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DIAGONAL SUMS IN NEGATIVE TRINOMIAL TABLE

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Abstract. We study the negative trinomial table T' of $(x^2+x+1)^{-n}$ and its t/u-slope diagonals for any $t, u > 0$. We investigate recurrence formula of the t/u-slope diagonal sums of T' and find interrelationships with t/u-slope diagonal sums of the trinomial table T.

1. introduction

The Pascal table P and the negative Pascal table P' are well known arithmetic tables of $(x+1)^{\pm n}$ respectively for $n \geq 0$. Each diagonal sum over P makes a Fibonacci number F_n, and it is not hard to see that certain diagonal sums over P' makes F_{-n} by comparing the tables P and P' ([1], [6], [7]). In fact, each diagonals and rows in P can be found as a type of diagonals in P'. As a generalization, there have been researches about the trinomial table T and the negative trinomial table T' of $(x^2+x+1)^{\pm n}$ respectively ([3], [4]).

T

0|	1					
1|	1	1	1			
2|	1	2	3	**2**	1	
3|	1	3	**6**	7	6	3
4|	1	**4**	10	16	19	16
5|	**1**	5	15	30	45	51

and

T'

1|	1	−1	0	1	−1	0
2|	1	−2	1	2	−4	2
3|	1	−3	3	2	−9	9
4|	1	−4	6	0	−15	24
5|	1	−5	10	−5	−20	49
6|	1	−6	15	−14	−21	84

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Each diagonal sum over T makes a tribonacci number ([2], [5]). However unlike P and P', interrelationships between components of T and T' may not be seen easily by only looking at the tables. For example, the marked diagonal $\{1, 4, 6, 2\}$ in T may not be appeared in any type of diagonals in T'.

In this work we investigate sequences of certain diagonal sums in T', and find their interrelationships. We consider various diagonals of any slope t/u that moves u steps in x-axis and t steps in y-axis over both T and T'. And we study sequential properties of t/u-slope diagonal sums. Throughout the work, let $P = [u_{i,j}]$ and $P' = [u'_{i,j}]$ be (negative) Pascal tables, while $T = [e_{i,j}]$ and $T' = [e'_{i,j}]$ be the (negative) trinomial tables for $i, j \geq 0$.

2. Certain slope diagonal sums of Negative trinomial table

For integers $t, u > 0$, a t/u-slope diagonal (abbr. diag.) over an arithmetic table means a diagonal that moves u steps toward x-axis and t steps toward y-axis. In particular if $u = 1$ then we simply say it a t-slope diagonal. So the 1-slope diag. is the ordinary diagonal. Over the negative trinomial table T', by $S_n^{(t/u)\uparrow}$ we mean the t/u-slope ascending diag. sum starting from $e'_{n,0}$. We also denote by $S_n^{(t/u)\downarrow}$ the t/u-slope descending diag. sum from $e'_{1,n}$. So for instance, $S_i^{(t/1)\uparrow} = e'_{i,0} + e'_{i-t,1} + e'_{i-2t,2} + \cdots$ and $S_j^{(1/t)\downarrow} = e'_{1,j} + e'_{2,j-t} + e'_{3,j-2t} + \cdots$.

Like $u_{i,j} + u_{i,j+1} = u_{i+1,j+1}$ in P, the recurrence rules over T and T'

$$e_{i,j-1} + e_{i,j} + e_{i,j+1} = e_{i+1,j+1} \text{ and } e'_{i,j+1} - e'_{i+1,j-1} - e'_{i+1,j} = e'_{i+1,j+1} \quad (\star)$$

are well known. We explore some entries in T' to get diagonal sums.

Theorem 1. *$T' = [e'_{i,j}]$ satisfies the followings.*

$$(1) \begin{cases} e'_{i,0} = e_{i,0} = 1 \\ e'_{i,1} = -e_{i,1} = -i \\ e'_{i,2} = e_{i-1,2} = \frac{(i-1)i}{2} \end{cases} \qquad (2)\ e'_{1,j} = \begin{cases} 1 & \text{if } j \equiv 0 \pmod 3 \\ -1 & \text{if } j \equiv 1 \pmod 3 \\ 0 & \text{if } j \equiv 2 \pmod 3 \end{cases}$$

So $e'_{1,j} + e'_{1,j+1} + e'_{1,j+2} = 0$ for $j \geq 0$.

Proof. Clearly $e'_{i+1,0} = 1 = e_{i+1,0}$, We notice

$e'_{3,0} = 1 = e_{3,0}$, $e'_{3,1} = -3 = -e_{3,1}$, $e'_{3,2} = 3 = e_{2,2}$,
and $e'_{4,0} = 1 = e_{4,0}$, $e'_{4,1} = -4 = -e_{4,1}$, $e'_{4,2} = 6 = e_{3,2}$.

Assume the identities (1) are true for some i. Then the recurrence rule ($\star$) of T' with induction hypothesis shows

$$\begin{aligned} e'_{i+1,1} &= e'_{i,1} - e'_{i+1,0} = -e_{i,1} - e_{i+1,0} = -e_{i+1,1} = -(i+1), \\ e'_{i+1,2} &= e'_{i,2} - e'_{i+1,0} - e'_{i+1,1} = e_{i-1,2} - e_{i+2,0} + e_{i+1,1} \\ &= e_{i-1,2} - e_{i,0} + (e_{i,0} + e_{i,1}) = e_{i-1,2} + e_{i,1} = e_{i,2}, \end{aligned}$$

and $e'_{i+1,2} = e'_{i,2} - e'_{i+1,0} - e'_{i+1,1} = \frac{(i-1)i}{2} - 1 + (i+1) = \frac{i(i+1)}{2}$.

Observe the first few entries $\{1, -1, 0, 1, -1, 0, 1, -1, 0, \cdots\}$ in the 1th row. In fact, from $e'_{1,0} = 1$ and $e'_{1,1} = -1$ in (1), we have $e'_{1,2} = e'_{0,2} - e'_{1,0} - e'_{1,1} = 0$ and $e'_{1,3} = e'_{0,3} - e'_{1,1} - e'_{1,2} = 1$. If we assume the identities (2) for $j < 3k$ ($k \in \mathbb{Z}$) then (1) implies

$$e'_{1,j} = e'_{0,j} - e'_{1,j-2} - e'_{1,j-1} = \begin{cases} 0 - (-1) - (0) = 1 & \text{if } j = 3k \\ 0 - (0) - 1 = -1 & \text{if } j = 3k+1 \\ 0 - 1 - (-1) = 0 & \text{if } j = 3k+2 \end{cases} \qquad \square$$

Let us begin to consider 1-slope diag. sums $S_j^{(1)\downarrow}$ in T'.

