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# SEVEN GENERALIZED TYPES OF SOFT SEMI-COMPACT SPACES

Tareq Mohammed Al-shami*, Mohammed E. El-Shafei, and Mohammed Abo-Elhamayel

Abstract. The soft compactness notion via soft topological spaces was first studied in [10,29]. In this work, soft semi-open sets are utilized to initiate seven new kinds of generalized soft semi-compactness, namely soft semi-Lindelöfness, almost (approximately, mildly) soft semi-compactness and almost (approximately, mildly) soft semi- Lindelöfness. The relationships among them are shown with the help of illustrative examples and the equivalent conditions of each one of them are investigated. Also, the behavior of these spaces under soft semi-irresolute maps are investigated. Furthermore, the enough conditions for the equivalence among the four sorts of soft semi-compact spaces and for the equivalence among the four sorts of soft semi-Lindelöf spaces are explored. The relationships between enriched soft topological spaces and the initiated spaces are discussed in different cases. Finally, some properties which connect some of these spaces with some soft topological notions such as soft semi-connectedness, soft semi $T_2$-spaces and soft subspaces are obtained.

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∗ Corresponding author.

## 1. Introduction

In the year 1999, Molodtsove [23] originated the concept of soft sets as a completely new approach for solving problems which contain incomplete information. After Molodtsove's work, Maji et al. ( [20], [21]) presented an application of soft sets to decision making problems and initiated some operation between two soft sets in 2002 and 2003, respectively. Among the valuable contributions to study operation between two soft sets was [1] which did in 2009.

In the year 2011, the concept of soft topological spaces was initiated by Shabir and Naz [28]. They studied the basic soft notions such as soft open sets, soft neighbourhoods and soft separation axioms. To continue Shabir and Naz' work, Min [22] studied in detail further properties for soft regular spaces and proved that soft $T_3$ implies soft $T_2$. Zorlutuna et al. [29] gave the first form of the soft point and then employed it to study some properties of soft neighborhood systems and soft interior points of a soft set. In 2012, [10, 29] introduced the concept of soft compact spaces and derived main properties. [10] also presented a notion of enriched soft topological spaces and illuminated its role to verified some results related to constant soft maps and soft compact spaces. Hida [16] introduced and explored another kind of soft compact spaces, namely SCPT1. The authors of [13, 24] modified simultaneously the previous form of soft point to be more effective for studying soft metric spaces and soft limit points of a soft point. Chen [11] was the first one who studied generalized soft open sets by defining and investigating soft semi-open sets. Based on them, Mahanta and Das [19] initiated the concepts of soft semi-irresolute maps, soft semi-compact and soft semi-connected spaces and soft semi $T_2$-spaces. Some amendments of the results obtained in [18,19] were done by [1,2]. Also, [4,15] explained the sufficient conditions to keep the relationships between compact and Hausdorff spaces on soft topologies. In 2014, Kandil et al. [18] introduced soft semi separation axioms and discussed main properties. Al-shami [3] introduced a concept of soft somewhere dense sets and and showed its relationship with the other types of generalized soft open sets. Roy and Samanta [27] in 2014, established the concepts of soft base and soft subbase and derived their main properties. Nazmul and Samanta [25] introduced a notion of pseudo constant soft sets and utilized it to define enriched soft topological spaces. We [14] defined two new soft relations and then we [7] utilized them to initiate new soft axioms on soft topological ordered spaces. Recently, [5, 6] formulated and investigated new forms of soft compact and soft Lindelöf spaces.

This study begins by presenting the fundamental concepts of soft set theory and soft topological spaces. The main aim of this work, is to formulate the concepts of seven sorts of generalized soft semi-compact spaces, namely soft semi-Lindelöf spaces, almost (approximately, mildly) soft semi-compact spaces and almost (approximately, mildly) soft semi-Lindelöf spaces. We present several examples to illustrate the relationships among these spaces and to point out some properties. Also, we give the equivalent conditions for each one of these concepts and derive some results which connect between enriched soft topological spaces and some of the given generalized soft semi-compact spaces. Moreover, we introduce some soft topological notions such as soft semi-hyperconnectedness, soft semi $T_2$-spaces and soft semi-partition spaces. Then we establish some properties which associated these notions with the initiated generalized soft semi-compact spaces. We investigate the enough conditions for these generalized soft semi-compact spaces to be soft hereditary properties. In the end, we point out that the soft semi-irresolute maps preserve all of the initiated generalized soft semi-compact spaces. The findings of this work extend and improve some findings that can be found in the literature.

## 2. Preliminaries

We recall some definitions and results which will be needed in the sequel.

**2.1. Soft set.** Since a parameters set is fixed on the soft topology, then we mention the definitions of operations between soft sets under a fixed parameters set.

Definition 2.1. [23] A pair $(G, K)$ is called a soft set over $X$ provided that $G$ is a map of $K$ a parameters set $K$ into the power set $P(X)$. It can be written as follows: $(G, K) = \{(k, G(k)) : k \in K \text{ and } G(k) \in P(X)\}$.

Definition 2.2. [14, 28] Let $(G, K)$ be a soft set cover $X$. Then we say that:

**(i):** $x \Subset (G, K)$ if $x \in G(k)$ for some $k \in K$; and $x \not\Subset (G, K)$ if $x \notin G(k)$ for each $k \in K$.

**(ii):** $x \in (G,K)$ if $x \in G(k)$ for each $k \in K$; and $x \notin (G,K)$ if $x \notin G(k)$ for some $k \in K$.

Definition 2.3. [26] $(G,K)$ is a soft subset of $(H,K)$, denoted by $(G,K)\widetilde{\subseteq}(H,K)$, provided that $G(k) \subseteq H(k)$ for each $k \in K$.

Definition 2.4. [9] The relative complement of a soft set $(G,K)$, denoted by $(G,K)^c$, is given by $(G,K)^c = (G^c,K)$, where a map $G^c : K \to P(X)$ is defined by $G^c(k) = X - G(k)$, for each $k \in K$.

Definition 2.5. [9] Let $(G,K)$ and $(F,K)$ be two soft sets. Then:

**(i):** $(G,K)\widetilde{\bigcup}(F,K) = (H,K)$, where $H(k) = G(k)\bigcup F(k)$ for each $k \in K$.

**(ii):** $(G,K)\widetilde{\bigcap}(F,K) = (H,K)$, where $H(k) = G(k)\bigcap F(k)$ for each $k \in K$.

Definition 2.6. [21] A soft set $(G,K)$ over $X$ is called:

**(i):** An absolute soft set if $G(k) = X$ for each $k \in K$. It is denoted by $\widetilde{X}$.

**(ii):** A null soft set if $G(k) = \emptyset$ for each $k \in K$. It is denoted by $\widetilde{\emptyset}$.

Definition 2.7. [25] A soft subset $(F,K)$ of an absolute soft set $\widetilde{X}$ is said to be pseudo constant provided that $F(k) = X$ or $\emptyset$ for each $k \in K$. The family of all pseudo constant soft sets is briefly denoted by $CS(X,K)$.

