Korean J. Math. **27** (2019), No. 4, pp. 1159–1180
http://dx.doi.org/10.11568/kjm.2019.27.4.1159

# $e$-FUZZY FILTERS OF MS-ALGEBRAS

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Abstract. In this article, we present the notion of $e$-fuzzy filters in an MS-Algebra and characterize in terms of equivalent conditions. The concept of $D$-fuzzy filters is studied and the set of equivalent conditions under which every $e$-fuzzy filter is an $D$-fuzzy filter are observed. Moreover we study some properties of the space of all prime $e$-fuzzy filters of an MS-algebra.

## 1. Introduction

MS-algebras introduced by Blyth and Varlet [2] as common abstraction of de Morgan algebras and MS-algebras. And also they [3] characterized the subvarieties of MS-algebras. Recently Roa [8] introduced $e$-filters of MS-algebras.
On the other hand, fuzzy set theory was introduced by Zadeh [11]. Next, fuzzy groups were studied by Rosenfield [7]. Many scholars have used this idea to different mathematical branches such as semi-group, ring, semi-ring, near-ring, lattice etc. For instance Yuan and Wu [10] introduced the notion of fuzzy sublattice and fuzzy ideals of lattice, Swamy and Raju [9] fuzzy ideals and congruences of lattices, Kumar [6], topologized the set of all fuzzy prime ideals of a commutative ring with unity and studied some properties of the space, Kumar [6], studied about the

Received February 12, 2019. Revised August 20, 2019. Accepted September 19, 2019.
2010 Mathematics Subject Classification: 06D99, 06D05, 06D30.
Key words and phrases: MS-algebras, $e$-fuzzy filters, $D$-fuzzy filter, prime $e$-fuzzy filters, maximal $e$-fuzzy filters.
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space of prime fuzzy ideals of a ring in different way and Hadji-Abadi and Zahedi [4] extended the result of Kumar. In this article our aim is to present $e$-fuzzy filters of an MS-algebra and that every $e$-fuzzy filter of an MS-algebra is an $D$-fuzzy filter. Finally we discuss the concept of topological space on the set all prime $e$-fuzzy filters.

## 2. Preliminaries

In this section, we recall basic definitions and results which will be used in this article. For in details in ordinary crisp theory of $e$-filters of MS-algebras, we refer to [8].

Definition 2.1. [2] An MS-algebra is an algebra $(L, \vee, \wedge, ^\circ, 0, 1)$ of type $(2, 2, 1, 0, 0)$ such that $(L, \vee, \wedge, 0, 1)$ is a bounded distributive lattice and $a \to a^\circ$ is a unary operation satisfying the conditions $a \leq a^{\circ\circ}$, $(a \wedge b)^\circ = a^\circ \vee b^\circ$ and $1^\circ = 0$ for all $a, b \in L$
A de Morgan algebra is an MS-algebra satisfying $a^{\circ\circ} = a$ for all $a \in L$.

Lemma 2.2. [2] *Let $L$ be any MS-algebra and $a, b \in L$. Then*
*(1)* $0^\circ = 1$
*(2)* $a \leq b \Rightarrow b^\circ \leq a^\circ$
*(3)* $a^{\circ\circ\circ} = a^\circ$
*(4)* $(a \vee b)^\circ = a^\circ \wedge b^\circ$
*(5)* $(a \vee b)^{\circ\circ} = a^{\circ\circ} \vee b^{\circ\circ}$
*(6)* $(a \wedge b)^{\circ\circ} = a^{\circ\circ} \wedge b^{\circ\circ}$

Definition 2.3. [8] For any filter $F$ of an MS-algebra $L$, define $F^e$ as the set $F^e = \{x \in L / x^\circ \leq a^\circ \text{ for some } a \in F\}$

Definition 2.4. [8] A filter $F$ of an MS-algebra $L$ is called an $e$-filter of $L$ if $F = F^e$

An element $a$ of an MS-algebra L is called a dense element if $a^\circ = 0$. The set of all dense elements in MS-algebra L is denoted by $D$.

Definition 2.5. [8] A filter $F$ of an MS-algebra L is called a D-filter of L if $D \subseteq F$.

Remember that, for any set $S$ a function $\mu : S \longrightarrow ([0, 1], \wedge, \vee)$ is called a fuzzy subset of $S$, where $[0, 1]$ is a unit interval, $\alpha \wedge \beta = min\{\alpha, \beta\}$ and $\alpha \vee \beta = max\{\alpha, \beta\}$ for all $\alpha, \beta \in [0, 1]$.

Let $\mu : S \to [0,1]$. For every $\alpha \in [0,1]$, the level subset $\mu$ of $S$ is $\mu_\alpha = \{x \in L : \alpha \leq \mu(x)\}$.

DEFINITION 2.6. Let $x \in S$, $0 < \alpha \leq 1$. A fuzzy point $x_\alpha$ of $S$ is a fuzzy subset of $S$ defined as

$$x_\alpha(z) = \begin{cases} \alpha & if\ z = x \\ 0 & otherwise \end{cases}$$

.

We define the binary operations " + " and "." on all fuzzy subsets of a lattice $L$ as: $(\mu + \theta)(x) = \sup\{\mu(a) \wedge \theta(b) : a, b \in L, a \vee b = x\}$ and $(\mu.\theta)(x) = \sup\{\mu(a) \wedge \theta(b) : a, b \in L, a \wedge b = x\}$.

The intersection of fuzzy filters of $L$ is a fuzzy filter. However the union of fuzzy filters may not be fuzzy filter. The least upper bound of a fuzzy filters $\mu$ and $\theta$ of $L$ is denoted as $\mu \vee \theta = \cap\{\sigma \in FF(L) : \mu \cup \theta \subseteq \sigma\}$.

If $\mu$ and $\theta$ are fuzzy filters of $L$, then $\mu.\theta = \mu \vee \theta$ and $\mu + \theta = \mu \cap \theta$

Let $\mu$ be a fuzzy subset of a lattice $L$. The smallest fuzzy filter of $L$ containing $\mu$ is called a fuzzy filter of $L$ induced by $\mu$ and denoted by $[\mu)$ and $[\mu) = \cap\{\theta : \theta \text{ is a fuzzy filter of } L,\ \mu \subseteq \theta\}$

DEFINITION 2.7. [9] A fuzzy subset $\mu$ of a bounded lattice $L$ is said to be a fuzzy ideal of L, if for all $x, y \in L$,

1. $\mu(0) = 1$,
2. $\mu(x \vee y) \geq \mu(x) \wedge \mu(y)$
3. $\mu(x \wedge y) \geq \mu(x) \vee \mu(y)\ for\ all\ x, y \in L.$

In [9], Swamy and Raju observed that, a fuzzy subset $\mu$ of a a bounded lattice $L$ is a fuzzy ideal of $L$ if and only if $\mu(0) = 1$ and $\mu(x \vee y) = \mu(x) \wedge \mu(y)\ for\ all\ x, y \in L.$

DEFINITION 2.8. [9] A fuzzy subset $\mu$ of a bounded lattice $L$ is said to be a fuzzy filter of L, if for all $x, y \in L$,

1. $\mu(1) = 1$,
2. $\mu(x \vee y) \geq \mu(x) \wedge \mu(y)$
3. $\mu(x \wedge y) \geq \mu(x) \vee \mu(y)\ for\ all\ x, y \in L.$

In [9] a fuzzy subset $\mu$ of a a bounded lattice $L$ is a fuzzy filter of $L$ if and only if $\mu(1) = 1$ and $\mu(x \vee y) = \mu(x) \wedge \mu(y)$ *for all* $x, y \in L$.

THEOREM 2.9. [9] *Let $\mu$ be a fuzzy subset of L. Then $\mu$ is a fuzzy ideal of L if and only if, for any $\alpha \in [0,1]$, $\mu_\alpha$ is an ideal of L.*

DEFINITION 2.10. [9] A proper fuzzy ideal $\mu$ of $L$ is called prime fuzzy ideal of $L$ if for any two fuzzy ideals $\lambda, \nu$ of $L$, $\lambda \cap \nu \subseteq \mu \Rightarrow \lambda \subseteq \mu$ or $\nu \subseteq \mu$.

$\mu$ is a prime fuzzy ideal of $L$ if and only if $Im\mu = \{1, \beta\}$ , $\beta \in [0,1)$ and $\mu_* = \{x \in L : \mu(x) = 1\}$ is a prime ideal of $L$.

Throughout in the next sections $L$ stands for an MS-algebra unless otherwise mentioned.

## 3. $e$-Fuzzy Filters of MS-algebras

In this section, the concept of $e$- fuzzy filters is introduced and some basic properties of $e$-fuzzy filter are observed. The concept of $D$-fuzzy filter is introduced and we obtain a set of equivalent conditions for any $e$-fuzzy filter to become an $D$-fuzzy filter. We prove that the class of $e$-fuzzy filters $\mathcal{FF}^e(L)$ is a complete distributive lattice with relation $\subseteq$.