Theorem 2. *$S_j^{(1)\downarrow} = -S_{j-2}^{(1)\downarrow}$, so $S_{j-3}^{(1)\downarrow} - S_{j-2}^{(1)\downarrow} + S_{j-1}^{(1)\downarrow} = S_j^{(1)\downarrow}$.*

Proof. By Theorem 1 and the recurrence rule ($\star$) of T', we have

$S_0^{(1)\downarrow} = e'_{1,0} = 1$, $S_1^{(1)\downarrow} = e'_{1,1} + e'_{2,0} = -1 + 1 = 0$,

$S_2^{(1)\downarrow} = e'_{1,2} + e'_{2,1} + e'_{3,0} = -1$ and $S_3^{(1)\downarrow} = e'_{1,3} + e'_{2,2} + e'_{3,1} + e'_{4,0} = 0$,

etc. So the first few values are $\{S_j^{(1)\downarrow} \mid 0 \le j \le 7\} = \{1, 0, -1, 0, 1, 0, -1, 0\}$, where these satisfy $S_j^{(1)\downarrow} = -S_{j-2}^{(1)\downarrow}$ and $S_j^{(1)\downarrow} = S_{j-3}^{(1)\downarrow} - S_{j-2}^{(1)\downarrow} + S_{j-1}^{(1)\downarrow}$.

In general, the 1-slope descending diag. sum starting from $e'_{1,j}$ is

$$S_j^{(1)\downarrow} = e'_{1,j} + e'_{2,j-1} + \cdots + e'_{j-1,2} + e'_{j,1} + e'_{j+1,0},$$

and each component can be expressed by the recurrence ($\star$) of T' that

$$\begin{aligned} e'_{1,j} &= e'_{1,j} \\ e'_{2,j-1} &= e'_{1,j-1} - e'_{2,j-3} - e'_{2,j-2} \\ e'_{j-1,2} &= e'_{j-2,2} - e'_{j-1,0} - e'_{j-1,1} \\ e'_{j,1} &= e'_{j-1,1} \qquad\quad - e'_{j,0} \\ e'_{j+1,0} &= e'_{j,0} \end{aligned}$$

Hence by taking columnwise sum from the above table, we have

$$\begin{aligned} S_j^{(1)\downarrow} = e'_{1,j} &+ \underbrace{(e'_{1,j-1} + \cdots + e'_{j-1,1} + e'_{j,0})}_{S_{j-1}^{(1)\downarrow}} \\ &- \underbrace{(e'_{2,j-3} + \cdots + e'_{j-1,0})}_{S_{j-2}^{(1)\downarrow} - e'_{1,j-2}} - \underbrace{(e'_{2,j-2} + \cdots + e'_{j-1,1} + e'_{j,0})}_{S_{j-1}^{(1)\downarrow} - e'_{1,j-1}} \end{aligned}$$

$= (e'_{1,j} + e'_{1,j-1} + e'_{1,j-2}) + S^{(1)\downarrow}_{j-1} - S^{(1)\downarrow}_{j-2} - S^{(1)\downarrow}_{j-1}.$

But since $e'_{1,j} + e'_{1,j-1} + e'_{1,j-2} = 0$ by Theorem 1, we have
$S^{(1)\downarrow}_{j} = S^{(1)\downarrow}_{j-1} - S^{(1)\downarrow}_{j-2} - S^{(1)\downarrow}_{j-1} = -S^{(1)\downarrow}_{j-2}$, so $S^{(1)\downarrow}_{j-3} - S^{(1)\downarrow}_{j-2} + S^{(1)\downarrow}_{j-1} = S^{(1)\downarrow}_{j}$. □

THEOREM 3. *$S^{(1/2)\downarrow}_{j} = -S^{(1/2)\downarrow}_{j-1}$, so $S^{(1/2)\downarrow}_{j-3} + S^{(1/2)\downarrow}_{j-2} - S^{(1/2)\downarrow}_{j-1} = S^{(1/2)\downarrow}_{j}$.*

Proof. Each 1/2-slope descending diagonal starting from $e'_{1,j}$ ends at either 0th or 1th column according to even or odd j. So if $j = 2k + r$ $(r = 0, 1)$ then

$S^{(1/2)\downarrow}_{j} = e'_{1,j} + e'_{2,j-2} + \cdots + e'_{k,r+2} + e'_{k+1,r}.$

The first few 1/2-slope descending diag. sums $\{S^{(1/2)\downarrow}_{j} \mid 0 \le j \le 5\}$ of T' are $\{1, -1, 1, -1, 1, -1\}$, and it satisfies $S^{(1/2)\downarrow}_{j} = -S^{(1/2)\downarrow}_{j-1}$ for $j \le 5$.

Assume $S^{(1/2)\downarrow}_{j} = -S^{(1/2)\downarrow}_{j-1}$ is true for all $j < 2k$ $(k \in \mathbb{Z})$. If $j = 2k$ then

$S^{(1/2)\downarrow}_{j} = e'_{1,j} + e'_{2,j-2} + \cdots + e'_{k,2} + e'_{k+1,0}$

From the recurrence rule $(\star)$ in T', since

$$\begin{array}{llll} e'_{1,j} &= e'_{1,j} & & \\ e'_{2,j-2} &= e'_{1,j-2} & -e'_{2,j-4} & -e'_{2,j-3} \\ \cdots & & & \\ e'_{k,2} &= e'_{k-1,2} & -e'_{k,0} & -e'_{k,1} \\ e'_{k+1,0} &= e'_{k,0} & & \end{array}$$

the columnwise sum of the above table gives rise to

$$S^{(1/2)\downarrow}_{j} = e'_{1,j} + \underbrace{(e'_{1,j-2} + \cdots + e'_{k-1,2} + e'_{k,0})}_{S^{(1/2)\downarrow}_{j-2}} - \underbrace{(e'_{2,j-4} + \cdots + e'_{k,0})}_{S^{(1/2)\downarrow}_{j-2} - e'_{1,j-2}}$$
$$- \underbrace{(e'_{2,j-3} + \cdots e'_{k,1})}_{S^{(1/2)\downarrow}_{j-1} - e'_{1,j-1}} = S^{(1/2)\downarrow}_{j-2} - S^{(1/2)\downarrow}_{j-2} - S^{(1/2)\downarrow}_{j-1} = -S^{(1/2)\downarrow}_{j-1},$$

because $e'_{1,j} + e'_{1,j-1} + e'_{1,j-2} = 0$ by Theorem 1.