Definition 2.8. [13,24] A soft set $(P,K)$ over $X$ is called soft point if there is $k \in K$ and $x \in X$ satisfies that $P(k) = \{x\}$ and $P(e) = \emptyset$ for each $e \in K \setminus \{k\}$. A soft point will be shortly denoted by $P_k^x$.

Definition 2.9. [13] A soft set $(H,K)$ over $X$ is called:

**(i):** A countable (resp. finite) soft set if $H(k)$ is countable (resp. finite) for each $k \in K$.

**(ii):** An uncountable (resp. infinite) soft set if $H(k)$ is uncountable (resp. infinite) for some $k \in K$.

Definition 2.10. [14] A soft set $(G,K)$ over $X$ is called stable if there is a subset $S$ of $X$ such that $G(k) = S$ for each $k \in K$ and it is denoted by $\widetilde{S}$.

Definition 2.11. The collection $\Lambda$ of soft sets over $X$ is said to have:

**(i):** The finite intersection property if any finite sub-collection of $\Lambda$ has a non-null soft intersection.
**(ii):** The countable intersection property if any countable sub-collection of $\Lambda$ has a non-null soft intersection.

### 2.2. Soft topology.

DEFINITION 2.12. [28] The collection $\tau$ of soft sets over $X$ with a fixed set of parameters $K$ is called a soft topology on $X$ if it satisfies the following three axioms:

**(i):** The null and absolute soft sets are members of $\tau$.
**(ii):** The soft union of an arbitrary number of soft sets in $\tau$ is also a member of $\tau$.
**(iii):** The soft intersection of a finite number of soft sets in $\tau$ is also a member of $\tau$.

The triple $(X, \tau, K)$ is called a soft topological space (STS, in short). Every member of $\tau$ is called soft open and its relative complement is called soft closed.

PROPOSITION 2.13. [28] *Let $(X, \tau, K)$ be an STS. Then the family $\tau_k = \{G(k) : (G, K) \in \tau\}$ defines a topology on $X$, for each $k \in K$.*

PROPOSITION 2.14. [28] *Let $(L, K)$ be a soft subset of an STS $(X, \tau, K)$. Then:*

**(i):** $(cl(L), K)\widetilde{\subseteq}cl(L, K)$,
**(ii):** $(cl(L), K) = cl(L, K)$ *if and only if* $(cl(L), K)^c$ *is soft closed.*

DEFINITION 2.15. [25] A soft topology $\tau$ on $X$ is said to be an enriched soft topology if **(i)** of Definition (2.12) is replaced by the following condition: $(G, K) \in \tau$ for all $(G, K) \in CS(X, K)$. In such a case, $(X, \tau, K)$ is called an enriched STS over $X$.

DEFINITION 2.16. [28] Let $(Y, K)$ be a non-null soft subset of $(X, \tau, K)$. Then $\tau_{(Y,K)} = \{(Y, K)\widetilde{\bigcap}(G, K) : (G, K) \in \tau\}$ is said to be a relative soft topology on $(Y, K)$ and $((Y, K), \tau_{(Y,K)}, K)$ is called a soft subspace of $(X, \tau, K)$.

DEFINITION 2.17. [11] A soft subset $(A, K)$ of $(X, \tau, K)$ is said to be soft semi-open if $(A, K)\widetilde{\subseteq}cl(int(A, K))$ and its relative complement is called soft semi-closed.

Definition 2.18. [11] Let $(A, K)$ be a soft subset of $(X, \tau, K)$. Then $int_s(A, K)$ is the union of all soft semi-open subsets of $(A, K)$ and $cl_s(A, K)$ is the intersection of all soft semi-closed supersets of $(A, K)$.

Proposition 2.19. [11] *The union of an arbitrary family of soft semi-open sets is soft semi-open and the intersection an arbitrary family of soft semi-closed sets is soft semi-closed.*

Theorem 2.20. [12, 17] *An STS $(X, \tau, K)$ is soft semi-connected if and only if the only soft semi-open and soft semi-closed subsets of $(X, \tau, K)$ are $\widetilde{\emptyset}$ and $\widetilde{X}$.*

Definition 2.21. [19]

- **(i):** The collection $\{(G_i, K) : i \in I\}$ of soft semi-open sets is called soft semi-open cover of an STS $(X, \tau, K)$ if $\widetilde{X} = \widetilde{\bigcup}_{i \in I}(G_i, K)$.
- **(ii):** An STS $(X, \tau, K)$ is called soft semi-compact if every soft semi-open cover of $\widetilde{X}$ has a finite sub-cover of $\widetilde{X}$.

Definition 2.22. [19] A soft map $g : (X, \tau, K) \to (Y, \theta, K)$ is called soft semi-irresolute if the inverse image of each soft semi-open subset of $\widetilde{Y}$ is a soft semi-open subset of $\widetilde{X}$.

Definition 2.23. An STS $(X, \tau, K)$ is said to be:

- **(i):** Soft semi $T_2$-space [18] if for every $x \neq y \in X$, there are two disjoint soft semi-open sets $(G, K)$ and $(F, K)$ such that $x \in (G, K)$ and $y \in (F, K)$.
- **(ii):** Soft semi $T_2'$-space [19] if for every $P_k^x \neq P_k^y$ such that $x \neq y$, there are two disjoint soft semi-open sets $(G, K)$ and $(F, K)$ such that $P_k^x \in (G, K)$ and $P_k^y \in (F, K)$.

Definition 2.24. [17] An STS $(X, \tau, K)$ is said to be soft hyperconnected if it does not contain disjoint soft open sets.

Definition 2.25. [14] An STS $(X, \tau, K)$ is called stable provided that all soft open sets in $\tau$ are stable.

Proposition 2.26. [6] *Consider $((U, K), \tau_{(U,K)}, K)$ is a soft subspace of $(X, \tau, K)$ and let $cl_U$ and $int_U$ stand for the soft closure and soft interior operators, respectively, in $((U, K), \tau_{(U,K)}, K)$. Then, for each $(A, K) \widetilde{\subseteq} (U, K)$, we have the following results:*

- **(i):** $cl_U(A, K) = cl(A, K) \widetilde{\bigcap} (U, K)$.

**(ii):** $int(A,K) = int_U(A,K)\widetilde{\bigcap} int(U,K)$.

Throughout this work, $(X,\tau,K)$ and $(Y,\theta,K)$ denote soft topological spaces and $S$ denotes a countable set.

## 3. Soft semi-Lindelöf spaces

DEFINITION 3.1. An STS $(X,\tau,K)$ is called soft semi-Lindelöf if every soft semi-open cover of $\widetilde{X}$ has a countable sub-cover of $\widetilde{X}$.

For the sake of brevity, the proofs of the following three propositions will be omitted.

