and V. Radu [17] applied this method to prove the stability theorems of *the Cauchy functional equation:*

$$f(x+y) - f(x) - f(y) = 0 \tag{1.1}$$

in random normed spaces. We call solutions of (1.1) by *additive mappings.*

In this paper, using the fixed point method, we will prove the stability for the *quadratic-additive type functional equation:*

$$2f(x+y) + f(x-y) + f(y-x) - f(2x) - f(2y) = 0 \tag{1.2}$$

in random normed spaces. It is easy to see that the mappings $f(x) = ax^2 + bx$ is a solution of the functional equation (1.2). The solution of the quadratic-additive type functional equation (1.2) is said to be *a quadratic-additive mapping.*

2. Preliminaries

In this section, we state the usual terminology, notations and conventions of the theory of random normed spaces, as in [21,22]. Firstly, the space of all probability distribution functions is denoted by

$$\Delta^+ := \{F : \mathbb{R} \cup \{-\infty, \infty\} \to [0,1] \big| F \text{ is left-continuous and nondecreasing on } \mathbb{R}, \text{ where } F(0) = 0 \text{ and } F(+\infty) = 1\}.$$

And let the subset $D^+ \subseteq \Delta^+$ be the set $D^+ := \{F \in \Delta^+ | l^- F(+\infty) = 1\}$, where $l^- f(x)$ denotes the left limit of the function f at the point x. The space Δ^+ is partially ordered by the usual pointwise ordering of functions, that is, $F \leq G$ if and only if $F(t) \leq G(t)$ for all $t \in \mathbb{R}$. The maximal element for Δ^+ in this order is the distribution function $\varepsilon_0 : \mathbb{R} \cup \{0\} \to [0, \infty)$ given by

$$\varepsilon_0(t) = \begin{cases} 0, & \text{if } t \leq 0, \\ 1, & \text{if } t > 0. \end{cases}$$

Definition 2.1. ([21]) A mapping $\tau : [0,1] \times [0,1] \to [0,1]$ is called *a continuous triangular norm* (briefly, *a continuous t-norm*) if τ satisfies the following conditions:

(a) τ is commutative and associative;
(b) τ is continuous;
(c) $\tau(a, 1) = a$ for all $a \in [0, 1]$;
(d) $\tau(a, b) \leq \tau(c, d)$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0, 1]$.

Typical examples of continuous t-norms are $\tau_P(a, b) = ab$, $\tau_M(a, b) = \min(a, b)$ and $\tau_L(a, b) = \max(a + b - 1, 0)$.

Definition 2.2. ([22]) *A random normed space* (briefly, *RN-space*) is a triple (X, Λ, τ), where X is a vector space, τ is a continuous t-norm, and Λ is a mapping from X into D^+ such that the following conditions hold:

(RN1) $\Lambda_x(t) = \varepsilon_0(t)$ for all $t > 0$ if and only if $x = 0$,
(RN2) $\Lambda_{\alpha x}(t) = \Lambda_x(t/|\alpha|)$ for all x in X, $\alpha \neq 0$ and all $t \geq 0$,
(RN3) $\Lambda_{x+y}(t+s) \geq \tau(\Lambda_x(t), \Lambda_y(s))$ for all $x, y \in X$ and all $t, s \geq 0$.

If $(X, \|\cdot\|)$ is a normed space, we can define a mapping $\Lambda : X \to D^+$ by

$$\Lambda_x(t) = \frac{t}{t + \|x\|}$$

for all $x \in X$ and $t > 0$. Then (X, Λ, τ_M) is a random normed space, which is called *the induced random normed space.*

Definition 2.3. *Let (X, Λ, τ) be an RN-space.*
(i) *A sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$ if, for every $t > 0$ and $\varepsilon > 0$, there exists a positive integer N such that $\Lambda_{x_n - x}(t) > 1 - \varepsilon$ whenever $n \geq N$.*
(ii) *A sequence $\{x_n\}$ in X is called a Cauchy sequence if, for every $t > 0$ and $\varepsilon > 0$, there exists a positive integer N such that $\Lambda_{x_n - x_m}(t) > 1 - \varepsilon$ whenever $n \geq m \geq N$.*
(iii) *An RN-space (X, Λ, τ) is said to be complete if and only if every Cauchy sequence in X is convergent to a point in X.*

Theorem 2.4. ([21]) *If (X, Λ, τ) is an RN-space and $\{x_n\}$ is a sequence such that $x_n \to x$, then $\lim_{n\to\infty} \Lambda_{x_n}(t) = \Lambda_x(t)$.*

3. Main results

We recall the fundamental result in the fixed point theory.