On the other hand, when $j = 2k + 1$, due to the following table

$$\begin{array}{llll} e'_{1,j} &= e'_{1,j} & & \\ e'_{2,j-2} &= e'_{1,j-2} & -e'_{2,j-4} & -e'_{2,j-3} \\ \cdots & & & \\ e'_{k,3} &= e'_{k-1,3} & -e'_{k,1} & -e'_{k,2} \\ e'_{k+1,1} &= e'_{k,1} & & -e'_{k+1,0} \end{array}$$

we have

$$S^{(1/2)\downarrow}_{j} = e'_{1,j} + \underbrace{(e'_{1,j-2} + \cdots + e'_{k-1,3} + e'_{k,1})}_{S^{(1/2)\downarrow}_{j-2}} - \underbrace{(e'_{2,j-4} + \cdots + e'_{k,1})}_{S^{(1/2)\downarrow}_{j-2} - e'_{1,j-2}}$$

$$-\underbrace{(e'_{2,j-3}+\cdots+e'_{k+1,0})}_{S^{(1/2)\downarrow}_{j-1}-e'_{1,j-1}}=S^{(1/2)\downarrow}_{j-2}-S^{(1/2)\downarrow}_{j-2}-S^{(1/2)\downarrow}_{j-1}=-S^{(1/2)\downarrow}_{j-1}.$$

This implies $S^{(1/2)\downarrow}_{j-3}+S^{(1/2)\downarrow}_{j-2}-S^{(1/2)\downarrow}_{j-1}=S^{(1/2)\downarrow}_{j}$. □

THEOREM 4. $S^{(1/3)\downarrow}_{j-3}-S^{(1/3)\downarrow}_{j-2}-S^{(1/3)\downarrow}_{j-1}=S^{(1/3)\downarrow}_{j}$.

Proof. Note that 1/3-slope descending diag. starting from $e'_{1,j}$ ends at 0, 1 or 2th column according to $j \pmod 3$. So when $j=3k+r$ $(r=0,1,2)$,

$S^{(1/3)\downarrow}_{j}=e'_{1,j}+e'_{2,j-3}+\cdots+e'_{k,r+3}+e'_{k+1,r}$

We easily see $\{S^{(1/3)\downarrow}_{j}\mid 0\le j\le 10\}=\{1,-1,0,2,-3,1,4,-8,5,7,-20\}$ and notice a recurrence $S^{(1/3)\downarrow}_{j-3}-S^{(1/3)\downarrow}_{j-2}-S^{(1/3)\downarrow}_{j-1}=S^{(1/3)\downarrow}_{j}$ for $0\le j\le 10$. We now assume $S^{(1/3)\downarrow}_{j-3}-S^{(1/3)\downarrow}_{j-2}-S^{(1/3)\downarrow}_{j-1}=S^{(1/3)\downarrow}_{j}$ is true for $j<3k$ $(k\in\mathbb{Z})$. If $j=3k$ then by making a table

$$\begin{aligned}
e'_{1,j} &= e'_{1,j}\\
e'_{2,j-3} &= e'_{1,j-3}-e'_{2,j-5}-e'_{2,j-4}\\
&\cdots\\
e'_{k,3} &= e'_{k-1,3}-e'_{k,1}\quad -e'_{k,2}\\
e'_{k+1,0} &= e'_{k,0}
\end{aligned}$$

we have

$$S^{(1/3)\downarrow}_{j}=e'_{1,j}+\underbrace{(e'_{1,j-3}+\cdots+e'_{k-1,3}+e'_{k,0})}_{S^{(1/3)\downarrow}_{j-3}}-\underbrace{(e'_{2,j-5}+\cdots+e'_{k,1})}_{S^{(1/3)\downarrow}_{j-2}-e'_{1,j-2}}$$
$$-\underbrace{(e'_{2,j-4}+\cdots+e'_{k,2})}_{S^{(1/3)\downarrow}_{j-1}-e'_{1,j-1}}=S^{(1/3)\downarrow}_{j-3}-S^{(1/3)\downarrow}_{j-2}-S^{(1/3)\downarrow}_{j-1}.$$

Analogously if $j=3k+1$ the with the similar table above we have

$$S^{(1/3)\downarrow}_{j}=e'_{1,j}+\underbrace{(e'_{1,j-3}+\cdots+e'_{k-1,4}+e'_{k,1})}_{S^{(1/3)\downarrow}_{j-3}}-\underbrace{(e'_{2,j-5}+\cdots+e'_{k,2})}_{S^{(1/3)\downarrow}_{j-2}-e'_{1,j-2}}$$
$$-\underbrace{(e'_{2,j-4}+\cdots+e'_{k,3}+e'_{k+1,0})}_{S^{(1/3)\downarrow}_{j-1}-e'_{1,j-1}}=S^{(1/3)\downarrow}_{j-3}-S^{(1/3)\downarrow}_{j-2}-S^{(1/3)\downarrow}_{j-1}.$$

Finally when $j=3k+2$ we also have

$$S^{(1/3)\downarrow}_{j}=e'_{1,j}+\underbrace{(e'_{1,j-3}+\cdots+e'_{k-1,5}+e'_{k,2})}_{S^{(1/3)\downarrow}_{j-3}}-\underbrace{(e'_{2,j-5}+\cdots+e'_{k,3}+e'_{k+1,0})}_{S^{(1/3)\downarrow}_{j-2}-e'_{1,j-2}}$$

$$- \underbrace{(e'_{2,j-4} + \cdots + e'_{k,4} + e'_{k+1,1})}_{S^{(1/3)\downarrow}_{j-1} - e'_{1,j-1}} = S^{(1/3)\downarrow}_{j-3} - S^{(1/3)\downarrow}_{j-2} - S^{(1/3)\downarrow}_{j-1}. \quad \square$$

Theorem 5. *$S^{(1/4)\downarrow}_{j-4} - S^{(1/4)\downarrow}_{j-2} - S^{(1/4)\downarrow}_{i-1} = S^{(1/4)\downarrow}_{j}$ for all $j \geq 4$.*