PROPOSITION 3.2. *Every soft semi-compact space is soft semi-Lindelöf.*

PROPOSITION 3.3. *Every soft semi-compact (resp. soft semi-Lindelöf) space is soft compact (resp. soft Lindelöf).*

PROPOSITION 3.4. *A finite (resp. countable) union of soft semi-compact (resp. soft semi-Lindelöf) subsets of $(X,\tau,K)$ is soft semi-compact (resp. soft semi-Lindelöf).*

The converse of Proposition (3.3) and Proposition (3.2) are not true, in general, as explained in the example below.

EXAMPLE 3.5. Consider $K$ is any set of parameters and let $\tau = \{\widetilde{\emptyset}, (G,K)\widetilde{\subseteq}\widetilde{\mathcal{R}} : 1 \not\in (G_i,K)\}$ be a soft topology on the set of real numbers $\mathcal{R}$. Obviously, $(\mathcal{R},\tau,K)$ is soft compact. On the other hand, the collection $\{(G,K) : G(k) = \{1,x\}$ for each $k \in K\}$ forms a soft semi-open cover of $\widetilde{\mathcal{R}}$. Since this collection has not a countable sub-cover of $\widetilde{R}$, then $(\mathcal{R},\tau,K)$ is not soft semi-Lindelöf.
If we replace the set of real numbers $\mathcal{R}$ by the set of natural numbers $\mathcal{N}$, then $(\mathcal{N},\tau,K)$ is soft semi-Lindelöf, but not soft semi-compact.

PROPOSITION 3.6. *Every soft semi-closed subset $(D,K)$ of a soft semi-compact (resp. soft semi-Lindelöf) space $(X,\tau,K)$ is soft semi-compact (resp. soft semi-Lindelöf).*

*Proof.* We prove the proposition when $(X,\tau,K)$ is soft semi-Lindelöf and one can prove the other case similarly.
Let $(D,K)$ be a soft semi-closed subset of $\widetilde{X}$ and $\{(H_i,K) : i \in I\}$ be a soft semi-open cover of $(D,K)$. Then $(D^c,K)$ is soft semi-open

and $(D,K)\widetilde{\subseteq}\widetilde{\bigcup}_{i\in I}(H_i,K)$. Therefore $\widetilde{X}=\widetilde{\bigcup}_{i\in I}(H_i,K)\widetilde{\bigcup}(D^c,K)$. Since $(X,\tau,K)$ is soft semi-Lindelöf, then $\widetilde{X}=\widetilde{\bigcup}_{i\in S}(H_i,K)\widetilde{\bigcup}(D^c,K)$. This implies that $(D,K)\widetilde{\subseteq}\widetilde{\bigcup}_{i\in S}(H_i,K)$. Hence $(D,K)$ is soft semi-Lindelöf. □

In the following example, we show that the converse of the above proposition is not necessarily true.

Example 3.7. Let $K=\{k_1,k_2\}$ be a set of parameters and consider the following two soft sets over $X=\{x,y\}$:

$(G,K)=\{(k_1,\{y\}),(k_2,\{x\})\}$ and

$(H,K)=\{(k_1,\{x\}),(k_2,\{y\})\}$.

Then $\tau=\{\widetilde{\emptyset},\widetilde{X},(G,K),(H,K)\}$ is a soft topology on $X$. Obviously, $(X,\tau,K)$ is soft semi-compact. On the other hand, a soft set $(F,K)$, where $F(k_1)=\{x\}$ and $F(k_2)=\{x\}$, is soft semi-compact, but it is not soft semi-closed.

Proposition 3.8. *If $(G,K)$ is a soft semi-compact (resp. soft semi-Lindelöf) subset of $\widetilde{X}$ and $(D,K)$ is a soft semi-closed subset of $\widetilde{X}$, then $(G,K)\widetilde{\bigcap}(D,K)$ is soft semi-compact (resp. soft semi-Lindelöf).*

*Proof.* For the proof, let $\Lambda=\{(H_i,K):i\in I\}$ be a soft semi-open cover of $(G,K)\widetilde{\bigcap}(D,K)$. Then $(G,K)\widetilde{\subseteq}\widetilde{\bigcup}_{i\in I}(H_i,K)\widetilde{\bigcup}(D^c,K)$. Because $(G,K)$ is soft semi-compact, then $(G,K)\widetilde{\subseteq}\widetilde{\bigcup}_{i=1}^{i=n}(H_i,K)\widetilde{\bigcup}(D^c,K)$. So $(G,K)\widetilde{\bigcap}(D,K)\widetilde{\subseteq}\widetilde{\bigcup}_{i=1}^{i=n}(H_i,K)$. Hence $(G,K)\widetilde{\bigcap}(D,K)$ is soft semi-compact.

A similar proof is given in the case of a soft semi-Lindelöf space. □

Theorem 3.9. *An STS $(X,\tau,K)$ is soft semi-compact (resp. soft semi-Lindelöf) if and only if every collection of soft semi-closed subsets of $(X,\tau,K)$, satisfying the finite (resp. countable) intersection property, has, itself, a non-null soft intersection.*

*Proof.* We only prove the theorem when $(X,\tau,K)$ is soft semi-Lindelöf, the other case can be made similarly.

Let $\Lambda=\{(F_i,K):i\in I\}$ be a soft semi-closed subsets of $\widetilde{X}$. Suppose that $\widetilde{\bigcap}_{i\in I}(F_i,K)=\widetilde{\emptyset}$. Then $\widetilde{X}=\widetilde{\bigcup}_{i\in I}(F_i^c,K)$. As $(X,\tau,K)$ is soft semi-Lindelöf, then $\widetilde{\bigcup}_{i\in S}(F_i^c,K)=\widetilde{X}$. Therefore $\widetilde{\bigcap}_{i\in S}(F_i,K)=\widetilde{\emptyset}$. Hence, the *necessary* part holds.

Conversely, let $\Lambda = \{(H_i, K) : i \in I\}$ be a soft semi-open cover of $\widetilde{X}$. Suppose that $\Lambda$ has no a countable soft sub-collection which cover $\widetilde{X}$. Then $\widetilde{X} \backslash \bigcup_{i \in S}(H_i, K) \neq \widetilde{\emptyset}$ for any countable set $S$. Now, $\widetilde{\bigcap}_{i \in S}(H_i^c, K) \neq \widetilde{\emptyset}$ implies that $\{(H_i^c, K) : i \in I\}$ is a soft collection of soft semi-closed subsets of $\widetilde{X}$ which has the countable intersection property. Therefore $\widetilde{\bigcap}_{i \in I}(H_i^c, K) \neq \widetilde{\emptyset}$. Thus $\widetilde{X} \neq \widetilde{\bigcup}_{i \in I}(H_i, K)$. But this contradicts that $\Lambda$ is a soft semi-open cover of $\widetilde{X}$. Hence $(X, \tau, K)$ is soft semi-Lindelöf. □

Proposition 3.10. *The soft semi-irresolute image of a soft semi-compact (resp. soft semi-Lindelöf) set is soft semi-compact (resp. soft semi-Lindelöf).*