Theorem 3.1. ([16] or [20]) *Suppose that a complete generalized metric space (X, d), which means that the metric d may assume infinite values, and a strictly contractive mapping $J : X \to X$ with the Lipschitz constant $0 < L < 1$ are given. Then, for each given element $x \in X$, either*

$$d(J^n x, J^{n+1} x) = +\infty, \ \forall n \in \mathbb{N} \cup \{0\},$$

or there exists a nonnegative integer k such that:
(1) *$d(J^n x, J^{n+1} x) < +\infty$ for all $n \geq k$;*
(2) *the sequence $\{J^n x\}$ is convergent to a fixed point y^* of J;*
(3) *y^* is the unique fixed point of J in $Y := \{y \in X, d(J^k x, y) < +\infty\}$;*
(4) *$d(y, y^*) \leq (1/(1-L))d(y, Jy)$ for all $y \in Y$.*

Let X and Y be vector spaces. We use the following abbreviation for a given mapping $f : X \to Y$

$$Df(x, y) := 2f(x + y) + f(x - y) + f(y - x) - f(2x) - f(2y)$$

for all $x, y \in X$. Now we will establish the stability for the functional equation (1.2) in random normed spaces via fixed point method.

THEOREM 3.2. *Let X be a linear space, (Z, Λ', τ_M) be an RN-space, (Y, Λ, τ_M) be a complete RN-space and $f : X \to Y$ be a mapping with $f(0) = 0$ for which there is $\varphi : X^2 \to Z$ such that*

$$\Lambda_{Df(x,y)}(t) \geq \Lambda'_{\varphi(x,y)}(t) \tag{3.1}$$

for all $x, y \in X$ and $t > 0$. If for all $x, y \in X$ and $t > 0$ φ satisfies one of the following conditions:
(i) $\Lambda'_{\alpha\varphi(x,y)}(t) \leq \Lambda'_{\varphi(2x,2y)}(t)$ *for some* $0 < \alpha < 2$,
(ii) $\Lambda'_{\varphi(2x,2y)}(t) \leq \Lambda'_{\alpha\varphi(x,y)}(t)$ *for some* $4 < \alpha$
then there exists a unique quadratic-additive mapping $F : X \to Y$ such that

$$\Lambda_{f(x)-F(x)}(t) \geq \begin{cases} M(x, (2-\alpha)t) & \text{if } \varphi \text{ satisfies (i)}, \\ M(x, (\alpha-4)t) & \text{if } \varphi \text{ satisfies (ii)} \end{cases} \tag{3.2}$$

for all $x \in X$ and $t > 0$, where

$$M(x,t) := \tau_M\big\{\Lambda'_{\varphi(x,0)}(t), \Lambda'_{\varphi(-x,0)}(t)\big\}.$$

Moreover if $\alpha < 1$ and $\Lambda'_{\varphi(x,y)}$ is continuous in x,y under the condition (i), *then f is a quadratic-additive mapping.*

Proof. We will prove the theorem in two cases, φ satisfies the condition (i) or (ii).

Case 1. Assume that φ satisfies the condition (i). Let S be the set of all functions $g : X \to Y$ with $g(0) = 0$ and introduce a generalized metric on S by

$$d(g,h) := \inf\big\{u \in \mathbb{R}^+ : \Lambda_{g(x)-h(x)}(ut) \geq M(x,t) \text{ for all } x \in X\big\}.$$

Consider the mapping $J : S \to S$ defined by

$$Jf(x) := \frac{f(2x) - f(-2x)}{4} + \frac{f(2x) + f(-2x)}{8}$$

then we have

$$J^n f(x) = \frac{1}{2}\left(4^{-n}\left(f(2^n x) + f(-2^n x)\right) + 2^{-n}\left(f(2^n x) - f(-2^n x)\right)\right)$$

for all $x \in X$ and $n \in \mathbb{N}$. Let $f, g \in S$ and let $u \in [0, \infty]$ be an arbitrary constant with $d(g, f) \leq u$. From the definition of d, (RN2), and (RN3), for the given $0 < \alpha < 2$ we have