Proof. The $S^{(1/4)\downarrow}_{j} = \{1, -1, 0, 1, 0, -2, 2, 1, -3, 0, 5, -4, -4\}$ satisfy $S^{(1/4)\downarrow}_{j-4} - S^{(1/4)\downarrow}_{j-2} - S^{(1/4)\downarrow}_{i-1} = S^{(1/4)\downarrow}_{j}$ for $0 \leq j \leq 12$. Any 1/4-slope descending diag. starting from $e'_{1,j}$ ends at j (mod 4)th column. In fact, when $j = 4k + r$ ($r = 0, 1, 2, 3$) we have

$S^{(1/4)\downarrow}_{j} = e'_{1,j} + e'_{2,j-4} + \cdots + e'_{k,r+4} + e'_{k+1,r}$,

and each component satisfies

$$\begin{aligned}
e'_{1,j} &= e'_{1,j} \\
e'_{2,j-4} &= e'_{1,j-4} - e'_{2,j-6} - e'_{2,j-5} \\
&\cdots \\
e'_{k,r+4} &= e'_{k-1,r+4} - e'_{k,r+2} - e'_{k,r+3} \\
e'_{k+1,r} &= e'_{k,r} - e'_{k+1,r-2} - e'_{k+1,r-1}
\end{aligned}$$

Hence if $j = 4k$ then

$$S^{(1/4)\downarrow}_{j} = e'_{1,j} + \underbrace{(e'_{1,j-4} + \cdots + e'_{k-1,4} + e'_{k,0})}_{S^{(1/4)\downarrow}_{j-4}} - \underbrace{(e'_{2,j-6} + \cdots + e'_{k,2})}_{S^{(1/4)\downarrow}_{j-2} - e'_{1,j-2}}$$

$$- \underbrace{(e'_{2,j-5} + \cdots + e'_{k,3})}_{S^{(1/4)\downarrow}_{j-1} - e'_{1,j-1}} = S^{(1/4)\downarrow}_{j-4} - S^{(1/4)\downarrow}_{j-2} - S^{(1/4)\downarrow}_{j-1}.$$

If $j = 4k + 1$ then we also have

$$S^{(1/4)\downarrow}_{j} = e'_{1,j} + \underbrace{(e'_{1,j-4} + \cdots + e'_{k-1,5}) + e'_{k,1}}_{S^{(1/4)\downarrow}_{j-4}} - \underbrace{(e'_{2,j-6} + \cdots + e'_{k,3})}_{S^{(1/4)\downarrow}_{j-2} - e'_{1,j-2}}$$

$$- \underbrace{(e'_{2,j-5} + \cdots + e'_{k,4} + e'_{k+1,0})}_{S^{(1/4)\downarrow}_{j-1} - e'_{1,j-1}} = S^{(1/4)\downarrow}_{j-4} - S^{(1/4)\downarrow}_{j-2} - S^{(1/4)\downarrow}_{j-1}.$$

Analogously, the recurrence $S^{(1/4)\downarrow}_{j} = S^{(1/4)\downarrow}_{j-4} - S^{(1/4)\downarrow}_{j-2} - S^{(1/4)\downarrow}_{j-1}$ holds for any $j = 4k + r$ with any $0 \leq r \leq 3$. $\square$

The $1/t$-slope descending diag. sum $S^{(1/t)\downarrow}_{j}$ ($t = 5, 6$) are observed that

$\{S^{(1/5)\downarrow}_{j}\} = \{1, -1, 0, 1, -1, 1, -1, 0, 2, -3, 2, 0, -2, 4\}$

$\{S_j^{(1/6)\downarrow}\} = \{1, -1, 0, 1, -1, 0, 2, -3, 1, 3, -5, 2, 5, -10\}$
and notice recurrences $S_{j-5}^{(1/5)\downarrow} - S_{j-2}^{(1/5)\downarrow} - S_{j-1}^{(1/5)\downarrow} = S_j^{(5)\downarrow}$ and $S_{j-6}^{(1/6)\downarrow} - S_{j-2}^{(1/6)\downarrow} - S_{j-1}^{(1/6)\downarrow} = S_j^{(1/6)\downarrow}$ for some j. A generalization is as follows.

Theorem 6. *$S_{j-t}^{(1/t)\downarrow} - S_{j-2}^{(1/t)\downarrow} - S_{j-1}^{(1/t)\downarrow} = S_j^{(1/t)\downarrow}$ for all $j \geq t \geq 3$.*

Proof. The first few $S_j^{(1/t)\downarrow}$ are

$$\begin{array}{l|l|l} S_0^{(1/t)\downarrow} = e'_{1,0} & S_{t-1}^{(1/t)\downarrow} = e'_{1,t-1} & S_{2t-2}^{(1/t)\downarrow} = e'_{1,2t-2} + e'_{2,t-2} \\ S_1^{(1/t)\downarrow} = e'_{1,1} & S_{t}^{(1/t)\downarrow} = e'_{1,t} + e'_{2,0} & S_{2t-1}^{(1/t)\downarrow} = e'_{1,2t-1} + e'_{2,t-1} \\ S_2^{(1/t)\downarrow} = e'_{1,2} & S_{t+1}^{(1/t)\downarrow} = e'_{1,t+1} + e'_{2,1} & S_{2t}^{(1/t)\downarrow} = e'_{1,2t} + e'_{2,t} + e'_{3,0} \end{array}$$

Since $e'_{1,t+1} + e'_{1,t} + e'_{1,t-1} = 0$ in Theorem 1, we have

$$\begin{aligned} S_{t+1}^{(1/t)\downarrow} + S_{t}^{(1/t)\downarrow} + S_{t-1}^{(1/t)\downarrow} &= (e'_{1,t+1} + e'_{2,1}) + (e'_{1,t} + e'_{2,0}) + e'_{1,t-1} \\ &= e'_{2,1} + e'_{2,0} = e'_{1,1} = S_1^{(t)\downarrow}. \end{aligned}$$

And $e'_{1,2t} + e'_{1,2t-1} + e'_{1,2t-2} = 0$ in Theorem 1 imply

$$\begin{aligned} &S_{2t}^{(1/t)\downarrow} + S_{2t-1}^{(1/t)\downarrow} + S_{2t-2}^{(1/t)\downarrow} \\ &\quad = (e'_{1,2t} + e'_{2,t} + e'_{3,0}) + (e'_{1,2t-1} + e'_{2,t-1}) + (e'_{1,2t-2} + e'_{2,t-2}) \\ &\quad = (e'_{1,2t} + e'_{1,2t-1} + e'_{1,2t-2}) + (e'_{2,t} + e'_{2,t-1}) + e'_{2,t-2}) + e'_{3,0} \\ &\quad = (e'_{2,t} + e'_{2,t-1}) + e'_{2,t-2}) + e'_{3,0} = e'_{1,t} + e'_{2,0} = S_t^{(t)\downarrow}. \end{aligned}$$