*Proof.* For the proof, let $g : X \to Y$ be a soft semi-irresolute map and let $(D, K)$ be a soft semi-Lindelöf subset of $\widetilde{X}$. Suppose that $\{(H_i, K) : i \in I\}$ is a soft semi-open cover of $g(D, K)$. Then $g(D, K) \widetilde{\subseteq} \widetilde{\bigcup}_{i \in I}(H_i, K)$. Now, $(D, K) \widetilde{\subseteq} \widetilde{\bigcup}_{i \in I} g^{-1}(H_i, K)$ and $g^{-1}(H_i, K)$ is soft semi-open for each $i \in I$. By hypotheses, $(D, K)$ is soft semi-Lindelöf, then $(D, K) \widetilde{\subseteq} \widetilde{\bigcup}_{i \in S} g^{-1}$ $(H_i, K)$. Therefore $g(D, K) \widetilde{\subseteq} \widetilde{\bigcup}_{i \in S} g(g^{-1}(H_i, K)) \widetilde{\subseteq} \widetilde{\bigcup}_{i \in S}(H_i, K)$. Thus $g(D, K)$ is soft semi-Lindelöf.
A similar proof is given in the case of a soft semi-compact space. □

Lemma 3.11. *If $H$ is a semi-open subset of $(X, \tau_k)$, then there is a soft semi-open subset $(F, K)$ of $(X, \tau, K)$ such that $F(k) = H$.*

*Proof.* Without loss of generality, consider $K = \{k_1, k_2\}$ and let $H(k_1)$ be a semi-open subset of $(X, \tau_1)$. Then there exists a soft open subset $G(k_1)$ of $(X, \tau_1)$ such that $G(k_1) \subseteq H(k_1) \subseteq cl[G(k_1)]$. Since $G(k_2)$ is an open subset of $(X, \tau_2)$, then we choose a subset $H(k_2)$ of $(X, \tau_2)$ to satisfies that $G(k_2) \subseteq H(k_2) \subseteq cl[G(k_2)]$. So $(G, K) \widetilde{\subseteq} (H, K) \subseteq (cl(G), K)$. From Proposition (**??**), we obtain $(cl(G), K) \widetilde{\subseteq} cl(G, K)$. Since $(G, K)$ is soft open, then the proof is complete. □

Theorem 3.12. *If $(X, \tau, K)$ is an enriched soft semi-compact (resp. enriched soft semi-Lindelöf) space, then $(X, \tau_k)$ is semi-compact (resp. semi-Lindelöf) for each $k \in K$.*

*Proof.* We prove the theorem in the case of an enriched soft semi-Lindelöf space and the other case is proven similarly.
Let $\{H_j(k) : j \in J\}$ be a semi-open cover of $(X, \tau_k)$. We construct a soft semi-open cover of $(X, \tau, K)$ consisting of the following soft sets:

**(i):** From the above lemma, we can choose all soft semi-open sets $(F_j, K)$ in which $F_j(k) = H_j(k)$ for each $j \in J$.

**(ii):** Since $(X, \tau, K)$ is enriched, then we chose a soft open set $(G, K)$ which satisfies that $G(k) = \emptyset$ and $G(k_i) = X$ for all $k_i \neq k$.

Obviously, $\{(F_j, K)\widetilde{\bigcup}(G, K) : j \in J\}$ is a soft semi-open cover of $(X, \tau, K)$. As $(X, \tau, K)$ is soft semi-Lindelöf, then $\widetilde{X} = \bigcup_{j \in S}(F_j, K)\widetilde{\bigcup}(G, K)$. So $X = \bigcup_{j \in S} F_j(k) = \bigcup_{j \in S} H_j(k)$. Hence $(X, \tau_k)$ is semi-Lindelöf. □

Proposition 3.13. *If $(X, \tau, K)$ is an enriched soft semi-compact (resp. enriched soft semi-Lindelöf) space, then $K$ is finite (resp. countable).*

*Proof.* Let $(X, \tau, K)$ be soft semi-compact (resp. soft semi-Lindelöf). Since $(X, \tau, K)$ is enriched, then the collection $\{(G, K) : G(k) = X$ and $G(\alpha) = \emptyset$ for each $\alpha \neq k\}$ forms a soft semi-open cover of $(X, \tau, K)$. Hence it must be that $K$ is finite (resp. countable). □

Proposition 3.14. *If $(U, K)$ is soft pre-open and $(H, K)$ is soft semi-open subsets of $(X, \tau, K)$, then $(U, K)\widetilde{\bigcap}(H, K)$ is a soft semi-open subset of $((U, K), \tau_{(U,K)}, K)$.*

*Proof.* Since $(U, K)$ is soft pre-open and $(H, K)$ is soft semi-open subsets of $(X, \tau, K)$, then $(U, K)\widetilde{\bigcap}(H, K)\widetilde{\subseteq}\ int(cl(U, K))\widetilde{\bigcap}cl(int(H, K))\widetilde{\subseteq}$ $cl[int(cl(U, K))\widetilde{\bigcap}int(H, K)]\widetilde{\subseteq}cl[cl(U, K)\widetilde{\bigcap}int(H, K)]\widetilde{\subseteq}cl[(U, K)\widetilde{\bigcap}int(H, K)]$. So $(U, K)\widetilde{\bigcap}(H, K)\widetilde{\subseteq}cl[(U, K)\widetilde{\bigcap}int(H, K)]\widetilde{\bigcap}(U, K) = cl_U[(U, K)\widetilde{\bigcap}int(H, K)]$. Since $int(H, K)$ is soft open subset of $(X, \tau, K)$, then $(U, K)\widetilde{\bigcap}int(H, K)$ is soft open subset of $((U, K), \tau_{(U,K)}, K)$. Thus $(U, K)\widetilde{\bigcap}(H, K)\ \widetilde{\subseteq}cl_U[int_U$ $((U, K)\widetilde{\bigcap}int(H, K))]\widetilde{\subseteq}cl_U[int_U((U, K)\widetilde{\bigcap}(H, K))]$. Hence the proof is complete. □

Corollary 3.15. *If $(U, K)$ is soft open and $(H, K)$ is soft semi-open subsets of $(X, \tau, K)$, then $(U, K)\widetilde{\bigcap}(H, K)$ is a soft semi-open subset of $((U, K), \tau_{(U,K)}, K)$.*

Proposition 3.16. *$(A, K)\widetilde{\bigcap}cl_s(B, K)\widetilde{\subseteq}cl_s((A, K)\widetilde{\bigcap}(B, K))$ for each soft open set $(A, K)$ and soft set $(B, K)$ in $(X, \tau, K)$.*