$$\begin{aligned}
\Lambda_{Jg(x)-Jf(x)}\left(\frac{\alpha u}{2}t\right) =&\Lambda_{\frac{3(g(2x))-f(2x))}{8}-\frac{g(-2x)-f(-2x)}{8}}\left(\frac{\alpha u}{2}t\right)\\
\geq&\tau_M\left\{\Lambda_{\frac{3(g(2x)-f(2x))}{8}}\left(\frac{3\alpha ut}{8}\right), \Lambda_{\frac{g(-2x)-f(-2x)}{8}}\left(\frac{\alpha ut}{8}\right)\right\}\\
\geq&\tau_M\left\{\Lambda_{g(2x)-f(2x)}(\alpha ut), \Lambda_{g(-2x)-f(-2x)}(\alpha ut)\right\}\\
\geq&\tau_M\left\{\Lambda'_{\varphi(2x,0)}(\alpha t), \Lambda'_{\varphi(-2x,0)}(\alpha t)\right\}\\
\geq&M(x,t)
\end{aligned}$$

for all $x \in X$, which implies that

$$d(Jf, Jg) \leq \frac{\alpha}{2}d(f, g).$$

That is, J is a strictly contractive self-mapping of S with the Lipschitz constant $\frac{\alpha}{2}$. Moreover, by (3.1), we see that

$$\begin{aligned}
\Lambda_{f(x)-Jf(x)}\left(\frac{t}{2}\right) =&\Lambda_{\frac{3Df(x,0)}{8}-\frac{Df(-x,0)}{8}}\left(\frac{t}{2}\right)\\
\geq&\tau_M\left\{\Lambda_{\frac{3Df(x,0)}{8}}\left(\frac{3t}{8}\right), \Lambda_{\frac{Df(-x,0)}{8}}\left(\frac{t}{8}\right)\right\}\\
\geq&\tau_M\left\{\Lambda_{Df(x,0)}(t), \Lambda_{Df(-x,0)}(t)\right\}\\
\geq&\tau_M\left\{\Lambda'_{\varphi(x,0)}(t), \Lambda'_{\varphi(-x,0)}(t)\right\}
\end{aligned}$$

for all $x \in X$. It means that $d(f, Jf) \leq \frac{1}{2} < \infty$ by the definition of d. Therefore according to Theorem 3.1, the sequence $\{J^n f\}$ converges to the unique fixed point $F : X \to Y$ of J in the set $T = \{g \in S | d(f, g) < \infty\}$, which is represented by

$$F(x) := \lim_{n\to\infty}\left(\frac{f(2^n x) + f(-2^n x)}{2 \cdot 4^n} + \frac{f(2^n x) - f(-2^n x)}{2^{n+1}}\right)$$

for all $x \in X$. Since

$$d(f, F) \leq \frac{1}{1 - \frac{\alpha}{2}}d(f, Jf) \leq \frac{1}{2 - \alpha}$$

the inequality (3.2) holds. Next we will show that F is a quadratic-additive mapping. Let $x, y \in X$. Then by (RN3) we have

$$\begin{aligned}\Lambda_{DF(x,y)}(t) \geq \tau_M \Big\{ & \Lambda_{2(F-J^n f)(x+y)}\left(\frac{t}{10}\right), \Lambda_{(F-J^n f)(x-y)}\left(\frac{t}{10}\right), \\ & \Lambda_{(F-J^n f)(y-x)}\left(\frac{t}{10}\right), \Lambda_{(J^n f-F)(2x)}\left(\frac{t}{10}\right), \\ & \Lambda_{(J^n f-F)(2y)}\left(\frac{t}{10}\right), \Lambda_{DJ^n f(x,y)}\left(\frac{t}{2}\right) \Big\}\end{aligned} \tag{3.3}$$

for all $x, y \in X$ and $n \in \mathbb{N}$. The first five terms on the right hand side of the above inequality tend to 1 as $n \to \infty$ by the definition of F. Now consider that