DEFINITION 3.1. Let $\mu$ be any fuzzy filter of an MS-algebra $L$, an extension of $\mu$ define as the fuzzy subset $\mu^e(x) = sup\{\mu(a) : x^\circ \leq a^\circ,\ a \in L\}$ for all $x \in L$.

The following Lemma reveals some basic properties of $\mu^e$

LEMMA 3.2. *Let $\mu$ and $\nu$ be any two fuzzy filters of an MS-algebra L. Then*

*(1)* $\mu^e$ *is a fuzzy filter of L*
*(2)* $\mu \subseteq \mu^e$,
*(3)* $\mu \subseteq \nu \Rightarrow \mu^e \subseteq \nu^e$,
*(4)* $(\mu \cap \nu)^e = \mu^e \cap \nu^e$,
*(5)* $(\mu^e)^e = \mu^e$.

*Proof.* For elements $x, y, a, b \in L$,

(1) $\mu^e(1) = sup\{\mu(a) : 1^\circ \leq a^\circ,\ a \in L\} \geq \mu(1) = 1$. Hence $\mu^e(1) = 1$. Next,

$$\begin{aligned} \mu^e(x) \wedge \mu^e(y) &= sup\{\mu(a) : x^\circ \leq a^\circ\} \wedge sup\{\mu(b) : y^\circ \leq b^\circ\} \\ &= sup\{\mu(a) \wedge \mu(b) : x^\circ \leq a^\circ,\ y^\circ \leq b^\circ\} \\ &\leq sup\{\mu(a \wedge b) : (x \wedge y)^\circ \leq (a \wedge b)^\circ\} \\ &= \mu^e(x \wedge y) \end{aligned}$$

and

$$\begin{aligned} \mu^e(x) \vee \mu^e(y) &= sup\{\mu(a) : x^\circ \leq a^\circ\} \vee sup\{\mu(b) : y^\circ \leq b^\circ\} \\ &= sup\{\mu(a) \vee \mu(b) : x^\circ \leq a^\circ,\ y^\circ \leq b^\circ\} \\ &\leq sup\{\mu(a \vee b) : (x \vee y)^\circ \leq (a \vee b)^\circ\} \\ &= \mu^e(x \vee y) \end{aligned}$$

Thus $\mu^e$ is a fuzzy filter of $L$.

(2) $\mu^e(x) = sup\{\mu(a) : x^\circ \leq a^\circ\} \geq \mu(x)$. Hence $\mu \subseteq \mu^e$.

(3) Suppose that $\mu \subseteq \nu$, then
$\nu^e(x) = sup\{\nu(a) : x^\circ \leq a^\circ\} \geq sup\{\mu(a) : x^\circ \leq a^\circ\} = \mu^e(x)$.
Hence $\mu^e \subseteq \nu^e$

(4) By (3) $(\mu \cap \nu)^e \subseteq \mu^e \cap \nu^e$.

Conversely,

$$\begin{aligned} (\mu^e \cap \nu^e)(x) &= \mu^e(x) \wedge \nu^e(x) \\ &= \sup\{\mu(a) : x^\circ \leq a^\circ\} \wedge sup\{\nu(b) : x^\circ \leq b^\circ\} \\ &\leq \sup\{\mu(a^{\circ\circ}) : x^{\circ\circ\circ} \leq a^{\circ\circ\circ}\} \wedge sup\{\nu(b^{\circ\circ}) : x^{\circ\circ\circ} \leq b^{\circ\circ\circ}\} \\ &= \sup\{\mu(a^{\circ\circ}) \wedge \nu(b^{\circ\circ}) : x^{\circ\circ\circ} \leq a^{\circ\circ\circ} \wedge b^{\circ\circ\circ}\} \\ &\leq \sup\{\mu(a^{\circ\circ} \vee b^{\circ\circ}) \wedge \nu(a^{\circ\circ} \vee b^{\circ\circ}) : x^{\circ\circ\circ} \leq a^{\circ\circ\circ} \wedge b^{\circ\circ\circ}\} \\ &\leq \sup\{(\mu \cap \nu)(a^{\circ\circ} \vee b^{\circ\circ}) : x^\circ \leq ((a^{\circ\circ} \vee b^{\circ\circ})^\circ\} \\ &= (\mu \cap \nu)^e(x) \end{aligned}$$

Hence $(\mu^e \cap \nu^e) = (\mu \cap \nu)^e$.

(5)

$$\begin{aligned} (\mu^e)^e(x) &= sup\{\mu^e(a) : x^\circ \leq a^\circ, a \in L\} \\ &= sup\{sup\{\mu(z) : a^\circ \leq z^\circ,\ z \in L\} : x^\circ \leq a^\circ, a, x \in L\} \\ &= sup\{\mu(z) : x^\circ \leq z^\circ,\ z \in L\} \\ &= \mu^e(x) \end{aligned}$$

Hence $(\mu^e)^e = \mu^e$. □

Now we define $e$-fuzzy filter in an MS-algebra.

Definition 3.3. A fuzzy filter $\mu$ of an MS-algebra $L$ is called an $e$-fuzzy filter of $L$ if $\mu = \mu^e$.

Theorem 3.4. *$\mu$ is an $e$-fuzzy filter of an MS-algebra $L$ if and only if, $\forall\ \alpha \in [0,1]$, $\mu_\alpha$ is an $e$-filter of $L$.*

Theorem 3.5. *$F$ is an $e$-filter of an MS-algebra $L$ if and only if $\chi_F$ is an $e$-fuzzy filter of $L$.*

Lemma 3.6. *Let $D$ be the set of all dense elements of $L$. Then $\chi_D$ is an $e$-fuzzy filter.*

In the Lemma 3.2(4), we can mention that the intersection of two $e$-fuzzy filters of an MS-algebra is an $e$-fuzzy filter. But the union of two $e$-fuzzy filters may not be the $e$-fuzzy filter.

Example 3.7. Let $L$ be the following MS-algebra described in the diagram 1.

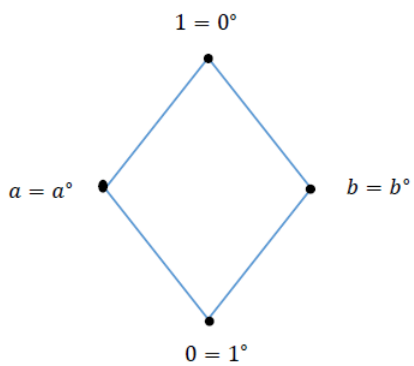



diagram 1

Consider $\mu$ and $\nu$ a fuzzy set of $L$ defined as $\mu(a) = \mu(0) = 0.5$, $\mu(b) = \mu(1) = 1$ and $\nu(0) = \nu(b) = 0.7$, $\nu(a) = 0.8$ and $\nu(1) = 1$. It can be easily verified that $\mu$ and $\nu$ are $e$-fuzzy filters of $L$. But $\mu \cup \nu$ is not an $e$-fuzzy filter of $L$. Since $\mu \cup \nu$ is not an fuzzy filter of $L$ i.e

$$\begin{aligned}(\mu \cup \nu)(a \wedge b) = max\{\mu(a \wedge b), \nu(a \wedge b)\} &= max\{\mu(0), \nu(0)\} \\ &= max\{0.5, 0.7\} = 0.7\end{aligned}$$

and

$$\begin{aligned}(\mu\cup\nu)(a)\wedge(\mu\cup\nu)(b) &= max\{\mu(a),\nu(a)\}\wedge max\{\mu(b),\nu(b)\}\\ &= max\{0.5,0.8\}\wedge max\{1,0.7\}=0.8\wedge 1=0.8\end{aligned}$$

Thus $(\mu\cup\sigma)(a\wedge b)=0.7\neq 0.8=(\mu\cup\sigma)(a)\wedge(\mu\cup\sigma)(b)$

COROLLARY 3.8. *Let $\{\mu_i : i\in\Omega\}$ be a family of $e$-fuzzy filters of an MS-algebra $L$. Then $\cap_{i\in\Omega}\mu_i$ is an $e$-fuzzy filter of $L$.*

In the following, we characterize the $e$-fuzzy filters

THEOREM 3.9. *Let $\mu$ be a fuzzy filter of an MS-algebra $L$. Then, the following are equivalent.*
(1) *$\mu$ is an e-fuzzy filter,*
(2) *$\mu(x)=\mu(x^{\circ\circ})$,*
(3) *For $x,y\in L$ , $x^\circ=y^\circ$ implies $\mu(x)=\mu(y)$.*

*Proof.* (1) $\Rightarrow$ (2). Suppose that $\mu$ is an $e$-fuzzy filter of $L$. For $x,a\in L$, $\mu(x)=\mu^e(x)=sup\{\mu(a) : x^\circ\leq a^\circ\}=sup\{\mu(a) : x^{\circ\circ\circ}=x^\circ\leq a^\circ\}=\mu^e(x^{\circ\circ})=\mu(x^{\circ\circ})$.

(2) $\Rightarrow$ (3). Suppose that condition (2) holds. Let $x,y\in L, x^\circ=y^\circ$. Then $x^{\circ\circ}=y^{\circ\circ}$. Thus $\mu(x)=\mu(x^{\circ\circ})=\mu(y^{\circ\circ})=\mu(y)$. Hence $\mu(x)=\mu(y)$.