*Proof.* Let $P_k^x \in (A, K)\widetilde{\bigcap}cl_s(B, K)$. Then $P_k^x \in (A, K)$ and $P_k^x \in cl_s(B, K)$. Therefore for each soft semi-open set $(U, K)$ containing $P_k^x$, we have $(U,K)\widetilde{\bigcap}(B,K) \neq \widetilde{\emptyset}$. Since $(U,K)\widetilde{\bigcap}(A,K)$ is a non-null soft semi-open set and $P_k^x \in (U,K)\widetilde{\bigcap}(A,K)$, then $((U,K)\widetilde{\bigcap}(A,K))\widetilde{\bigcap}(B,K) \neq \widetilde{\emptyset}$. Now, $(U,K)\widetilde{\bigcap}((A,K)\widetilde{\bigcap}(B,K)) \neq \widetilde{\emptyset}$ implies that $P_k^x \in cl_s((A,K)\widetilde{\bigcap}(B,K))$. Hence $(A,K)\widetilde{\bigcap}cl_s(B,K)\widetilde{\subseteq}cl_s((A,K)\widetilde{\bigcap}(B,K))$. □

Lemma 3.17. *If $(U,K)$ is a soft open subset of $(X,\tau,K)$ and $(H,K)$ is soft semi-open subset of $((U,K),\tau_{(U,K)},K)$, then $(H,K)$ is soft semi-open subset of $(X,\tau,K)$.*

*Proof.* Since $(H,K)$ is soft semi-open subset of $((U,K),\tau_{(U,K)},K)$, then $(H,K)\widetilde{\subseteq}cl_U(\ int_U(H,K)) = cl(int_U(H,K))\widetilde{\bigcap}(U,K)\widetilde{\subseteq}cl[int(H,K))\widetilde{\bigcap}(U,K)] = \widetilde{\subseteq}cl(int[(H,K)\widetilde{\bigcap}(U,K)])$. So $(H,K)$ is a soft semi-open subset of $(X,\tau,K)$. □

Now, we are in a position to verify the following result.

Theorem 3.18. *A soft open subset $(A,K)$ of $(X,\tau,K)$ is soft semi-compact (resp. soft semi-Lindelöf) if and only if a soft subspace $((A,K),\tau_{(A,K)},K)$ is soft semi-compact (resp. soft semi-Lindelöf).*

*Proof.* We prove the theorem in the case of soft semi-compactness and the case between parentheses can be proven similarly.
*Necessity:* Let $\{(H_i,K) : i \in I\}$ be a soft semi-open cover of $((A,K),\tau_{(A,K)},K)$. Since $(A,K)$ is a soft open set containing $(H_i,K)$, then it follows, by the above lemma, that $(H_i,K)$ is soft semi-open subsets of $(X,\tau,K)$. By hypotheses, $(A,K)\widetilde{\subseteq}\widetilde{\bigcup}_{i=1}^{i=n}(H_i,K)$. Thus a soft subspace $((A,K),\tau_{(A,K)},K)$ is soft semi-compact.
*Sufficiency:* Let $\{(G_i,K) : i \in I\}$ be a soft semi-open cover of $(A,K)$ in $(X,\tau,K)$. Now, $(A,K)\widetilde{\bigcap}(G_i,K)$ is a soft semi-open subset of $(X,\tau,K)$. By Corollary (3.15), we find that $(A,K)\widetilde{\bigcap}(G_i,K)$ is soft semi-open subset of $((A,K),\tau_{(A,K)},K)$. As a soft subspace $((A,K),\tau_{(A,K)},K)$ is soft semi-compact, then $(A,K)\widetilde{\subseteq}\widetilde{\bigcup}_{i=1}^{i=n}((A,K)\widetilde{\bigcap}(G_i,K))$. This implies that $(A,K)\widetilde{\subseteq}\widetilde{\bigcup}_{i=1}^{i=n}(G_i,K)$. Thus $(A,K)$ is a soft semi-compact subset of $(X,\tau,K)$. □

Lemma 3.19. *The soft intersection of finite soft semi-open subsets of a soft hyperconnected space is soft semi-open.*

*Proof.* Let $(H,K)$ and $(F,K)$ be two soft semi-open subsets of $\widetilde{X}$. If $(H,K)$ or $(F,K)$ are null soft semi-open sets, then the proof is trivial.

So we suppose that $(H,K)$ and $(F,K)$ are two non-null soft semi-open sets. Since $(X,\tau,K)$ is soft hyperconnected, then $int[(H,K)\widetilde{\bigcap}(F,K)] = int(H,K)\widetilde{\bigcap}int(F,K) \neq \widetilde{\emptyset}$. This implies that $cl[int[(H,K)\widetilde{\bigcap}(F,K)]] = \widetilde{X}$. So $(H,K)\widetilde{\bigcap}(F,K)\widetilde{\subseteq}cl[int[(H,K)\widetilde{\bigcap}(F,K)]]$. Hence $(H,K)\widetilde{\bigcap}(F,K)$ is a soft semi-open set. □

PROPOSITION 3.20. *If $(A,K)$ is a soft semi-compact subset of a soft hyperconnected soft semi $T_2'$-space $(X,\tau,K)$, then $(A,K)$ is soft semi-closed.*

*Proof.* Let the given conditions be satisfied and let $P_k^x \in (A,K)^c$. Then for each $P_l^y \in (A,K)$, there are two disjoint soft semi-open sets $(G_i,K)$ and $(W_i,K)$ such that $P_k^x \in (G_i,K)$ and $P_l^y \in (W_i,K)$. It follows that $\{(W_i,K) : i \in I\}$ forms a soft semi-open cover of $(A,K)$. Consequently, $(A,K)\widetilde{\subseteq}\widetilde{\bigcup}_{i=1}^{i=n}(W_i,K)$. Since $(X,\tau,K)$ is soft hyperconnected, then $\widetilde{\bigcap}_{i=1}^{i=n}(G_i,K) = (H,K)$ is a soft semi-open set and since $(H,K)\widetilde{\bigcap}[\widetilde{\bigcup}_{i=1}^{i=n}(W_i,K)] = \widetilde{\emptyset}$, then $(H,K)\widetilde{\subseteq}(A,K)^c$. Since $P_k^x$ is chosen arbitrary, then $(A,K)^c$ is a soft semi-open set. Hence $(A,K)$ is soft semi-closed. □

COROLLARY 3.21. *If $(A,K)$ is a stable soft semi-compact subset of a soft hyperconnected soft semi $T_2$-space $(X,\tau,K)$, then $(A,K)$ is soft semi-closed.*

*Proof.* Since $(A,K)$ is stable, then $P_k^x \in (A,K)$ if and only if $x \in (A,K)$. So by using the similar technique given in the above proof, the corollary holds. □

## 4. Almost soft semi-compact spaces

DEFINITION 4.1. An STS $(X,\tau,K)$ is called almost soft semi-compact (resp. almost soft semi-Lindelöf) if every soft semi-open cover of $\widetilde{X}$ has a finite (resp. countable) sub-cover such that the soft semi-closures of whose members cover $\widetilde{X}$.

For the sake of brevity, the proofs of the following three propositions will be omitted.