Now we assume $S_{j-t}^{(1/t)\downarrow} - S_{j-2}^{(1/t)\downarrow} - S_{j-1}^{(1/t)\downarrow} = S_j^{(1/t)\downarrow}$ for $j < kt$ $(k \in \mathbb{Z})$. Let $t = kt + r$ $(0 \leq r < t)$. Then by making use of the table

$$\begin{array}{lcllll} e'_{1,j} &=& e'_{1,j} & & \\ e'_{2,j-t} &=& e'_{1,j-t} & -e'_{2,j-t-2} & -e'_{2,j-t-1} \\ & \cdots & & & \\ e'_{k,t+r} &=& e'_{k-1,t+r} & -e'_{k,t+r-2} & -e'_{k,t+r-1} \\ e'_{k+1,r} &=& e'_{k,r} & -e'_{k+1,r-2} & -e'_{k+1,r-1} \end{array}$$

we have

$$\begin{aligned} S_j^{(1/t)\downarrow} &= e'_{1,j} + e'_{2,j-t} + \cdots + e'_{k,t+r} + e'_{k+1,r} \\ &= e'_{1,j} + \underbrace{(e'_{1,j-t} + \cdots + e'_{k-1,t+r} + e'_{k,r})}_{S_{j-t}^{(1/t)\downarrow}} - \underbrace{(e'_{2,j-t-2} + \cdots + e'_{k+1,r-2})}_{S_{j-2}^{(1/t)\downarrow} - e'_{1,j-2}} \\ &\quad - \underbrace{(e'_{2,j-t-1} + \cdots + e'_{k+1,r-1})}_{S_{j-1}^{(1/t)\downarrow} - e'_{1,j-1}} = S_{j-t}^{(1/t)\downarrow} - S_{j-2}^{(1/t)\downarrow} - S_{j-1}^{(1/t)\downarrow}. \end{aligned}$$

□

3. Reflected sequence of diagonal sums

Table 1 is about sequences of $1/t$-slope descending diag. sums $S_n^{(1/t)\downarrow}$ of T' satisfying $S_{j-t}^{(1/t)\downarrow} - S_{j-2}^{(1/t)\downarrow} - S_{j-1}^{(1/t)\downarrow} = S_j^{(1/t)\downarrow}$ for all $j \geq t \geq 3$.

Table 1. $S_n^{(1/t)\downarrow}$ $(3 \leq t \leq 8)$

$t\backslash n$	0	1	2	3	4	5	6	7	8	9	10	11	12	13
3	1	−1	0	2	−3	1	4	−8	5	7	−20	18	9	−47
4	1	−1	0	1	0	−2	2	1	−3	0	5	−4	−4	8
5	1	−1	0	1	−1	1	−1	0	2	−3	2	0	−2	4
6	1	−1	0	1	−1	0	2	−3	1	3	−5	2	5	−10
7	1	−1	0	1	−1	0	1	0	−2	2	1	−4	3	2

Refer A077889, A247920 OEIS to $\{S_j^{(1/t)\downarrow}\}$ with $t = 4, 5$. If we display the numbers in $\{S_n^{(1/3)\downarrow}\}$ in reverse order then $\{\cdots, 5, -8, 4, 1, -3, 2, 0, -1, 1\}$ corresponds to the negative indexed part of the extended tribonacci sequence $\{\cdots, \mathit{5}, \mathit{-8}, \mathit{4}, \mathit{1}, \mathit{-3}, \mathit{2}, \mathit{0}, \mathit{-1}, \mathit{1}, 0, 0, 1, 1, 2, 4, 7, \cdots\}$. The rearranged sequence of $\{S_n^{(1/t)\downarrow}\}$ $(t \geq 3)$ in reverse order will be called the reflected sequence and denoted by $\{\hat{S}_n^{(1/t)\downarrow} \mid n \in \mathbb{Z}\}$.

Table 2. $\hat{S}_n^{(1/t)\downarrow}$ $(3 \leq t \leq 6)$

$t\backslash n$	−5	−4	−3	−2	−1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
3	1	−3	2	0	−1	1	0	0	1	1	2	4	7	13	24	44	81	149	274	504	927	1705	3136	5768
4	−2	0	1	0	−1	1	0	0	0	1	0	1	1	2	2	4	5	8	11	17	24	36	52	77
5	1	−1	1	0	−1	1	0	0	0	0	1	0	0	1	1	1	1	2	3	3	4	6	8	10
6	0	−1	1	0	−1	1	0	0	0	0	0	1	0	0	0	1	1	1	0	1	2	3	2	2

So the reflected sequence $\{\hat{S}_n^{(1/3)\downarrow} \mid n \in \mathbb{Z}\}$ is the extended tribonacci sequence satisfying $\hat{S}_{n-3}^{(1/3)\downarrow} + \hat{S}_{n-2}^{(1/3)\downarrow} + \hat{S}_{n-1}^{(1/3)\downarrow} = \hat{S}_n^{(1/3)\downarrow}$ for $n \in \mathbb{Z}$.

Theorem 7. *For $t \geq 3$, a recurrence rule is $\hat{S}_{n+t}^{(1/t)\downarrow} = \hat{S}_{n+2}^{(1/t)\downarrow} + \hat{S}_{n+1}^{(1/t)\downarrow} + \hat{S}_n^{(1/t)\downarrow}$, and the limit of $\frac{\hat{S}_n^{(1/t)\downarrow}}{\hat{S}_{n-1}^{(1/t)\downarrow}}$ in $\{\hat{S}_n^{(1/t)\downarrow} \mid n \in \mathbb{Z}\}$ is a real root of $x^t - x^2 - x - 1 = 0$.*

Proof. From the recurrence $S_{j-t}^{(1/t)\downarrow} = S_{j-2}^{(1/t)\downarrow} + S_{j-1}^{(1/t)\downarrow} + S_j^{(1/t)\downarrow}$, if we consider $j = -n$ $(n > 0)$ then $S_{-(n+t)}^{(1/t)\downarrow} = S_{-(n+2)}^{(1/t)\downarrow} + S_{-(n+1)}^{(1/t)\downarrow} + S_{-n}^{(1/t)\downarrow}$, so we have

$\hat{S}_{n+t}^{(1/t)\downarrow} = \hat{S}_{n+2}^{(1/t)\downarrow} + \hat{S}_{n+1}^{(1/t)\downarrow} + \hat{S}_n^{(1/t)\downarrow}$ for any $n \in \mathbb{Z}$.