(3) $\Rightarrow$ (1). Suppose that condition (3) holds. $\mu^e(x)=sup\{\mu(a) : x^\circ\leq a^\circ\}=sup\{\mu(a\wedge x) : x^\circ\leq a^\circ\}\leq\mu(x)$. Since by (3) $a^\circ=x^\circ\vee a^\circ=(a\wedge x)^\circ$ and $\mu(x\wedge a)\leq\mu(x)$. This implies $\mu^e\subseteq\mu$. Clearly $\mu\subseteq\mu^e$. Hence $\mu$ is an $e$-filter of $L$. □

THEOREM 3.10. *For any fuzzy filter $\mu$ of an MS-algebra $L$, a fuzzy subset $\mu^\circ(x)=sup\{\mu(b) : x^\circ\wedge b=0\ , b\in L\}\ \forall\ x\in L$ is an e-fuzzy filter.*

*Proof.* For any $x,y\in L$,

$$\mu^\circ(1)=sup\{\mu(b) : 1^\circ\wedge b=0,\ b\in L\}\geq\mu(1)=1$$

and

$$\begin{aligned}\mu^\circ(x\wedge y) &= sup\{\mu(b) : (x\wedge y)^\circ\wedge b=0,\ b\in L\}\\ &= sup\{\mu(b) : (x^\circ\vee y^\circ)\wedge b=0,\ b\in L\}\\ &= sup\{\mu(b)\wedge\mu(b) : (x^\circ\wedge b)\vee(y^\circ\wedge b)=0,\ b\in L\}\\ &= sup\{\mu(b) : x^\circ\wedge b=0,\ b\in L\}\wedge sup\{\mu(b) : y^\circ\wedge b=0,\ b\in L\}\\ &= \mu^\circ(x)\wedge\mu^\circ(y)\end{aligned}$$

This implies $\mu^{\circ}$ is a fuzzy filter of L. Next we prove that $\mu$ is an $e$-fuzzy filter. Now

$\mu^{\circ}(x^{\circ\circ}) = sup\{\mu(c) : x^{\circ\circ\circ} \wedge c = 0,\ c \in L\} = sup\{\mu(c) : x^{\circ} \wedge c = 0,\ c \in L\} = \mu^{\circ}(x)$.

Therefor $\mu^{\circ}$ is an $e$-fuzzy filter of $L$. □

DEFINITION 3.11. A fuzzy filter $\mu$ of an MS-algebra $L$ is called a $D$-fuzzy filter of $L$ if $\chi_D \subseteq \mu$ .

THEOREM 3.12. *A fuzzy filter $\mu$ of an MS-algebra $L$ is a D-fuzzy filter of $L$ if and only if, $\forall\ \alpha \in [0, 1]$ , $\mu_{\alpha}$ is a D-filter.*

*Proof.* Suppose that $\mu$ is a $D$-fuzzy filter of $L$, then $\chi_D(x) \leq \mu(x)\ \forall\ x \in L$. Let $x \in D$. Then $\chi_D(x) = 1 \leq \mu(x)$. This implies $\mu(x) = 1$. This implies $x \in \mu_1 \subseteq \mu_{\alpha}\ \forall \alpha \in [0, 1]$, and so $D \subseteq \mu_{\alpha}$. Hence $\mu_{\alpha}$ is a $D$-filter. Conversely, suppose that $\mu_{\alpha}$ is a $D$-filter of $L$, $\forall\ \alpha \in [0, 1]$. If $x \notin D$, then $\chi_D(x) = 0 \leq \mu(x)$. If $x \in D$, then $x \in D \subseteq \mu_{\alpha}, \forall\ \alpha \in [0, 1]$. This implies $x \in D \subseteq \mu_1,\ for\ \alpha = 1$. Thus $\mu(x) \geq 1$. This implies $\chi_D(x) = 1 \leq \mu(x)$. Therefore for all $x \in L$, $\chi_D(x) \leq \mu(x)$. Hence the result. □

THEOREM 3.13. *$F$ is a D-filter of $L$ if and only if $\chi_F$ is a D-fuzzy filter of $L$.*

THEOREM 3.14. *Any e-fuzzy filter of an MS-algebra $L$ is D-fuzzy filter.*

*Proof.* Let $\mu$ be any $e$-fuzzy filter of $L$. For any $x \in L$. If $\chi_D(x) = 0$, then $\chi_D(x) \leq \mu(x)$. If $\chi_D(x) = 1$, then $x \in D$. Thus $\mu^e(x) = \sup\{\mu(a) : x^{\circ} \leq a^{\circ},\ a \in L\} = \sup\{\mu(a) : x^{\circ\circ\circ} \leq a^{\circ},\ a \in L\} = \mu^e(x^{\circ\circ}) = \mu(x^{\circ\circ}) = \mu(1) = 1$. This implies $\chi_D \subseteq \mu^e = \mu$. Hence $\mu$ is a $D$-fuzzy filter. □

COROLLARY 3.15. *$\chi_D$ is the smallest e-fuzzy filter of an MS-algebra.*

We denote the class of all $e$-fuzzy filters of an MS-algebra $L$ by $\mathcal{FF}^e(L)$

THEOREM 3.16. *The class $\mathcal{FF}^e(L)$ is a complete distributive lattice with relation $\subseteq$.*

*Proof.* Since $\chi_D$, $\chi_L \in \mathcal{FF}^e(L)$, $\mathcal{FF}^e(L) \neq \emptyset$. Clearly $(\mathcal{FF}^e(L), \subseteq)$ is a partially order set. Now for any $\mu, \sigma \in \mathcal{FF}^e(L)$, define $\mu \wedge \sigma = \mu \cap \sigma$ and $\mu \sqcup \sigma = (\mu \vee \sigma)^e$, where $(\mu \vee \sigma)^e(x) = sup\{\mu(a) \wedge \mu(b) : x^{\circ} \leq (a \wedge b)^{\circ},\ a, b \in L\}\ \forall\ x \in L$. It can be easily verified that $\mu \cap \sigma, (\mu \vee \sigma)^e \in \mathcal{FF}^e(L)$ and $\mu \cap \sigma$ is the greatest lower bound of $\mu$ and $\sigma$. We need to show $\mu \sqcup \sigma$ is the least upper bound of $\mu$ and $\sigma$. Since $\mu, \sigma \subseteq \mu \vee \sigma \subseteq (\mu \vee \sigma)^e$, $(\mu \vee \sigma)^e$ is an upper bound of $\mu$ and $\sigma$. Let $\beta$ be any $e$-fuzzy filter of $L$ such that $\mu \subseteq \beta$ and $\sigma \subseteq \beta$.

$$\begin{aligned}(\mu \vee \sigma)^e(x) &= Sup\{\mu(a) \wedge \mu(b) : (x)^\circ \leq (a \wedge b)^\circ \,; a, b \in L\} \\ &\leq Sup\{\beta(a) \wedge \beta(b) : (x)^\circ \leq (a \wedge b)^\circ,\ a, b \in L\} \\ &= Sup\{\beta(a \wedge b) : (x)^\circ \leq (a \wedge b)^\circ,\ a, b \in L\} \\ &= \beta^e(x) = \beta(x)\end{aligned}$$

Hence $(\mu \vee \sigma)^e = sup\{\mu, \sigma\}$. Thus $(\mathcal{FF}^e(L), \subseteq)$ is a lattice. Since $\chi_{\{D\}}$ and $\chi_L$ are the smallest and the greatest $e$-fuzzy filters of $\mathcal{FF}^e(L)$, $(\mathcal{FF}^e(L), \cap, \sqcup, \chi_{\{D\}}, \chi_L)$ is a bounded lattice. By Corollary 3.8 any subfamily of $e$-fuzzy filters of $\mathcal{FF}^e(L)$ has infimum in $\mathcal{FF}^e(L)$ and $\mathcal{FF}^e(L)$ has greatest element. Hence $(\mathcal{FF}^e(L), \cap, \sqcup, \chi_{\{D\}}, \chi_L)$ is a complete bounded lattice. For any $\mu, \sigma$, and $\theta \in \mathcal{FF}^e(L)$, we obtain $(\mu \sqcup \sigma) \cap (\mu \sqcup \theta) = (\mu \vee \sigma)^e \cap (\mu \vee \theta)^e = ((\mu \vee \sigma) \cap (\mu \vee \theta))^e = (\mu \vee (\sigma \cap \theta))^e = \mu \sqcup (\sigma \cap \theta)$. Therefore $(\mathcal{FF}^e(L), \cap, \sqcup, \chi_{\{D\}}, \chi_L)$ is a bounded and complete distributive lattice. □

## 4. Prime $e$-Fuzzy Filters and Maximal $e$-fuzzy Filters of MS-algebras

In this section, we introduce prime $e$-fuzzy filters and maximal $e$-fuzzy filters of MS-algebras and we discuss some properties of them.