$$\begin{aligned}\Lambda_{DJ^n f(x,y)}\left(\frac{t}{2}\right) \geq \tau_M \Big\{ & \Lambda_{\frac{Df(2^n x, 2^n y)}{2 \cdot 4^n}}\left(\frac{t}{8}\right), \Lambda_{\frac{Df(-2^n x, -2^n y)}{2 \cdot 4^n}}\left(\frac{t}{8}\right), \\ & \Lambda_{\frac{Df(2^n x, 2^n y)}{2 \cdot 2^n}}\left(\frac{t}{8}\right), \Lambda_{\frac{Df(-2^n x, -2^n y)}{2 \cdot 2^n}}\left(\frac{t}{8}\right) \Big\} \\ \geq \tau_M \Big\{ & \Lambda_{Df(2^n x, 2^n y)}\left(\frac{4^n t}{4}\right), \Lambda_{Df(-2^n x, -2^n y)}\left(\frac{4^n t}{4}\right), \\ & \Lambda_{Df(2^n x, 2^n y)}\left(\frac{2^n t}{4}\right), \Lambda_{Df(-2^n x, -2^n y)}\left(\frac{2^n t}{4}\right) \Big\} \\ \geq \tau_M \Big\{ & \Lambda'_{\varphi(x,y)}\left(\frac{4^n t}{4\alpha^n}\right), \Lambda'_{\varphi(-x,-y)}\left(\frac{4^n t}{4\alpha^n}\right), \\ & \Lambda'_{\varphi(x,y)}\left(\frac{2^n t}{4\alpha^n}\right), \Lambda'_{\varphi(-x,-y)}\left(\frac{2^n t}{4\alpha^n}\right) \Big\}\end{aligned}$$

which tends to 1 as $n \to \infty$ by (RN3) and $\frac{2}{\alpha} > 1$ for all $x, y \in X$. Therefore it follows from (3.3) that

$$\Lambda_{DF(x,y)}(t) = 1$$

for each $x, y \in X$ and $t > 0$. By (RN1), this means that $DF(x,y) = 0$ for all $x, y \in X$. Assume that $\alpha < 1$ and $\Lambda'_{\varphi(x,y)}$ is continuous in x, y.

If m, a, b, c, d are any fixed integers with $a, c \neq 0$, then we have

$$\begin{aligned}\lim_{n\to\infty} \Lambda'_{\varphi((2^n a+b)x,(2^n c+d)y)}(t) &\geq \lim_{n\to\infty} \Lambda'_{\varphi\left(\left(a+\frac{b}{2^n}\right)x,\left(c+\frac{d}{2^n}\right)y\right)}\left(\frac{t}{\alpha^n}\right)\\ &= \lim_{n\to\infty} \Lambda'_{\varphi\left(\left(a+\frac{b}{2^n}\right)x,\left(c+\frac{d}{2^n}\right)y\right)}(mt)\\ &=\Lambda'_{\varphi(ax,cy)}(mt)\end{aligned}$$

for all $x, y \in X$ and $t > 0$. Since m is arbitrary, we have

$$\lim_{n\to\infty} \Lambda'_{\varphi((2^n a+b)x,(2^n c+d)y)}(t) \geq \lim_{m\to\infty} \Lambda'_{\varphi(ax,cy)}(mt) = 1$$

for all $x, y \in X$ and $t > 0$. From these, we get the inequality

$$\begin{aligned}\Lambda_{2(f-F)(x)}(5t) \geq& \lim_{n\to\infty} \tau_M\big\{\Lambda_{(Df-DF)((2^n+1)x,-2^n x)}(t),\\ &\Lambda_{(F-f)((2^{n+1}+1)x)}(t), \Lambda_{(F-f)(-(2^{n+1}+1)x)}(t),\\ &\Lambda_{(f-F)((2^{n+1}+2)x)}(t), \Lambda_{(f-F)(-2^{n+1}x)}(t)\big\}\\ \geq& \lim_{n\to\infty} \tau_M\big\{\Lambda'_{\varphi((2^n+1)x,-2^n x)}(t), M((2^{n+1}+1)x,(2-\alpha)t),\\ &M((2^{n+1}+2)x,(2-\alpha)t), M\left(-2^{n+1}x,(2-\alpha)t\right)\big\}\\ =&1\end{aligned}$$

for all $x \in X$. From the above equality and the fact $f(0) = 0 = F(0)$, we obtain $f \equiv F$.