By dividing the both sides of the recurrence by $\hat{S}_{n-1}^{(1/t)\downarrow}$ we have

$$\frac{\hat{S}_n^{(1/t)\downarrow}}{\hat{S}_{n-1}^{(1/t)\downarrow}} = \frac{1}{\frac{\hat{S}_{n-1}^{(1/t)\downarrow}}{\hat{S}_{n-t+2}^{(1/t)\downarrow}}} + \frac{1}{\frac{\hat{S}_{n-1}^{(1/t)\downarrow}}{\hat{S}_{n-t+1}^{(1/t)\downarrow}}} + \frac{1}{\frac{\hat{S}_{n-1}^{(1/t)\downarrow}}{\hat{S}_{n-t}^{(1/t)\downarrow}}}.$$

So if let $r = \lim_{n\to\infty} \frac{\hat{S}_n^{(1/t)\downarrow}}{\hat{S}_{n-1}^{(1/t)\downarrow}}$ then $r = \frac{1}{r^{t-3}} + \frac{1}{r^{t-2}} + \frac{1}{r^{t-1}}$, and r is a real root of the polynomial $x^t - x^2 - x - 1 = 0$. □

An interesting connection of $\hat{S}_n^{(1/t)\downarrow}$ with trinomial table T is as follows.

THEOREM 8. *Let r_k $(k \geq 0)$ be the kth row of T. Then inner product of r_k and $2k+1$ consecutive terms $\{\hat{S}_n^{(1/t)\downarrow}\}$ yields $(\hat{S}_{n-2k}^{(1/t)\downarrow}, \cdots, \hat{S}_{n-1}^{(1/t)\downarrow}, \hat{S}_n^{(1/t)\downarrow}) \circ r_k = \hat{S}_{n+(t-2)k}^{(1/t)\downarrow}$.*

Proof. Let $t = 3$. Clearly $(\hat{S}_{n-2}^{(1/3)\downarrow}, \hat{S}_{n-1}^{(1/3)\downarrow}, \hat{S}_n^{(1/3)\downarrow}) \circ r_1 = \hat{S}_{n+1}^{(1/3)\downarrow}$, for $r_1 = (1,1,1)$.

Since $r_2 = (1,2,3,2,1) = (1,1,1,0,0) + (0,1,1,1,0) + (0,0,1,1,1)$ by $(\star)$, if we write it by $r_2 = (r_1,0,0) + (0,r_1,0) + (0,0,r_1)$ then

$$\begin{aligned}
&(\hat{S}_{n-4}^{(1/3)\downarrow}, \hat{S}_{n-3}^{(1/3)\downarrow}, \hat{S}_{n-2}^{(1/3)\downarrow}, \hat{S}_{n-1}^{(1/3)\downarrow}, \hat{S}_n^{(1/3)\downarrow}) \circ r_2 \\
&= (\hat{S}_{n-4}^{(1/3)\downarrow}, \hat{S}_{n-3}^{(1/3)\downarrow}, \hat{S}_{n-2}^{(1/3)\downarrow}) \circ r_1 + (\hat{S}_{n-3}^{(1/3)\downarrow}, \hat{S}_{n-2}^{(1/3)\downarrow}, \hat{S}_{n-1}^{(1/3)\downarrow}) \circ r_1 \\
&\quad + (\hat{S}_{n-2}^{(1/3)\downarrow}, \hat{S}_{n-1}^{(1/3)\downarrow}, \hat{S}_n^{(1/3)\downarrow}) \circ r_1 \\
&= \hat{S}_{n-1}^{(1/3)\downarrow} + \hat{S}_n^{(1/3)\downarrow} + \hat{S}_{n+1}^{(1/3)\downarrow} = \hat{S}_{n+2}^{(1/3)\downarrow}
\end{aligned}$$

by Theorem 7. Assume the identity in the theorem is true with respect to r_{k-1}. Since r_k equals $(r_{k-1},0,0) + (0,r_{k-1},0) + (0,0,r_{k-1})$, we have

$$\begin{aligned}
&(\hat{S}_{n-2k}^{(1/3)\downarrow}, \hat{S}_{n-2k+1}^{(1/3)\downarrow}, \cdots, \hat{S}_{n-1}^{(1/3)\downarrow}, \hat{S}_n^{(1/3)\downarrow}) \circ r_k \\
&= (\hat{S}_{n-2k}^{(1/3)\downarrow}, \cdots, \hat{S}_{n-2}^{(1/3)\downarrow}) \circ r_{k-1} + (\hat{S}_{n-2k+1}^{(1/3)\downarrow}, \cdots, \hat{S}_{n-1}^{(1/3)\downarrow}) \circ r_{k-1} \\
&\quad + (\hat{S}_{n-2k+2}^{(1/3)\downarrow}, \cdots, \hat{S}_n^{(1/3)\downarrow}) \circ r_{k-1} \\
&= \hat{S}_{n-2+(k-1)}^{(1/3)\downarrow} + \hat{S}_{n-1+(k-1)}^{(1/3)\downarrow} + \hat{S}_{n+(k-1)}^{(1/3)\downarrow} = \hat{S}_{n+(k-1)+1}^{(1/3)\downarrow} = \hat{S}_{n+(t-2)k}^{(1/3)\downarrow},
\end{aligned}$$

by the induction hypothesis and Theorem 7.