Definition 4.1. A proper $e$-fuzzy filter $\mu$ in MS-algebra $L$ is called a prime $e$-fuzzy filter if for any fuzzy filters $\lambda$ and $\nu$ of $L$, $\lambda \cap \nu \subseteq \mu \Rightarrow \lambda \subseteq \mu$ or $\nu \subseteq \mu$.

Theorem 4.2. *A proper filter $F$ is a prime $e$-filter of $L$ and $\alpha \in [0, 1)$ if and only if the fuzzy subset given by*

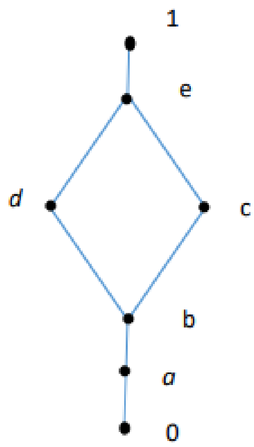
$$F^1_\alpha(x) = \begin{cases} 1 & if\ x \in F \\ \alpha & if\ \ x \notin F \end{cases}$$

*is a prime $e$-fuzzy filter of $L$.*

*Proof.* Suppose that a proper filter $F$ of $L$ is a prime $e$-filter of $L$ and $\alpha \in [0, 1)$. Clearly $F^1_\alpha$ is a proper fuzzy filter of $L$. Since $(F^1_\alpha)_1 = F$ and $(F^1_\alpha)_\alpha = L$ are $e$-filters of $L$. This implies by Theorem 3.4, $F^1_\alpha$ is a proper $e$-fuzzy filter of $L$. Now we prove that $F^1_\alpha$ is a prime $e$-fuzzy filter.

Let $\nu$ and $\theta$ be any fuzzy filters of $L$ such that $\nu \cap \theta \subseteq F^1_\alpha$. Suppose if possible that $\nu \nsubseteq F^1_\alpha$ and $\theta \nsubseteq F^1_\alpha$. Then there exist $x, y \in L$ such that $\nu(x) > F^1_\alpha(x)$ and $\theta(y) > F^1_\alpha(y)$. This indicates $F^1_\alpha(x) = F^1_\alpha(y) = \alpha$ and so $x \notin F$ and $y \notin F$. Since $F$ is prime, $x \vee y \notin F$ and so $F^1_\alpha(x \vee y) = \alpha$. Now, $(\nu \cap \theta)(x \vee y) = \nu(x \vee y) \wedge \theta(x \vee y) \geq \nu(x) \wedge \theta(y) > \alpha \wedge \alpha = \alpha = F^1_\alpha(x \vee y)$, which is a contradiction to our assumption $\nu \cap \theta \subseteq F^1_\alpha$. Hence $F^1_\alpha$ is a prime $e$-fuzzy filter. Conversely, suppose that $F^1_\alpha$ is a prime $e$-fuzzy filter. Clearly $F^1_\alpha$ is an $e$-fuzzy filter and $(F^1_\alpha)_1 = F$. Hence $F$ is an $e$-filter of $L$. Let $A$ and $B$ be any filters of $L$ such that $A \cap B \subseteq F$. Then $(A \cap B)^1_\alpha = A^1_\alpha \cap B^1_\alpha \subseteq F^1_\alpha$. Since $F^1_\alpha$ is prime, $A^1_\alpha \subseteq F^1_\alpha$ or $A^1_\alpha \subseteq F^1_\alpha$. This implies $B \subseteq F$ or $A \subseteq F$. Hence $F$ is a prime $e$-filter of $L$. □

Example 4.3. Let us consider an MS-algebra $L$ described in the diagram 2




| | 0 | a | b | c | d | e | 1 |
|---|---|---|---|---|---|---|---|
| ° | 1 | e | e | c | d | b | 0 |

diagram 2

In diagram 2, $A = \{1\}$, $B = \{1, e\}$, $C = \{1, e, c\}$, $D = \{1, e, d\}$, $E = \{1, e, d, c, b\}$, $F = \{1, e, d, c, b, a\}$ are filters of $L$ and all except $B$ are prime filters of $L$ and also $A, C$ and $F$ are prime $e$-filters of $L$.

In addition to this, it can be easily verified that $A^1_\alpha$,$C^1_\alpha$, and $F^1_\alpha$ are prime $e$-fuzzy filters of $L$.

Corollary 4.4. *A proper $e$-filter $F$ of $L$ is a prime if and only if $\chi_F$ is a prime $e$-fuzzy filter of $L$.*

THEOREM 4.5. *A proper e-fuzzy filter $\mu$ of $L$ is a prime e-fuzzy filter if and only if $Img(\mu) = \{1, \alpha\}$, where $\alpha \in [0, 1)$ and the set $\mu_* = \{x \in L : \mu(x) = 1\}$ is a prime e-filter of $L$.*

*Proof.* The converse part of this theorem follows from Lemma 4.2. Suppose that $\mu$ is a prime $e$-fuzzy filter. Clearly $1 \in Im(\mu)$ and since $\mu$ is proper, there is $x \in L$ such that $\mu(x) < 1$. We prove that $\mu(x) = \mu(y)$ for all $x, y \in L - \mu_*$. Suppose that $\mu(x) \neq \mu(y)$ for some $x, y \in L - \mu_*$. Without loss of generality we can assume that $\mu(y) < \mu(x) < 1$. Define fuzzy subsets $\theta$ and $\lambda$ as follows:

$$\theta(z) = \begin{cases} 1 & if\ z \in [x) \\ 0 & \text{otherwise.} \end{cases}$$

and

$$\lambda(z) = \begin{cases} 1 & if\ z \in \mu_* \\ \mu(x) & \text{otherwise.} \end{cases}$$

for all $z \in L$. Then it can be easily verified that both $\theta$ and $\lambda$ are fuzzy filters of $L$. Let $z \in L$. If $z \in \mu_*$, then $(\theta \cap \lambda)(z) \leq 1 = \mu(z)$. If $z \in [x) - \mu_*$, then $z = x \vee z$, and we have $(\theta \cap \lambda)(z) = \theta(z) \wedge \lambda(z) = 1 \wedge \mu(x) = \mu(x) \leq \mu(z)$.
Also if $z \notin [x)$, then $\theta(z) = 0$, so that $(\theta \cap \lambda)(z) = 0 \leq \mu(z)$. Therefore for all $x \in L$, $(\theta \cap \lambda)(x) \subseteq \mu(x)$. But we have $\theta(x) = 1 > \mu(x)$ and $\lambda(y) = \mu(x) > \mu(y)$. This implies $\lambda \nsubseteq \mu$ and $\theta \nsubseteq \lambda$, which is a contradiction. Thus $\mu(x) = \mu(y)$ for all $x, y \in L - \mu_*$ and hence $Im(\mu) = \{1, \alpha\}$ for some $\alpha \in [0, 1)$. Let $P = \{x \in L : \mu(x) = 1\}$. Since $\mu$ is proper, we get that $P$ is a proper $e$-filter of $L$ such that

$$\mu(z) = \begin{cases} 1 & if\ z \in P \\ \alpha & if\ z \notin P. \end{cases}$$

for $\alpha \neq 1$. Hence by Lemma 4.2, $P = \mu_*$. □

THEOREM 4.6. *If $\mu$ is a prime e-filter of $L$, then $\mu(x \vee y) = \mu(x)$ or $\mu(x \vee y) = \mu(y)$ for all $x, y \in L$.*

*Proof.* Suppose that $\mu$ is a prime $e$-filter of $L$, then there exists a prime $e$-filter $F$ of $L$ and $\alpha \in [0, 1)$ such that

$$\mu(x) = \begin{cases} 1 & if\ x \in F \\ \alpha & if\ \ x \notin F \end{cases}$$

for all $x \in L$. If $x, y \in F$, then $x \vee y \in F$ and so $1 = \mu(x) = \mu(y) = \mu(x \vee y)$. If $x \in F$ and $y \notin F$, then $x \vee y \in F$ and so $1 = \mu(x) = \mu(x \vee y)$. If $x \notin F$ and $y \notin F$, then $x \vee y \notin F$ and so $\alpha = \mu(x) = \mu(y) = \mu(x \vee y)$. Hence the Theorem holds. □

Definition 4.7. A proper fuzzy filter $\mu$ in MS-algebra $L$ is called a maximal fuzzy filter if $Img(\mu) = \{1, \alpha\}$, where $\alpha \in [0, 1)$. and the set $\mu_*$ is a maximal filter of $L$.

Definition 4.8. A proper $e$-fuzzy filter $\mu$ in MS-algebra $L$ is called a maximal $e$-fuzzy filter if $Img(\mu) = \{1, \alpha\}$, where $\alpha \in [0, 1)$. and the set $\mu_*$ is a maximal $e$-filter of $L$.