PROPOSITION 4.2. *Every almost soft semi-compact space is almost soft semi-Lindelöf.*

Proposition 4.3. *Every soft semi-compact (resp. soft semi-Lindelöf) space is almost soft semi-compact (resp. almost soft semi-Lindelöf).*

Proposition 4.4. *A finite (resp. countable) union of almost soft semi-compact (resp. almost soft semi-Lindelöf) subsets of* $(X, \tau, K)$ *is almost soft semi-compact (resp. almost soft semi-Lindelöf).*

The converse of Proposition (4.2) and Proposition (4.3) are not true as explained in the two example below.

Example 4.5. Let $\tau$ be the discrete soft topology on the set of integer numbers $\mathcal{Z}$ under a parameters set $K = \{k_1, k_2\}$. Then $(\mathcal{Z}, \tau, K)$ is soft semi-Lindelöf, but not almost soft semi-compact.

Example 4.6. Consider $K$ is any set of parameters and let $\tau = \{\widetilde{\emptyset}, (G_i, K)\widetilde{\subseteq}\widetilde{\mathcal{R}} : 1 \in (G_i, K)\}$ be a soft topology on the set of real numbers $\mathcal{R}$. Since the soft semi closure of any soft semi-open set is $\widetilde{\mathcal{R}}$, then $(\mathcal{R}, \tau, K)$ is almost soft semi-compact. On the other hand, the collection $\{(G, E) : G(k) = \{1, x\}$ for each $k \in K\}$ forms a soft semi-open cover of $\widetilde{X}$. Since this collection has not a countable sub-cover of $\widetilde{X}$, then $(\mathcal{R}, \tau, K)$ is not soft semi-Lindelöf.

Definition 4.7. A soft subset $(D, K)$ of $(X, \tau, K)$ is called soft semi-clopen provided that it is soft semi-open and soft semi-closed.

Proposition 4.8. *Every soft semi-clopen subset* $(D, K)$ *of an almost soft semi-compact (resp. almost soft semi-Lindelöf) space* $(X, \tau, K)$ *is almost soft semi-compact (resp. almost soft semi-Lindelöf).*

*Proof.* We only prove the proposition when $(X, \tau, K)$ is almost soft semi-compact. The other case is made similarly.
Let $(D, K)$ be a soft semi-clopen subset of $\widetilde{X}$ and $\{(H_i, K) : i \in I\}$ be a soft semi-open cover of $(D, K)$. Then $(D^c, K)$ is soft semi-clopen and $\widetilde{X} = \widetilde{\bigcup}_{i\in I}(H_i, K)\widetilde{\bigcup}(D^c, K)$. Since $\widetilde{X}$ is almost soft semi-compact, then $\widetilde{X} = \widetilde{\bigcup}_{i=1}^{i=n} cl_s(H_i, K)\,\widetilde{\bigcup}(D^c, K)$. This implies that $(D, K)\widetilde{\subseteq}\widetilde{\bigcup}_{i=1}^{i=n} cl_s(H_i, K)$. Hence $(D, K)$ is almost soft semi-compact. □

In Example (4.6), let $(H, K)$ be a soft subset of $(\mathcal{R}, \tau, K)$, where $H(k_1) = \{1, 4\}$ and $H(k_2) = \{4, 5\}$. Then $(H, K)$ is almost soft semi-compact, but it is not soft semi-clopen. So the converse of the above proposition is not necessarily true.

Proposition 4.9. *If $(G,K)$ is an almost soft semi-compact (resp. almost soft semi-Lindelöf) subset of $\widetilde{X}$ and $(D,K)$ is a soft semi-clopen subset of $\widetilde{X}$, then $(G,K)\widetilde{\bigcap}(D,K)$ is almost soft semi-compact (resp. almost soft semi-Lindelöf).*

*Proof.* For the proof, let $\Lambda = \{(H_i,K) : i \in I\}$ be a soft semi-open cover of $(G,K)\widetilde{\bigcap}(D,K)$. Then $(G,K)\widetilde{\subseteq}\widetilde{\bigcup}_{i\in I}(H_i,K)\widetilde{\bigcup}(D^c,K)$. Because $(G,K)$ is almost soft semi-compact, then $(G,K)\widetilde{\subseteq}\widetilde{\bigcup}_{i=1}^{i=n} cl_s(H_i,K)\widetilde{\bigcup}(D^c,K)$. So $(G,K)\widetilde{\bigcap}(D,K)\widetilde{\subseteq}\ \widetilde{\bigcup}_{i=1}^{i=n} cl_s(H_i,K)$. Hence $(G,K)\widetilde{\bigcap}(D,K)$ is almost soft semi-compact.
A similar proof is given in the case of an almost soft semi-Lindelöf space. □

Definition 4.10. The collection $\Lambda = \{(F_i,K) : i \in I\}$ of soft sets is said to have the first type of finite (resp. countable) semi-intersection property if $\widetilde{\bigcap}_{i\in M} int_s(F_i,K) \neq \widetilde{\emptyset}$ for any finite (resp. countable) set $M$.

It is clear that the collection which satisfies the first type of finite (resp. countable) semi-intersection property, it also satisfies the finite (resp. countable) intersection property.

Theorem 4.11. *An STS $(X,\tau,K)$ is almost soft semi-compact (resp. almost soft semi-Lindelöf) if and only if every collection of soft semi-closed subsets of $(X,\tau,K)$, satisfying the first type of finite (resp. countable) semi-intersection property, has, itself, a non-null soft intersection.*

*Proof.* We only prove the theorem when $(X,\tau,K)$ is almost soft semi-compact. The other case can be made similarly.
Let $\Lambda = \{(F_i,K) : i \in I\}$ be a soft semi-closed subsets of $\widetilde{X}$. Suppose that $\widetilde{\bigcap}_{i\in I}(F_i,K) = \widetilde{\emptyset}$. Then $\widetilde{X} = \widetilde{\bigcup}_{i\in I}(F_i^c,K)$. As $(X,\tau,K)$ is almost soft semi-compact, then $\widetilde{X} = \widetilde{\bigcup}_{i=1}^{i=n} cl_s(F_i^c,K)$. Therefore $\widetilde{\emptyset} = (\widetilde{\bigcup}_{i=1}^{i=n} cl_s(F_i^c,K))^c = \widetilde{\bigcap}_{i=1}^{i=n} int_s(F_i,K)$. Hence the *necessary* condition holds.
Conversely, let $\Lambda$ be the collection of soft semi-closed subsets of $\widetilde{X}$ which satisfies the first type of finite semi-intersection property. Then it also satisfies the finite intersection property. Since $\Lambda$ has a non-null soft intersection, then $(X,\tau,K)$ is a soft semi-compact space. It follows, by Proposition (4.3), that $(X,\tau,K)$ is almost soft semi-compact. □

Theorem 4.12. *Consider $g : (X, \tau, K) \to (Y, \theta, K)$ is a soft map. Then the following statements are equivalent:*