When $t = 4$, we also can see from Theorem 7 that

$$\begin{aligned}
&(\hat{S}_{n-2}^{(1/4)\downarrow}, \hat{S}_{n-1}^{(1/4)\downarrow}, \hat{S}_n^{(1/4)\downarrow}) \circ r_1 \\
&= \hat{S}_{n-2}^{(1/4)\downarrow} + \hat{S}_{n-1}^{(1/4)\downarrow} + \hat{S}_n^{(1/4)\downarrow} = \hat{S}_{n+2}^{(1/4)\downarrow} = \hat{S}_{n+(t-2)}^{(1/4)\downarrow},
\end{aligned}$$

and also

$$\begin{aligned}
&(\hat{S}_{n-4}^{(1/4)\downarrow}, \hat{S}_{n-3}^{(1/4)\downarrow}, \hat{S}_{n-2}^{(1/4)\downarrow}, \hat{S}_{n-1}^{(1/4)\downarrow}, \hat{S}_n^{(1/4)\downarrow}) \circ r_2 \\
&= (\hat{S}_{n-4}^{(1/4)\downarrow}, \hat{S}_{n-3}^{(1/4)\downarrow}, \hat{S}_{n-2}^{(1/4)\downarrow}) \circ r_1 + (\hat{S}_{n-3}^{(1/4)\downarrow}, \hat{S}_{n-2}^{(1/4)\downarrow}, \hat{S}_{n-1}^{(1/4)\downarrow}) \circ r_1 \\
&\quad + (\hat{S}_{n-2}^{(1/4)\downarrow}, \hat{S}_{n-1}^{(1/4)\downarrow}, \hat{S}_n^{(1/4)\downarrow}) \circ r_1 \\
&= \hat{S}_n^{(1/4)\downarrow} + \hat{S}_{n+1}^{(1/4)\downarrow} + \hat{S}_{n+2}^{(1/4)\downarrow} = \hat{S}_{n+4}^{(1/4)\downarrow} = \hat{S}_{n+(t-2)2}^{(1/4)\downarrow}.
\end{aligned}$$

Now assume $(\hat{S}^{(1/t)\downarrow}_{n-2(k-1)}, \cdots, \hat{S}^{(1/t)\downarrow}_{n-1}, \hat{S}^{(1/t)\downarrow}_{n}) \circ r_{k-1} = \hat{S}^{(1/t)\downarrow}_{n+(t-2)(k-1)}$ for any $t \geq 3$ and $k > 1$. Then

$$\begin{aligned}&(\hat{S}^{(1/t)\downarrow}_{n-2k}, \hat{S}^{(1/t)\downarrow}_{n-2k+1}, \cdots, \hat{S}^{(1/t)\downarrow}_{n-1}, \hat{S}^{(1/t)\downarrow}_{n}) \circ r_k\\ &= (\hat{S}^{(1/t)\downarrow}_{n-2k}, \cdots, \hat{S}^{(1/t)\downarrow}_{n-2}) \circ r_{k-1} + (\hat{S}^{(1/t)\downarrow}_{n-2k+1}, \cdots, \hat{S}^{(1/t)\downarrow}_{n-1}) \circ r_{k-1}\\ &\quad + (\hat{S}^{(1/t)\downarrow}_{n-2k+2}, \cdots, \hat{S}^{(1/t)\downarrow}_{n}) \circ r_{k-1}\\ &= \hat{S}^{(1/t)\downarrow}_{n-2+(t-2)(k-1)} + \hat{S}^{(1/t)\downarrow}_{n-1+(t-2)(k-1)} + \hat{S}^{(1/t)\downarrow}_{n+(t-2)(k-1)}\\ &= \hat{S}^{(1/t)\downarrow}_{n-2+(t-2)(k-1)+t} = \hat{S}^{(1/t)\downarrow}_{n+(t-2)k},\end{aligned}$$

by Theorem 7. This finishes the proof. □

Since $\{\hat{S}^{(1/3)\downarrow}_n \mid n \in \mathbb{Z}\}$ corresponds to the extended tribonacci sequence, the numbers $\hat{S}^{(1/3)\downarrow}_n$ $(n \geq 1)$ can be graphically explained by 1/1-slope ascending diag. sums of T, while $\hat{S}^{(1/3)\downarrow}_n$ $(n \leq 0)$ are 1/3-slope descending diag. sums of T'. Then it is natural to ask graphical description of $\hat{S}^{(1/t)\downarrow}_n$ $(n \geq 1)$ over T for any $t \geq 3$. For this purpose, similar to $S^{(t/u)\uparrow}_n$ and $S^{(t/u)\downarrow}_n$ over T', we shall use notations $\sigma^{(t/u)\uparrow}_i$ and $\sigma^{(t/u)\downarrow}_j$ over T. The former means the t/u-slope ascending diag. sum starting from $e_{i,0}$ while the latter is the descending diag. sum starting from $e_{0,j}$ over T. For instance $\sigma^{(t)\uparrow}_i = \sigma^{(t/1)\uparrow}_i = e_{i,0} + e_{i-t,1} + e_{i-2t,2} + \cdots$ and $\sigma^{(1/t)\downarrow}_j = e_{0,j} + e_{1,j-t} + e_{2,j-2t} + \cdots$.

Theorem 9. *$\hat{S}^{(1/3)\downarrow}_n = \sigma^{(1)\downarrow}_{n-3} = \sigma^{(1)\uparrow}_{n-3}$. And $\hat{S}^{(1/4)\downarrow}_n = \sigma^{(1/2)\downarrow}_{n-4}$.*

Proof. The 1-slope descending diag. sums over T clearly satisfy $\{\sigma^{(1)\downarrow}_{n-3} \mid n \geq 3\} = \{1, 1, 2, 4, 7, 13, \cdots\} = \{\sigma^{(1)\uparrow}_{n-3} \mid n \geq 3\}$, which is the tribonacci numbers. So by $\{\hat{S}^{(1/3)\downarrow}_n\} = \{1, 1, 2, 4, 7, \cdots\}$ in Table 2, the proof of the first identity is clear.

The first few numbers of 1/2-slope descending diag. sums over T are

$$\{\sigma^{(1/2)\downarrow}_{n-4} \mid n \geq 4\} = \{1, 0, 1, 1, 2, 2, 4, 5, 8, 11, 17, \cdots\},$$

where this equals $\{\hat{S}^{(1/4)\downarrow}_n\} = \{1, 0, 1, 1, 2, 2, 4, 5, 8, 11, 17, \cdots\}$ (see Table 2). In fact, $\sigma^{(1/2)\downarrow}_{10} = \underbrace{e_{0,10} + e_{1,8} + e_{2,6}}_{0} + \underbrace{e_{3,4} + e_{4,2} + e_{5,0}}_{17} = \hat{S}^{(1/4)\downarrow}_{14}$. Since the first few numbers in sequences $\{\hat{S}^{(1/4)\downarrow}_n\}$ and $\{\sigma^{(1/2)\downarrow}_{n-4}\}$ correspond each other, it is enough to show that $\{\sigma^{(1/2)\downarrow}_j\}$ satisfies the recurrence $\sigma^{(1/2)\downarrow}_j + \sigma^{(1/2)\downarrow}_{j+1} + \sigma^{(1/2)\downarrow}_{j+2} = \sigma^{(1/2)\downarrow}_{j+4}$, that is the same pattern of $\hat{S}^{(1/4)\downarrow}_n$ in Theorem 7. In fact,

$$\sigma_j^{(1/2)\downarrow} + \sigma_{j+1}^{(1/2)\downarrow} + \sigma_{j+2}^{(1/2)\downarrow}$$
$$= (e_{0,j} + e_{1,j-2} + e_{2,j-4} + \cdots) + (e_{0,j+1} + e_{1,j-1} + e_{2,j-3} + \cdots)$$
$$+ (e_{0,j+2} + e_{1,j} + e_{2,j-2} + \cdots).$$