Corollary 4.9. *Any maximal e-fuzzy filter of L is a prime e-fuzzy filter.*

*Proof.* Let $\mu$ be a maximal $e$-fuzzy filter of $L$. Then $Im\mu = \{1, \alpha\}$, and $\mu_*$ is a maximal $e$-filter of $L$. Since every maximal $e$-filter of $L$ is a prime $e$-filter of $L$. This implies $\mu_*$ is a prime $e$-filter of $L$. Hence $\mu$ is a prime $e$-filter of $L$. But the converse is not true, since in the Example 4.3, $A^1_\alpha$, $C^1_\alpha$ are prime $e$-fuzzy filters of $L$ but not maximal $e$-fuzzy filters of $L$. □

Theorem 4.10. *Every maximal fuzzy filter of an MS-algebra is an e-fuzzy filter.*

Corollary 4.11. *Every maximal fuzzy filter of an MS-algebra is prime e-fuzzy filter.*

Theorem 4.12. *If $\mu$ is minimal in the class of all prime fuzzy filters L containing a given e-fuzzy filter, then $\mu$ is an e-fuzzy filter of L.*

*Proof.* Suppose that $\mu$ is minimal in the class of all prime fuzzy filters containing an $e$-fuzzy filter $\theta$ of $L$. We prove that $\mu$ is an $e$-fuzzy filter. Since $\mu$ is a prime fuzzy filter of $L$, there exists a prime filter $P$ of $L$ such

$$\mu(z) = \begin{cases} 1 & if\ z \in P \\ \alpha & \text{otherwise.} \end{cases}$$

for some $\alpha \in [0, 1)$. Suppose that $\mu$ is not an $e$-fuzzy filter of $L$, then there exist $x, y \in L$, $x^\circ = y^\circ$ such that $\mu(x) \neq \mu(y)$. Without loss of generality, assume $\mu(x) = 1$ and $\mu(y) = \alpha$. Consider a fuzzy ideal $\phi$ of $L$ defined by

$$\phi(z) = \begin{cases} 1 & if\ z \in (L - P) \vee (x \vee y] \\ \alpha & \text{otherwise.} \end{cases}$$

Then $\theta \cap \phi \leq \alpha$. Otherwise there exists $a \in L$ such that $\phi(a) = 1$ and $\theta(a) > \alpha$. This implies $a \in (L - P) \vee (x \vee y]$.

$$\Longrightarrow \quad a = r \vee s \text{ for some } \ r \in L - P \text{ and } \ s \in (x \vee y]$$
$$\Longrightarrow \quad a = r \vee s = r \vee (s \wedge (x \vee y)) = (r \vee s) \wedge (r \vee x \vee y) \leq r \vee x \vee y$$

As $x^\circ = y^\circ$ implies $(r \vee x \vee y)^\circ = (r \vee y)^\circ$. Since $\theta$ is an $e$-fuzzy filter of $L$, $\alpha < \theta(a) = \theta(r \vee s) \leq \theta(r \vee x \vee y) = \theta(r \vee y) \leq \mu(r \vee y)$. This implies $1 = \mu(r \vee y)$.

Hence $\mu(y) = 1$ or $\mu(r) = 1$ , which is a contradiction. Thus $\theta \cap \phi \leq \alpha$.

This implies there exists a prime fuzzy filter $\eta$ such that $\eta \cap \phi \leq \alpha$ and $\theta \subseteq \eta$. Clearly $x \vee y \in (L - P) \vee (x \vee y]$. This implies $\phi(x \vee y) = 1$. Since $\phi \cap \eta \leq \alpha$, $\eta(x \vee y) \leq \alpha < \mu(x \vee y) = 1$. This implies $\mu \nsubseteq \eta$. This indicates $\mu$ is not minimal in the class of all prime fuzzy filters containing a given $e$-fuzzy filter, which is a contradiction. Therefore, $\mu$ is an $e$-fuzzy filter. □

THEOREM 4.13. *Let $L$ be an Ms-algebra. Then the following conditions are equivalent.*

(1) *$L$ is a de Morgan algebra,*
(2) *For all $x, y \in L$, $x^\circ = y^\circ$ implies $x = y$,*
(3) *Every fuzzy filter is an $e$-fuzzy filter,*
(4) *Every prime fuzzy filter is an $e$-fuzzy filter.*

*Proof.* The proof of (1) ⇒ (2), (2) ⇒ (3), and (3) ⇒ (4) are straightforward. Now prove that (4) ⇒ (1). Suppose that $x \neq x^{\circ\circ}$ for $x \in L$. ThIs implies $x < x^{\circ\circ}$. We have $(x] \cap [x^{\circ\circ}) = \emptyset$. We know that $\chi_{(x]}$ and $\chi_{[x^{\circ\circ})}$ are fuzzy ideal and fuzzy filter of L respectively such that $\chi_{(x]} \cap \chi_{[x^{\circ\circ})} = \chi_\emptyset$ (the constant fuzzy subset attaining, value 0), there exists a prime fuzzy filter $\theta$ of L such that $\chi_{[x^{\circ\circ})} \subseteq \theta$ and $\chi_{(x]} \cap \theta = \chi_\emptyset$. Since $\chi_{[x^{\circ\circ})} \subseteq \theta$, we get $\theta(x) = 1$. Also $\chi_{(x]}(x) \wedge \theta(x) = 0$. This implies $\theta(x) = 0$, which is a contradiction $\theta$ is an $e$-filter. Hence $x = x^{\circ\circ}$ and so $L$ is a de Morgan algebra. □

THEOREM 4.14. *Let $\mu$ be a prime fuzzy filter of an MS-algebra $L$, and $\mu(0) = 0$. Then a fuzzy subset $\ell(\mu)$ of $L$ defined as $\ell(\mu)(x) = \mu^{'}(x^{\circ})\ \forall x \in L$ is an $e$-fuzzy filter of $L$.*

*Proof.* $\ell(\mu)(1) = \mu^{'}(1^{\circ}) = 1 - \mu(1^{\circ}) = 1 - \mu(0) = 1$.

$$\begin{aligned}\ell(\mu)(x \wedge y) = \mu^{'}((x \wedge y)^{\circ}) &= 1 - \mu(x^{\circ} \vee y^{\circ}) \\ &= (1 - \mu(x^{\circ})) \wedge (1 - \mu(y^{\circ})) \\ &= \mu^{'}(x^{\circ}) \wedge \mu^{'}(y^{\circ}) = \ell(\mu)(x) \wedge \ell(\mu)(y)\end{aligned}$$

This implies $\ell(\mu)$ is a fuzzy filter of $L$. Next we prove that $\ell(\mu)$ is an $e$-fuzzy filter. Since $x \leq x^{\circ\circ}$, $x^{\circ\circ} = x \vee x^{\circ\circ}$,

$$\ell(\mu)(x^{\circ\circ}) = \ell(\mu)(x \vee x^{\circ\circ}) = \mu^{'}((x \vee x^{\circ\circ})^{\circ}) = \mu^{'}(x^{\circ} \wedge x^{\circ\circ\circ}) = \mu^{'}(x^{\circ}) = \ell(\mu)(x).$$

This implies $\ell(\mu)$ is an $e$-fuzzy filter of $L$ by Theorem 3.9 □

COROLLARY 4.15. *Let $\mu$ be a maximal fuzzy filter of an MS-algebra $L$ and $\mu(0) = 0$. Then $\ell(\mu)$ is an $e$-fuzzy filter of $L$.*

## 5. The space of prime $e$-fuzzy filters

In this section, we discuss some properties of prime $e$-fuzzy filters of an MS-algebra and topological properties of the collection of all prime $e$-fuzzy filters of an MS-algebra.

THEOREM 5.1. *Let $\alpha \in [0, 1)$, $\mu$ be an $e$-fuzzy filter and $\sigma$ be a fuzzy ideal of an MS-algebra $L$ such that $\mu \cap \sigma \leq \alpha$. Then there exists a prime $e$-fuzzy filter $\beta$ such that $\mu \subseteq \beta$ and $\beta \cap \sigma \leq \alpha$.*

*Proof.* Put $\xi = \{\theta \in \mathcal{FF}^{e}(L) : \mu \subseteq \theta \text{ and } \theta \cap \sigma \leq \alpha\}$. Clearly $(\xi, \subseteq)$ is a poset. Let $Q = \{\mu_i : i \in \Omega\}$ be a chain in $\xi$. We prove that $\cup_{i\in\Omega}\mu_i \in \xi$. Clearly $(\cup_{i\in\Omega}\mu_i)(1) = 1$. For any $x, y \in L$,

$$\begin{aligned}(\cup_{i\in\Omega}\mu_i)(x) \wedge (\cup_{i\in\Omega}\mu_i)(y) &= \sup\{\mu_i(x) : i \in \Omega\} \wedge \sup\{\mu_j(y) : j \in \Omega\} \\ &= \sup\{\mu_i(x) \wedge \mu_j(y) : i, j \in \Omega\} \\ &\leq \sup\{(\mu_i \cup \mu_j)(x) \wedge (\mu_i \cup \mu_j)(y) : i, j \in \Omega\}\end{aligned}$$