**(i):** *$g$ is soft semi-irresolute;*

**(ii):** *The inverse image of each soft semi-closed subset of $\widetilde{Y}$ is a soft semi-closed subset of $\widetilde{X}$;*

**(iii):** *$cl_s(g^{-1}(A, K)) \subseteq g^{-1}(cl_s(A, K))$ for each soft subset $(A, K)$ of $\widetilde{Y}$;*

**(iv):** *$g(cl_s(E, K)) \subseteq cl_s(g(E, K))$ for each soft subset $(E, K)$ of $\widetilde{X}$;*

**(v):** *$g^{-1}(int_s(A, K)) \subseteq int_s(g^{-1}(A, K))$ for each soft subset $(A, K)$ of $\widetilde{Y}$.*

*Proof.* **(i) ⇒ (ii)**: Suppose that $(F, K)$ is a soft semi-closed subset of $\widetilde{Y}$. Then $(F^c, K)$ is soft semi-open. Therefore $g^{-1}(F^c, K)$ is a soft semi-open subset of $\widetilde{X}$. From the fact that $g^{-1}(F^c, K) = \widetilde{X} \setminus g^{-1}(F, K)$. Hence $g^{-1}(F, K)$ is a soft semi-closed subset of $\widetilde{X}$.
**(ii) ⇒ (iii)**: For any soft subset $(A, K)$ of $\widetilde{Y}$, we get that $cl_s(A, K)$ is a soft semi-closed subset of $\widetilde{Y}$. Since $g^{-1}(cl_s(A, K))$ is a soft semi-closed subset of $\widetilde{X}$, then $cl_s(g^{-1}(A, K))\widetilde{\subseteq}\, cl_s(g^{-1}(cl_s(A, K))) = g^{-1}(cl_s(A, K))$.
**(iii) ⇒ (iv)**: For any soft subset $(E, K)$ of $\widetilde{X}$, we have $cl_s(E, K)\widetilde{\subseteq} cl_s(g^{-1}$ $(g(E, K))$. By **(iii)**, we find that $cl_s(g^{-1}(g(E, K)))\widetilde{\subseteq} g^{-1}(cl_s(g(E, K)))$. Hence $g(cl_s(E, K))\widetilde{\subseteq}\, g(g^{-1}(cl_s(g(E, K))))\, \widetilde{\subseteq} cl_s(g(E, K))$.
**(iv) ⇒ (v)**: Let $(A, K)$ be any soft subset of $\widetilde{Y}$. Then $g(cl_s(\widetilde{X} \setminus g^{-1}(A, K)))\widetilde{\subseteq} cl_s(g(X - g^{-1}(A, K)))$. Therefore $g(\widetilde{X} \setminus int_s(g^{-1}(A, K))) = g(cl_s(\widetilde{X} \setminus g^{-1}(A, K)))\widetilde{\subseteq} cl_s(\widetilde{Y} \setminus (A, K)) = \widetilde{Y} \setminus int_s(A, K)$. Thus $\widetilde{X} \setminus int_s(g^{-1}(A, K))\widetilde{\subseteq} g^{-1}(\widetilde{Y} \setminus int_s(A, K)) = g^{-1}(\widetilde{Y}) \setminus g^{-1}(int_s(A, K))$. Hence $g^{-1}(int_s(A, K))\widetilde{\subseteq} int_s(g^{-1}(A, K))$.
**(v) ⇒ (i)**: Suppose that $(A, K)$ is any soft semi-open subset of $\widetilde{Y}$. Since $g^{-1}(int_s(A, K))\, \widetilde{\subseteq} int_s(g^{-1}(A, K))$, then $g^{-1}(A, K)\widetilde{\subseteq} int_s(g^{-1}(A, K))$. Since $int_s(g^{-1}(A, K))\widetilde{\subseteq} g^{-1}(A, K)$, then $g^{-1}(A, K) = int_s(g^{-1}(A, K))$. Therefore $g^{-1}(A, K)$ is a soft semi-open set. Hence $g$ is a soft semi-irresolute map. □

Proposition 4.13. *The soft semi-irresolute image of an almost soft semi-compact (resp. almost soft semi-Lindelöf) set is almost soft semi-compact (resp. almost soft semi-Lindelöf).*

*Proof.* To prove the proposition in the case of almost soft semi- compactness, let $g : X \to Y$ be a soft semi-irresolute map and $(D, K)$ be

an almost soft semi-Lindelöf subset of $\widetilde{X}$. Suppose that $\{(H_i, K) : i \in I\}$ is a soft semi-open cover of $g(D, K)$. Then $g(D, K)\widetilde{\subseteq}\widetilde{\bigcup}_{i\in I}(H_i, K)$. Now, $(D, K)\widetilde{\subseteq}\widetilde{\bigcup}_{i\in I}g^{-1}(H_i, K)$ and $g^{-1}(H_i, K)$ is a soft semi-open set for each $i \in I$. By hypotheses, $(D, K)$ is almost soft semi-Lindelöf, then $(D, K)\widetilde{\subseteq}\widetilde{\bigcup}_{i\in S}cl_s(g^{-1}(H_i, K))$. So $g(D, K)\widetilde{\subseteq}\widetilde{\bigcup}_{i\in S}g(cl_s(g^{-1}(H_i, K)))$. From item **(iv)** of the above theorem, we obtain $g(cl_s(g^{-1}(H_i, K)))\widetilde{\subseteq}cl_s(g(g^{-1}(H_i, K)))\widetilde{\subseteq}cl_s(H_i, K)$. Thus $g(D, K)\widetilde{\subseteq}\widetilde{\bigcup}_{i\in S}cl_s(H_i, K)$. Hence $g(D, K)$ is almost soft semi-Lindelöf.
A similar proof is given in the case of an almost soft semi-compact space. □

Definition 4.14. An STS $(X, \tau, K)$ is said to be soft semi-hyperconnected if it does not contain disjoint soft semi-open sets.

Proposition 4.15. *Every soft semi-hyperconnected space $(X, \tau, K)$ is almost soft semi-compact.*

*Proof.* Since the soft semi closure of any soft semi-open subset of a soft semi-hyperconnected space $(X, \tau, K)$ is the absolute soft set $\widetilde{X}$, then the result holds. □

The converse of this proposition is not necessarily true as shown in the following example.

Example 4.16. Let $K = \{k_1, k_2\}$ be a set of parameters and $\tau = \{\widetilde{\emptyset},\ (G, K)\widetilde{\subseteq}\widetilde{\mathcal{R}} : \text{either } [1 \in (G, K) \text{ and } (G^c, K) \text{ is finite}] \text{ or } [1 \notin (G, K)]\}$ be a soft topology on the set of real numbers $\mathcal{R}$. On the one hand, the relative complement of any soft open set containing $\{1\}$ is finite. Then $(\mathcal{R}, \tau, K)$ is almost soft semi-compact. On the other hand, $(G, K)$ and $(H, K)$, where $G(k_1) = G(k_2) = \mathcal{R} \setminus \{5, 6\}$ and $H(k_1) = H(k_2) = \{5\}$ are two disjoint soft semi-open sets. Then $(\mathcal{R}, \tau, K)$ is not soft semi-hyperconnected.