Then by considering each columnwise sum, we have

$$\sigma_j^{(1/2)\downarrow} + \sigma_{j+1}^{(1/2)\downarrow} + \sigma_{j+2}^{(1/2)\downarrow} = e_{1,j+2} + e_{2,j} + e_{3,j-2} + e_{4,j-5} + \cdots$$
$$= e_{0,j+4} + (e_{1,j+2} + e_{2,j} + e_{3,j-2} + e_{4,j-5} + \cdots) = \sigma_{j+4}^{(1/2)\downarrow},$$

because $e_{0,j+4} = 0$ for all $j \geq 0$ and the recurrence $(\star)$ of T. □

Clearly $\sigma_{11}^{(1)\downarrow} = \underbrace{e_{0,11} + \cdots + e_{3,8} + e_{4,7} + e_{5,6} + \cdots + e_{11,0}}_{4+45+126+161+112+45+10+1=504} = \hat{S}_{14}^{(1/3)\downarrow}$.

Let $\sigma_{(a,b)}^{(1/2)\uparrow}$ and $\sigma_{(a,b)}^{(1/2)\downarrow}$ be 1/2-slope ascending and descending diag. sums starting from the component $e_{a,b}$ of T. The next theorem further explains $\hat{S}_n^{(1/4)\downarrow}$ in relation to certain 1/2-slope diag. in T.

THEOREM 10. $\hat{S}_n^{(1/4)\downarrow} = \sigma_{(0,n-4)}^{(1/2)\downarrow} = \begin{cases} \sigma_{(\frac{n-4}{2},0)}^{(1/2)\uparrow} & \text{if } n \equiv 0 \pmod 2 \\ \sigma_{(\frac{n-5}{2},1)}^{(1/2)\uparrow} & \text{if } n \equiv 1 \pmod 2 \end{cases}$.

Proof. Since $\sigma_{(0,n-4)}^{(1/2)\downarrow} = \sigma_{n-4}^{(1/2)\downarrow}$, the first equality is due to Theorem 9. Now we look at a 1/2-slope descending diag. sum in T, for example, $\sigma_{(0,12)}^{(1/2)\downarrow} = \underbrace{e_{0,12} + e_{1,10} + e_{2,8} + e_{3,6} + \cdots + e_{6,0}}_{1+19+15+1=36} = \hat{S}_{4+12}^{(1/4)\downarrow}$. Also it can be explained as the increasing diagonal sum $e_{6,0} + e_{5,2} + e_{4,4} + e_{3,6} = 36 = \sigma_{(6,0)}^{(1/2)\uparrow}$.

Table 3. $\sigma_{(\frac{n-4}{2},0)}^{(1/2)\uparrow}$

$n=4$	$\sigma_{(0,0)}^{(1/2)\uparrow} = 1 = \hat{S}_6^{(1/4)\downarrow}$
6	$\sigma_{(1,0)}^{(1/2)\uparrow} = 1 = \hat{S}_8^{(1/4)\downarrow}$
8	$\sigma_{(2,0)}^{(1/2)\uparrow} = 1+1 = \hat{S}_{10}^{(1/4)\downarrow}$
10	$\sigma_{(3,0)}^{(1/2)\uparrow} = 1+6+1 = \hat{S}_{12}^{(1/4)\downarrow}$
12	$\sigma_{(4,0)}^{(1/2)\uparrow} = 1+10+6 = \hat{S}_{14}^{(1/4)\downarrow}$

Table 4. $\sigma_{(\frac{n-5}{2},1)}^{(1/2)\uparrow}$

$n=5$	$\sigma_{(0,1)}^{(1/2)\uparrow} = 0$
7	$\sigma_{(1,1)}^{(1/2)\uparrow} = 1$
9	$\sigma_{(2,1)}^{(1/2)\uparrow} = 2$
11	$\sigma_{(3,1)}^{(1/2)\uparrow} = 3+2 = 5$
13	$\sigma_{(4,1)}^{(1/2)\uparrow} = 4+7 = 11$

In case of $n = 2k \geq 4$, the first few numbers $\sigma_{(\frac{n-4}{2},0)}^{(1/2)\uparrow}$ are in Table 3, where it shows $\{\sigma_{(\frac{n-4}{2},0)}^{(1/2)\uparrow}\} = \{1,1,2,4,8,17,36,77,165,\cdots\} = \{\hat{S}_n^{(1/4)\downarrow} \mid n : \text{even}\}$.

Similarly when $n = 2k+1 \geq 4$, the first few numbers $\sigma^{(1/2)\uparrow}_{(\frac{n-5}{2},1)}$ are in Table 4, where it shows that $\{\sigma^{(1/2)\uparrow}_{(\frac{n-5}{2},1)}\} = \{0, 1, 2, 5, 11, 24, 52, 112, 241, \cdots\} = \{\hat{S}^{(1/4)\downarrow}_n \mid n : \text{odd}\}$. This completes the proof □

In fact Theorem 10 corresponds to the following table.

$\hat{S}^{(1/4)\downarrow}_n$ with $\sigma^{(1/2)\uparrow}_{(a,b)}$

n	4	5	6	7	8	9	$\cdots$
$\hat{S}^{(1/4)\downarrow}_n$	$1 = \sigma^{(1/2)\uparrow}_{(0,0)}$	$0 = \sigma^{(1/2)\uparrow}_{(0,1)}$	$1 = \sigma^{(1/2)\uparrow}_{(1,0)}$	$1 = \sigma^{(1/2)\uparrow}_{(1,1)}$	$2 = \sigma^{(1/2)\uparrow}_{(2,0)}$	$2 = \sigma^{(1/2)\uparrow}_{(2,1)}$	

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