Since $Q$ is a chain, $\mu_i \subseteq \mu_j$ or $\mu_j \subseteq \mu_i$. Without loss of generality, assume $\mu_j \subseteq \mu_i$. This implies $\mu_i \cup \mu_j = \mu_i$. This shows,

$$\begin{aligned}(\cup_{i\in\Omega}\mu_i)(x) \wedge (\cup_{i\in\Omega}\mu_i)(y) &\leq \sup\{\mu_i(x) \wedge \mu_i(y),\ i \in \Omega\}\\ &= \sup\{\mu_i(x \wedge y),\ i \in \Omega\}\\ &= (\cup_{i\in\Omega}\mu_i)(x \wedge y)\end{aligned}$$

Again $(\cup_{i\in\Omega}\mu_i)(x) = \sup\{\mu_i(x) : i \in \Omega\} \leq \sup\{\mu_i(x \vee y) : i \in \Omega\} = (\cup_{i\in\Omega}\mu_i)(x \vee y)$. Similarly $(\cup_{i\in\Omega}\mu_i)(y) \leq (\cup_{i\in\Omega}\mu_i)(x \vee y)$. This implies $(\cup_{i\in\Omega}\mu_i)(x) \vee (\cup_{i\in\Omega}\mu_i)(y) \leq (\cup_{i\in\Omega}\mu_i)(x \vee y)$. Hence $\cup_{i\in\Omega}\mu_i$ is a fuzzy filter of $L$. Now prove that $(\cup_{i\in\Omega}\mu_i)$ is $e$-fuzzy filter.

$$\begin{aligned}(\cup_{i\in\Omega}\mu_i)^e(x) &= \sup\{(\cup_{i\in\Omega}\mu_i)(a) : x^\circ \leq a^\circ,\ a \in L\}\\ &= \sup\{\sup\{\mu_i(a) : i \in \Omega\} : x^\circ \leq a^\circ,\ a \in L\}\\ &= \sup\{\sup\{\mu_i(a) : x^\circ \leq a^\circ,\ a \in L\} : i \in \Omega\}\\ &= \sup\{\mu_i^e(x), i \in \Omega\} = \sup\{\mu_i(x), i \in \Omega\}\\ &= (\cup_{i\in\Omega}\mu_i)(x)\end{aligned}$$

Thus $\cup_{i\in\Omega}\mu_i$ is an $e$-fuzzy filter of $L$. Since $\mu_i \cap \sigma \leq \alpha$ for each $i \in \Omega$,

$$\begin{aligned}((\cup_{i\in\Omega}\mu_i) \cap \sigma)(x) &= (\cup_{i\in\Omega}\mu_i)(x) \wedge \sigma(x)\\ &= \sup\{\mu_i(x),\ i \in \Omega\} \wedge \sigma(x)\\ &= \sup\{\mu_i(x) \wedge \sigma(x),\ i \in \Omega\}\\ &= \sup\{(\mu_i \cap \sigma)(x),\ i \in \Omega\} \leq \alpha\end{aligned}$$

Thus $(\cup_{i\in\Omega}\mu_i) \cap \theta) \leq \alpha$. Hence $\cup_{i\in\Omega}\mu_i \in \xi$. By applying Zorn's Lemma, $\xi$ has a maximal element, say $\delta$, i.e, $\delta$ is an $e$-fuzzy filter of $L$ such that $\mu \subseteq \delta$ and $\delta \cap \sigma \leq \alpha$. Next we show that $\delta$ is a prime $e$-fuzzy filter of $L$. Assume that $\delta$ is not a prime $e$-fuzzy filter. Let $\lambda_1,\ \lambda_2 \in FF(L)$, and $\lambda_1 \cap \lambda_2 \subseteq \delta$ such that$\lambda_1 \nsubseteq \delta$ and $\lambda_2 \nsubseteq \delta$. If we put $\delta_1 = (\lambda_1 \vee \delta)^e$ and $\delta_2 = (\lambda_2 \vee \delta)^e$, then both $\delta_1, \delta_2$ are $e$-fuzzy filters of $L$ properly containing $\delta$. Since $\delta$ is a maximal in $\xi$, we have $\delta_1, \delta_2 \notin \xi$. Thus we show that $\delta_1 \cap \sigma \nleq \alpha$ and $\delta_2 \cap \sigma \nleq \alpha$. This implies there exist $x, y \in L$ such that $(\delta_1 \cap \sigma)(x) > \alpha$ and $(\delta_2 \cap \sigma)(y) > \alpha$.

Now we have,

$$\begin{aligned}
\alpha &< (\delta_1 \cap \sigma)(x) \wedge (\delta_2 \cap \sigma)(y) \\
&= \delta_1(x) \wedge \delta_2(y) \wedge \sigma(x) \wedge \sigma(y) \\
&\leq \delta_1(x \vee y) \wedge \delta_2(x \vee y) \wedge \sigma(x \vee y) \\
&= (\delta_1 \cap \sigma)(x \vee y) \wedge (\delta_2 \cap \sigma)(x \vee y) \\
&= (((\delta_1 \cap \sigma) \cap (\delta_2 \cap \sigma))(x \vee y) \\
&= (((\delta_2 \cap \delta_2) \cap \sigma)(x \vee y) \\
&= (((\lambda_1 \vee \delta)^e \cap (\lambda_2 \vee \delta)^e) \cap \sigma)(x \vee y) \\
&= ((\lambda_1 \cap \lambda_2) \vee \delta)^e \cap \sigma)(x \vee y) \\
&= (\delta^e \cap \sigma)(x \vee y) \\
&= (\delta \cap \sigma)(x \vee y).
\end{aligned}$$

Which is a contradiction $\delta \cap \sigma \leq \alpha$. This implies $\delta$ is a prime $e$-fuzzy filter of $L$. □

Corollary 5.2. *Let $\mu$ be an e-fuzzy filter and $\sigma$ be a fuzzy ideal of an MS-algebra $L$ such that $\mu \cap \sigma = 0$. Then there exists a prime e-fuzzy filter $\beta$ such that $\mu \subseteq \beta$ and $\beta \cap \sigma = 0$.*

Corollary 5.3. *Let $\alpha \in [0, 1)$, $\mu$ be an e-fuzzy filter of an MS-algebra $L$ and $\mu(x) \leq \alpha$. Then there exists a prime e-fuzzy filter $\theta$ of $L$ such that $\mu \subseteq \theta$ and $\theta(x) \leq \alpha$.*

*Proof.* Put $\xi = \{\theta \in \mathcal{FF}^e(L) : \mu \subseteq \theta \textit{ and } \theta(x) \leq \alpha\}$. Clearly $(\xi, \subseteq)$ is a poset. Let $Q = \{\mu_i : i \in \Omega\}$ be a chain in $\xi$. We prove that $\cup_{i\in\Omega}\mu_i \in \xi$. By Theorem 5.1, $(\cup_{i\in\Omega}\mu_i)$ is an $e$-fuzzy filter of $L$. Since $\mu_i \subseteq \theta$ for each $i \in \Omega$ and $\theta(x) \leq \alpha$, $(\cup_{i\in\Omega}\mu_i)(x) = \sup\{\mu_i(x),\ i \in \Omega\} \leq \theta(x) \leq \alpha$.
Hence $\cup_{i\in\Omega}\mu_i \in \xi$. By applying Zorn's Lemma, $\xi$ has a maximal element, say $\delta$, i.e, $\delta$ is an $e$-fuzzy filter of $L$ such that $\mu \subseteq \delta$ and $\delta(x) \leq \alpha$. Next we show that $\delta$ is a prime $e$-fuzzy filter of $L$. Assume that $\delta$ is not a prime $e$-fuzzy filter. Let $\lambda_1,\ \lambda_2 \in FF(L)$, and $\lambda_1 \cap \lambda_2 \subseteq \delta$ such that $\lambda_1 \nsubseteq \delta$ and $\lambda_2 \nsubseteq \delta$. If we put $\delta_1 = (\lambda_1 \vee \delta)^e$ and $\delta_2 = (\lambda_2 \vee \delta)^e$, then both $\delta_1, \delta_2$ are $e$-fuzzy filters of $L$ properly containing $\delta$. Since $\delta$ is a maximal in $\xi$, we get $\delta_1, \delta_2 \notin \xi$. This implies $\delta_1(x) > \alpha$ and $\delta_2(x) > \alpha$. We have

$\delta_1(x) \wedge \delta_2(x) \geq (\delta_1 \cap \delta_2)(x) > \alpha$. Which implies

$$\begin{aligned} \alpha &\leq \delta_1(x) \wedge \delta_2(x) \\ &= ((\lambda_1 \vee \delta)^e \cap (\lambda_2 \vee \delta)^e)(x) \\ &= ((\lambda_1 \cap \lambda_2) \vee \delta)^e(x) \\ &= \delta^e(x), \text{ becouse } \lambda_1 \subseteq \delta \textit{ and } \lambda_2 \subseteq \delta \\ &= \delta(x). \end{aligned}$$

Which is a contradiction $\delta(x) \leq \alpha$. Thus $\delta$ is a prime $e$-fuzzy filter of $L$. □

COROLLARY 5.4. *For any e-fuzzy filter $\mu$ of an MS-algebra $L$, we have $\mu = \cap\{\sigma : \sigma$ is a prime e-fuzzy filter of L, $\mu \subseteq \sigma\}$.*