Case 2. We take $\alpha > 4$ and suppose that φ satisfies the condition (ii). Let the set (S, d) be as in the proof of Case 1. Now we consider the mapping $J : S \to S$ defined by

$$Jg(x) := g\left(\frac{x}{2}\right) - g\left(-\frac{x}{2}\right) + 2\left(g\left(\frac{x}{2}\right) + g\left(-\frac{x}{2}\right)\right)$$

for all $g \in S$ and $x \in X$. Notice that

$$J^n g(x) = 2^{n-1}\left(g\left(\frac{x}{2^n}\right) - g\left(-\frac{x}{2^n}\right)\right) + \frac{4^n}{2}\left(g\left(\frac{x}{2^n}\right) + g\left(-\frac{x}{2^n}\right)\right)$$

for all $x \in X$ and $n \in \mathbb{N}$. Let $f, g \in S$ and let $u \in [0, \infty]$ be an arbitrary constant with $d(g, f) \leq u$. From the definition of d, (RN2), and (RN3), we have

$$\begin{aligned}
\Lambda_{Jg(x)-Jf(x)}\left(\frac{4u}{\alpha}t\right) =&\Lambda_{3(g(\frac{x}{2})-f(\frac{x}{2}))+g(-\frac{x}{2})-f(-\frac{x}{2})}\left(\frac{4u}{\alpha}t\right)\\
\geq&\tau_M\left\{\Lambda_{3(g(\frac{x}{2})-f(\frac{x}{2}))}(\frac{3u}{\alpha}t),\Lambda_{g(-\frac{x}{2})-f(-\frac{x}{2})}\left(\frac{u}{\alpha}t\right)\right\}\\
\geq&\tau_M\left\{\Lambda_{g(\frac{x}{2})-f(\frac{x}{2})}\left(\frac{u}{\alpha}t\right),\Lambda_{g(-\frac{x}{2})-f(-\frac{x}{2})}\left(\frac{u}{\alpha}t\right)\right\}\\
\geq&\tau_M\left\{\Lambda'_{\varphi(\frac{x}{2},0)}\left(\frac{t}{\alpha}\right),\Lambda'_{\varphi(-\frac{x}{2},0)}\left(\frac{t}{\alpha}\right)\right\}\\
\geq&M(x,t)
\end{aligned}$$

for all $x\in X$, which implies that

$$d(Jf,Jg)\leq\frac{4}{\alpha}d(f,g).$$

That is, J is a strictly contractive self-mapping of S with the Lipschitz constant $0<\frac{4}{\alpha}<1$. Moreover, by (3.1), we see that

$$\Lambda_{f(x)-Jf(x)}\left(\frac{t}{\alpha}\right)=\Lambda_{Df(\frac{x}{2},0)}\left(\frac{t}{\alpha}\right)\geq\Lambda'_{\varphi(\frac{x}{2},0)}\left(\frac{t}{\alpha}\right)\geq\Lambda'_{\varphi(x,0)}(t)$$

for all $x\in X$. It means that $d(f,Jf)\leq\frac{1}{\alpha}<\infty$ by the definition of d. Therefore according to Theorem 3.1, the sequence $\{J^nf\}$ converges to the unique fixed point $F:X\to Y$ of J in the set $T=\{g\in S|d(f,g)<\infty\}$, which is represented by

$$F(x):=\lim_{n\to\infty}\left(2^{n-1}\left(f\left(\frac{x}{2^n}\right)-f\left(-\frac{x}{2^n}\right)\right)+\frac{4^n}{2}\left(f\left(\frac{x}{2^n}\right)+f\left(-\frac{x}{2^n}\right)\right)\right)$$

for all $x\in X$. Since

$$d(f,F)\leq\frac{1}{1-\frac{4}{\alpha}}d(f,Jf)\leq\frac{1}{\alpha-4}$$

the inequality (3.2) holds. Next we will show that F is quadratic-additive. Let $x,y\in X$. Then by (RN3) we have the inequality (3.3) for all $x, y \in X$ and $n \in \mathbb{N}$. The first five terms on the right hand side of the inequality (3.3) tend to 1 as $n \to \infty$ by the definition of F. Now consider that