*Proof.* Let $\mu$ be any $e$-fuzzy filter of $L$. Put $\eta = \cap\{\theta : \theta$ is a prime $e$-fuzzy filter such that $\mu \subseteq \theta\}$. Now, we prove that $\mu = \eta$. Clearly $\mu \subseteq \eta$. Suppose that $\eta(a) > \mu(a)$ for some $a \in L$. Put $\alpha = \mu(a)$. This implies $\mu \subseteq \mu$ and $\mu(a) \leq \alpha$. Thus by the Corollary 5.3, there exists a prime $e$-fuzzy filter $\delta$ such that $\mu \subseteq \delta$ and $\delta(a) \leq \alpha$. Since $\eta \subseteq \delta$, $\eta(a) \leq \alpha$. Which is a contradiction for $\eta(a) > \alpha$. Hence $\eta \subseteq \mu$. Hence $\mu = \eta$. This implies every proper $e$-fuzzy filter of $L$ is the intersection of all prime $e$-fuzzy filters containing it. □

COROLLARY 5.5. *Let $L$ be an MS-algebra. Then the intersection of all prime e-fuzzy filters of $L$ is equal to $\chi_D$.*

Let $L$ be an MS-algebra and $X^e$ denotes the set of all prime $e$-fuzzy filters of $L$. For a fuzzy subset $\theta$ of $L$, define $H^e(\theta) = \{\mu \in X^e : \theta \subseteq \mu\}$, and $X^e(\theta) = \{\mu \in X^e : \theta \nsubseteq \mu\}$.

LEMMA 5.6. *For any fuzzy filters $\lambda$ and $\nu$ of $L$, we have*

*1. $\lambda \subseteq \nu \Rightarrow X^e(\lambda) \subseteq X^e(\nu)$,*
*2. $X^e(\lambda \vee \nu) = X^e(\lambda) \cup X^e(\nu)$,*
*3. $X^e(\lambda \cap \nu) = X^e(\lambda) \cap X^e(\nu)$*

*Proof.* (1) Let $\mu \in X^e(\lambda)$. Then $\lambda \nsubseteq \mu$ and so $\nu \nsubseteq \mu$. Thus $\mu \in X^e(\nu)$. Hence $X^e(\lambda) \subseteq X^e(\nu)$.

(2) By (1) $X^e(\lambda) \subseteq X^e(\lambda \vee \nu)$ and $X^e(\nu) \subseteq X^e(\lambda \vee \nu)$. We have $X^e(\nu) \cup X^e(\lambda) \subseteq X^e(\lambda \vee \nu)$. Conversely, If $\mu \in X^e(\lambda \vee \nu)$, then $\lambda \vee \nu \nsubseteq \mu$. Since $\mu$ is a prime $e$-fuzzy filter, $\lambda \nsubseteq \mu$ or $\nu \nsubseteq \mu$, and so $\mu \in X^e(\lambda)$ or $\mu \in X^e(\nu)$. Hence $\mu \in X^e(\lambda) \cup X^e(\nu)$. Thus $X^e(\lambda \vee \nu) = X^e(\lambda) \cup X^e(\nu)$.

(3) Clearly $X^e(\lambda \cap \nu) \subseteq X^e(\lambda) \cap X^e(\nu)$. Again $\mu \in X^e(\lambda) \cap X^e(\nu)$, then $\lambda \nsubseteq \mu$ and $\nu \nsubseteq \mu$. Since $\mu$ is a prime $e$-fuzzy filter, we have $\lambda \cap \mu \nsubseteq \mu$. Thus $\mu \in X^e(\lambda \cap \nu)$ and so $X^e(\lambda) \cap X^e(\nu) \subseteq X^e(\lambda \cap \nu)$. Hence $X^e(\lambda) \cap X^e(\nu) = X^e(\lambda \cap \nu)$. □

LEMMA 5.7. *Let $\lambda$ be a fuzzy subset of $L$. Then $X^e(\lambda) = X^e([\lambda))$.*

*Proof.* Since $\lambda \subseteq [\lambda)$, $X^e(\lambda) \subseteq X^e([\lambda))$. Let $\mu \in X^e([\lambda))$, Then $[\lambda) \nsubseteq \mu$. This implies $\lambda \nsubseteq \mu$. Otherwise, if $\lambda \subseteq \mu$, then $[\lambda) \subseteq \mu$. Which is impossible. So that $\mu \in X^e(\lambda)$ and so $X^e(\lambda) = X^e([\lambda))$. □

LEMMA 5.8. *Let $x, y \in L$, and $\alpha \in (0, 1]$. Then*

*(1)* $\cup_{x\in L,\, \alpha\in(0,1]} X^e(x_\alpha) = X^e$,
*(2)* $X^e(x_\alpha) \cap X^e(y_\alpha) = X^e((x \vee y)_\alpha)$,
*(3)* $X^e(x_\alpha) \cup X^e(y_\alpha) = X^e((x \wedge y)_\alpha)$,
*(4)* $X^e(x_\alpha) = \emptyset \Leftrightarrow x \in D$,

*Proof.* (1) Clearly $\cup_{x\in L,\, \alpha\in(0,1]} X^e(x_\alpha) \subseteq X^e$. Let $\mu \in X^e$. Then $Im\mu = \{1, r\}$, $r \in [0, 1)$. This implies there is $x \in L$ such that $\mu(x) = r$. Let us take some $\alpha \in (0, 1]$ such that $\alpha > r$. This implies $\mu \in X^e(x_\alpha)$, and so $\mu \in \cup_{x\in L,\, \alpha\in(0,1]} X^e(x_\alpha)$. Thus $X^e \subseteq \cup_{x\in L,\, \alpha\in(0,1]} X^e(x_\alpha)$. Hence $X^e = \cup_{x\in L,\, \alpha\in(0,1]} X^e(x_\alpha)$.

(2) Let,

$$
\begin{aligned}
\mu \in X^e(x_\alpha) \cap X^e(y_\alpha) &\Rightarrow \mu \in X^e(x_\alpha) \textit{ and } \mu \in X^e(y_\alpha) \\
&\Rightarrow x_\alpha \nsubseteq \mu \textit{ and } y_\alpha \nsubseteq \mu \\
&\Rightarrow \alpha > \mu(x) \textit{ and } \alpha > \mu(y) \\
&\Rightarrow \alpha > \mu(x) \vee \mu(y) = \mu(x \vee y) \\
&\Rightarrow (x \vee y)_\alpha \nsubseteq \mu \\
&\Rightarrow \mu \in X^e((x \vee y)_\alpha \\
&\Rightarrow X^e(x_\alpha) \cap X^e(y_\alpha) \subseteq X^e(((x \vee y)_\alpha))
\end{aligned}
$$

Conversely, let

$$\begin{aligned}
\mu \in X^e((x \vee y)_\alpha) &\Rightarrow (x \vee y)_\alpha \nsubseteq \mu \\
&\Rightarrow \alpha > \mu(x \vee y) = \mu(x) \vee \mu(y) \textit{ as } \mu \textit{ is prime} \\
&\Rightarrow \alpha > \mu(x) \textit{ and } \alpha > \mu(y) \\
&\Rightarrow x_\alpha \nsubseteq \mu \textit{ and } y_\alpha \nsubseteq \mu \\
&\Rightarrow \mu \in X^e(x_\alpha) \textit{ and } \mu \in X^e(y_\alpha) \\
&\Rightarrow \mu \in X^e(x_\alpha) \cap X^e(y_\alpha) \\
&\Rightarrow X^e((x \vee y)_\alpha) \subseteq X^e(x_\alpha) \cap X^e(y_\alpha)
\end{aligned}$$

Hence $X^e(x_\alpha) \cap X^e(y_\alpha) = X^e((x \vee y)_\alpha)$.

(3) The prove is similar to (2).

(4)

$$\begin{aligned}
X^e(x_\alpha) = \emptyset &\Leftrightarrow x_\alpha \subseteq \mu \;\forall\; \mu \in X^e \\
&\Leftrightarrow x_\alpha \subseteq \cap_{\mu \in X^e} \mu = \chi_D \\
&\Leftrightarrow \chi_D(x) = 1 \\
&\Leftrightarrow x \in D.
\end{aligned}$$

□

LEMMA 5.9. *Let* $\alpha_1, \alpha_2 \in (0, 1]$, $\alpha = min\{\alpha_1, \alpha_2\}$ *and any* $x, y \in L$. *Then* $X^e(x_{\alpha_1}) \cap X^e(y_{\alpha_2}) = X^e((x \vee y)_\alpha)$.

*Proof.* Let $\mu \in X^e(x_{\alpha_1}) \cap X^e(y_{\alpha_2})$. Then $x_{\alpha_1} \nsubseteq \mu$ and $y_{\alpha_2} \nsubseteq \mu$. This implies $\alpha_1 > \mu(x)$ and $\alpha_2 > \mu(y)$. Since $\mu_*$ is a prime filter of $L$ and $x, y \notin \mu_*$, we have $x \vee y \notin \mu_*$ and $\mu(x) = \mu(y) = \mu(x \vee y)$. This shows $\alpha = \alpha_1 \wedge \alpha_2 > \mu(x \vee y)$, Whence $(x \vee y)_\alpha \nsubseteq \mu$ and so $\mu \in X^e((x \vee y)_\alpha)$. Thus $X^e(x_{\alpha_2}) \cap X^e(y_{\alpha_2}) \subseteq X^e((x \vee y)_\alpha)$. Conversely, let $\mu \in X^e((x \vee y)_\alpha)$. Then $(x \vee y)_\alpha \nsubseteq \mu$. This implies $\alpha > \mu(x \vee y) = \mu(x) \vee \mu(y)$. This show $\alpha_1 > \mu(x)$ and $\alpha_2 > \mu(y)$ and $x_{\alpha_2} \nsubseteq \mu$ and $y_{\alpha_2} \nsubseteq \mu$. Then we have $\mu \in X^e(x_{\alpha_2}) \cap X^e(y_{\alpha_2})$. Hence $X^e(x_{\alpha_2}) \cap X^e(y_{\alpha_2}) = X^e((x \vee y)_\alpha)$. □

LEMMA 5.10. *The collection* $\mathcal{T} = \{X^e(\theta) : \theta$ *is a fuzzy filter of* $L\}$ *is a topology on* $X^e$.

*Proof.* Consider the fuzzy subsets $\lambda_1, \lambda_2$ of $L$ defined as : $\lambda_1(x) = 0$ and $\lambda_2(x) = 1$ for all $x \in L$. Clearly $[\lambda_1)$ and $\lambda_2$ are fuzzy filters of $L$. $[\lambda_1) \subseteq \mu$ for all $\mu \in X^e$. Thus $X^e([\lambda_1)) = \emptyset$. Since each $\mu \in X^e$ is non-constant, $\lambda_2 \nsubseteq \mu$ for all $\mu \in X^e$. Thus $X^e(\lambda_2) = X^e$. This implies $\emptyset, X^e \in \mathcal{T}$. Also for any fuzzy filters $\lambda_1$ and $\lambda_2$ of $L$, by Lemma

5.6(3) we have $X^e(\lambda_1) \cap X^e(\lambda_2) = X^e(\lambda_1 \cap \lambda_2)$ . This show that $\mathcal{T}$ is closed under finite intersections. Next, let $\{\lambda_i, i \in \Omega\}$ be any family of fuzzy filters of $L$. Now we prove that $\cup_{i\in\Omega} X^e(\lambda_i) = X^e([\cup_{i\in\Omega}\lambda_i))$. Let $\mu \in X^e([\cup_{i\in\Omega}\lambda_i))$, then $[\cup_{i\in\Omega}\lambda_i) \nsubseteq \mu$, which implies that $\lambda_i \nsubseteq \mu$ for some $i \in \Omega$. Otherwise if $\lambda_i \subseteq \mu$ for each $i \in \Omega$, it will be true that $[\cup_{i\in\Omega}\lambda_i) \subseteq \mu$. Thus $\mu \in \cup_{i\in\Omega} X^e(\lambda_i)$ Whence $X^e([\cup_{i\in\Omega}\lambda_i)) \subseteq \cup_{i\in\Omega} X^e(\lambda_i)$. Clearly $\cup_{i\in\Omega} X^e(\lambda_i) \subseteq X^e([\cup_{i\in\Omega}\lambda_i))$. Hence $\cup_{i\in\Omega} X^e(\lambda_i) = X^e([\cup_{i\in\Omega}\lambda_i))$. Therefore, $\mathcal{T}$ is closed under arbitrary unions and hence, it is Topology on $X^e$. □

Definition 5.11. The topological space $(X^e, \mathcal{T})$ is called the prime $e$-fuzzy filter Spectrum of $L$ and it is denoted by $F - Spac^e_F(L)$.

Theorem 5.12. *Let $\mathcal{B} = \{X^e(x_\alpha) : x \in L, \alpha \in (0, 1]\}$. Then $\mathcal{B}$ forms a base for some topology on $\tau$.*

*Proof.* Clearly by (1) and (2) from Lemma 5.8, it follow that $\mathcal{B}$ forms a base for some topology on $X^e$. □

Theorem 5.13. *The space $X^e$ is a $T_0$-space.*

*Proof.* Let $\mu, \theta \in X^e$ such that $\mu \neq \theta$. Then either $\mu \nsubseteq \theta$ or $\theta \nsubseteq \mu$. Without loss of generality, we can assume that $\mu \nsubseteq \theta$. Then $\theta \in X^e(\mu)$ and $\mu \notin X^e(\mu)$. Thus $X^e$ is a $T_0$-space. □

Theorem 5.14. *For any fuzzy filter $\mu$ of $L$, $X^e(\mu) = X^e(\mu^e)$.*

*Proof.* Clearly $\mu \subseteq \mu^e$ for any fuzzy filter $\mu$ of $L$. Then $X^e(\mu) \subseteq X^e(\mu^e)$. Conversely, let $\theta \in X^e(\mu^e)$. Then $\mu^e \nsubseteq \theta$. Suppose $\theta \notin X^e(\mu)$, then $\mu \subseteq \theta$. This implies $\mu^e \subseteq \theta^e = \theta$. Which is impossible. Thus $\theta \in X^e(\mu)$ and so $X^e(\mu^e) \subseteq X^e(\mu)$. Hence $X^e(\mu) = X^e(\mu^e)$. □

Theorem 5.15. *For any fuzzy filter $\mu$ of $L$, $X^e(\mu) = \cup_{x_\alpha \in \mu} X^e(x_\alpha)$.*

Theorem 5.16. *The lattice $\mathcal{FF}^e(L)$ is isomorphic with the lattice of all open sets $X^e$.*

*Proof.* The lattice of all open sets in $X^e$ is $(\mathcal{T}, \cap, \cup)$. Define the mapping $f : \mathcal{FF}^e(L) \longrightarrow \mathcal{T}$ by $f(\mu) = X^e(\mu)$ for all $\mu \in \mathcal{FF}^e(L)$. Let $\mu, \theta \in F^eF(L)$. Then $f(\mu \sqcup \theta) = f((\mu \vee \theta)^e) = X^e(\mu \vee \theta) = X^e(\mu) \cup X^e(\theta) = f(\mu) \cup f(\theta)$, and $f(\mu \cap \theta) = X^e(\mu \cap \theta) = X^e(\mu) \cap X^e(\theta) = f(\mu) \cap f(\theta)$. This shows $f$ is homomorphism. Since $X^e(\mu) = X^e(\mu^e)$ and $\mu^e \in \mathcal{FF}^e(L)$, $\forall X^e(\mu) \in T$, there exists $\mu^e \in \mathcal{FF}^e(L)$ such that $f(\mu^e) = X^e(\mu)$. Hence $f$ is onto. Next we prove that $f$ is one to one.

Let $f(\mu) = f(\theta)$. Suppose that $\mu \neq \theta$, then there exists $x \in L$ such that either $\mu(x) < \theta(x)$ or $\theta(x) < \mu(x)$. Without loss of generality, we can assume that $\mu(x) < \theta(x)$. Put $\theta(x) = \alpha$, then by Corollary 5.3, we can find a prime $e$-fuzzy filter $\delta$ of $L$ such that $\mu \subseteq \delta$ and $\delta(x) < \alpha$. This implies $\delta \notin X^e(\mu)$ and $\theta \nsubseteq \delta$. This show that $\delta \notin X^e(\mu)$ and $\delta \in X^e(\theta)$. This is a contradiction $f(\mu) = f(\theta)$. Thus $\mu = \theta$. Hence $f$ is an isomorphism. □

For any fuzzy subset $\theta$ of $L$, $X^e(\theta) = \{\mu \in X^e : \mu \nsubseteq \theta\}$ is open set of $X^e$ and $H^e(\theta) = X^e - X^e(\theta)$ is a closed set of $X^e$. Also every closed set in $X^e$ is the form of $H^e(\theta)$ for all fuzzy subset of $L$. Then we have the following:

Theorem 5.17. *The closure of any $A \subseteq X^e$ is given by $\overline{A} = H^e(\cap_{\mu\in A}\mu)$.*

*Proof.* Let $A \subseteq X^e$ and $\beta \in A$. Then $\cap_{\mu\in A}\mu \subseteq \beta$. Thus $\beta \in H^e(\beta) \subseteq H^e(\cap_{\mu\in A}\mu)$. Therefore, $H^e(\cap_{\mu\in A}\mu)$ is a closed set containing $A$. Let $C$ be any closed set containing $A$ in $X^e$ . Then $C = H^e(\theta)$ for some fuzzy subset of $\theta$ of $L$. Since $A \subseteq C = H^e(\theta)$, we have $\theta \subseteq \mu$ for all $\mu \in A$. Hence $\theta \subseteq \cap_{\mu\in A}\mu$ . Therefore, $H^e(\cap_{\mu\in A}\mu) \subseteq H^e(\theta) = C$. Hence $H^e(\cap_{\mu\in A}\mu)$ is the smallest closed set containing $A$. Therefore, $\overline{A} = H^e(\cap_{\mu\in A}\mu)$. □

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