$$\begin{aligned}\Lambda_{DJ^n f(x,y)}\left(\frac{t}{2}\right) \geq & \tau_M\left\{\Lambda_{2^{2n-1}Df(\frac{x}{2^n},\frac{y}{2^n})}\left(\frac{t}{8}\right), \Lambda_{2^{2n-1}Df(\frac{-x}{2^n},\frac{-y}{2^n})}\left(\frac{t}{8}\right),\right.\\ & \left.\Lambda_{2^{n-1}Df(\frac{x}{2^n},\frac{y}{2^n})}\left(\frac{t}{8}\right), \Lambda_{-2^{n-1}Df(\frac{-x}{2^n},\frac{-y}{2^n})}\left(\frac{t}{8}\right)\right\}\\ \geq & \tau_M\left\{\Lambda'_{\varphi(x,y)}\left(\frac{\alpha^n t}{4^{n+1}}\right), \Lambda'_{\varphi(-x,-y)}\left(\frac{\alpha^n t}{4^{n+1}}\right),\right.\\ & \left.\Lambda'_{\varphi(x,y)}\left(\frac{\alpha^n t}{2^{n+2}}\right), \Lambda'_{\varphi(-x,-y)}\left(\frac{\alpha^n t}{2^{n+2}}\right)\right\}\end{aligned}$$

which tends to 1 as $n \to \infty$ by (RN3) for all $x, y \in X$. Therefore it follows from (3.3) that

$$\Lambda_{DF(x,y)}(t) = 1$$

for each $x, y \in X$ and $t > 0$. By (RN1), this means that $DF(x, y) = 0$ for all $x, y \in X$. It completes the proof of Theorem 3.2. ☐

Now we have a generalized Hyers-Ulam stability of the quadratic-additive functional equation (1.2) in the framework of normed spaces. Let $\Lambda_x(t) = \frac{t}{t+\|x\|}$. Then (X, Λ, τ_M) is an induced random normed space, which leads us to get the following result.

Corollary 3.3. *Let X be a linear space, Y be a complete normed-space and $f : X \to Y$ be a mapping with $f(0) = 0$ for which there is $\varphi : X^2 \to [0, \infty)$ such that*

$$\|Df(x, y)\| \leq \varphi(x, y)$$

for all $x, y \in X$. If for all $x, y \in X$ φ satisfies one of the following conditions:

(i) $\alpha\varphi(x, y) \geq \varphi(2x, 2y)$ *for some* $0 < \alpha < 2$,
(ii) $\varphi(2x, 2y) \geq \alpha\varphi(x, y)$ *for some* $4 < \alpha$

then there exists a unique quadratic-additive mapping $F : X \to Y$ such that

$$\|f(x) - F(x)\| \leq \begin{cases} \frac{\Phi(x)}{2-\alpha} & \text{if } \varphi \text{ satisfies (i)},\\ \frac{\Phi(x)}{\alpha-4} & \text{if } \varphi \text{ satisfies (ii)}\end{cases}$$

for all $x \in X$, *where* $\Phi(x)$ *is defined by*

$$\Phi(x) = \max(\varphi(x,0), \varphi(-x,0)).$$

Moreover, if $0 < \alpha < 1$ *under the condition* (i), *then* f *is a quadratic-additive mapping.*

Now we have Hyers-Ulam-Rassias stability results of the quadratic-additive type functional equation (1.2).

COROLLARY 3.4. *Let* X *be a normed space,* $p \in \mathbb{R}^+ \backslash [1,2]$ *and* Y *a complete normed-space. If* $f : X \to Y$ *is a mapping such that*

$$\|Df(x,y)\| \leq \|x\|^p + \|y\|^p$$

for all $x, y \in X$ *with* $f(0) = 0$, *then there exists a unique quadratic-additive mapping* $F : X \to Y$ *such that*

$$\|f(x) - F(x)\| \leq \begin{cases} \frac{\|x\|^p}{2-2^p} & \text{if } 0 \leq p < 1, \\ \frac{\|x\|^p}{2^p-4} & \text{if } p > 2 \end{cases}$$

for all $x \in X$.

Proof. If we denote by $\varphi(x,y) = \|x\|^p + \|y\|^p$, then the induced random normed space (X, Λ_x, τ_M) holds the conditions of Theorem 3.3 with $\alpha = 2^p$. □

COROLLARY 3.5. *Let* X *be a normed space and* Y *a Banach space. Suppose that the mapping* $f : X \to Y$ *satisfies the inequality*

$$\|Df(x,y)\| \leq \theta \|x\|^p \|y\|^q$$

for all $x, y \in X$, *where* $\theta \geq 0$, $p, q > 0$ *and* $p + q \in (0,1) \cup (2, \infty)$. *Then* f *is itself a quadratic additive mapping.*

Proof. It follows from Theorem 3.2, by putting

$$\varphi(x,y) := \theta \|x\|^p \|y\|^q$$

for all $x, y \in X$ and $\alpha = 2^{p+q}$. □

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