Definition 4.17. Let $(F, K)$ be a soft subset of $(X, \tau, K)$. Then $(cl_s(F), K)$ is defined as $cl_s(F)(k) = cl_s(F(k))$, where $cl_s(F(k))$ is the semi-closure of $F(k)$ in $(X, \tau_k)$ for each $k \in K$.

Proposition 4.18. *Let $(L, E)$ be a soft subset of an enriched STS $(X, \tau, K)$. Then:*

**(i):** $(cl_s(L), K)\widetilde{\subseteq}cl_s(L, K)$.

**(ii):** $(cl_s(L), K) = cl_s(L, K)$ *if and only if* $(cl_s(L), K)$ *is soft semi-closed.*

*Proof.* **(i):** For any $k \in K$, $cl_s(L(k))$ is the smallest semi-closed subset of $(X, \tau_k)$ containing $L(k)$. Putting $cl_s(L, K) = (F, K)$ then $F(k)$ is a semi-closed subset of $(X, \tau_k)$ containing $L(k)$ as well. This means that $((cl_s(L))(k) = cl_s(L(k)) \subseteq F(k)$. Hence $(cl_s(L), K) \widetilde{\subseteq} cl_s(L, K)$.

**(ii):** If $(cl_s(L), K) = cl_s(L, K)$, then $(cl_s(L), K)$ is a soft semi-closed set. Conversely, let $(cl_s(L), K)$ be a soft semi-closed set. Obviously, $(cl_s(L), K)$ containing $(L, K)$. So from the definition of soft semi-closure of $(L, K)$, we infer that $cl_s(L, K) \widetilde{\subseteq} (cl_s(L), K)$ and from **(i)** above, we obtain that $(cl_s(L), K) \widetilde{\subseteq} cl_s(L, K)$. Hence $(cl_s(L), K) = cl_s(L, K)$.

□

LEMMA 4.19. *Let* $(H, K)$ *be a soft subset of an enriched soft topological space* $(X, \tau, K)$. *If* $H(k)$ *is a non-empty subset of* $(X, \tau_k)$ *and* $H(k_j) = \emptyset$ *for each* $k_j \neq k$, *then* $(cl_s(H), K)$ *is soft semi-closed and* $(cl_s(H), K) = cl_s(H, K)$.

*Proof.* Assume that $k = k_n$ and $(cl_s(H), K) = \{(k_i, cl_s(H(k_i)) : i \in I\}$. Let $P^x_{k_m} \in cl_s(H, K)$. As $(X, \tau, K)$ is enriched, then $k = k_n$. Now, for each soft semi-open set $(W, K)$ containing $P^x_{k_n}$, we have $(W, K) \widetilde{\bigcap} (H, K) \neq \widetilde{\emptyset}$. Therefore $W(k_n) \bigcap H(k_n) \neq \emptyset$. It follows, by Lemma (3.11), that for each semi-open set $L(k_n)$ in $(X, \tau_{k_n})$ containing $x$, we have that $L(k_n) \bigcap H(k_n) \neq \emptyset$. This implies that $x \in cl_s(H(k_n))$. Thus $P^x_{k_n} \in (cl_s(H), K)$. Hence $cl_s(H, K) \widetilde{\subseteq} (cl_s(H), K)$. It follows, from Proposition (4.18), that $cl_s(H, K) = (cl_s(H), K)$. □

THEOREM 4.20. *If* $(X, \tau, K)$ *is an enriched almost soft semi-compact (resp. enriched almost soft semi-Lindelöf) space, then* $(X, \tau_k)$ *is almost semi-compact (resp. almost semi-Lindelöf) for each* $k \in K$.

*Proof.* We prove the theorem in the case of an almost soft semi-compact space. The case between parenthesis is made similarly.
Let $\{H_j(k) : j \in J\}$ be a semi-open cover for $(X, \tau_k)$. We construct a soft semi-open cover for $\widetilde{X}$ like the soft semi-open cover which initiated in the proof of Theorem (3.12). Now, $(X, \tau, K)$ is almost soft semi-compact implies that $\widetilde{X} = \bigcup_{j=1}^{j=n} cl_s[(F_j, K) \widetilde{\bigcup} (G, K)] = \bigcup_{j=1}^{j=n} [(cl_s(F_j), K) \widetilde{\bigcup} (G, K)]$.

Therefore $X = \bigcup_{j=1}^{j=n} cl_s(F_j(k)) = \bigcup_{j=1}^{j=n} cl_s(H_j(k))$. Hence $(X, \tau_k)$ is almost semi-compact. □

PROPOSITION 4.21. *If $(X, \tau, K)$ is an enriched almost soft semi-compact (resp. enriched almost soft semi-Lindelöf) space, then $K$ is finite (resp. countable).*

*Proof.* Let $(X, \tau, K)$ be almost soft semi-compact (resp. almost soft semi-Lindelöf). Since $(X, \tau, K)$ is enriched, then the collection $\{(G, K) : G(k) = X$ and $G(\alpha) = \emptyset$ for each $\alpha \neq k\}$ forms a soft semi-open cover of $(X, \tau, K)$. Since every soft open set $(G, K)$ in this collection is soft closed, then $cl_s(G, K) = (G, K)$. Hence it must be that $K$ is finite (resp. countable). □

PROPOSITION 4.22. *Consider $((U, K), \tau_{(U,K)}, K)$ is a soft subspace of $(X, \tau, K)$. Let $cl_s$ and $int_s$ stand for the soft semi-closure and soft semi-interior operators, respectively, in $(X, \tau, K)$ and Let $cl_{sU}$ and $int_{sU}$ stand for the soft semi-closure and soft semi-interior operators, respectively, in $((U, K), \tau_{(U,K)}, K)$. Then, for each $(A, K)\widetilde{\subseteq}(U, K)$, we have the following results:*

**(i):** $cl_{sU}(A, K) = cl_s(A, K)\widetilde{\bigcap}(U, K)$.

**(ii):** $int_s(A, K) = int_{sU}(A, K))\widetilde{\bigcap} int_s(U, K)$.

THEOREM 4.23. *A soft open subset $(A, K)$ of $(X, \tau, K)$ is almost soft semi-compact (resp. almost soft semi-Lindelöf) if and only if a soft subspace $((A, K), \tau_{(A,K)}, K)$ is almost soft semi-compact (resp. almost soft semi-Lindelöf).*

*Proof. Necessity*: Let $\{(H_i, K) : i \in I\}$ be a soft semi-open cover of $((A, K), \tau_{(A,K)}, K)$. Since $(A, K)$ is soft open containing $(H_i, K)$, then it follows, from Lemma (3.17), that $(H_i, K)$ is a soft semi-open subset of $(X, \tau, K)$. By hypotheses, $(A, K)\widetilde{\subseteq}\widetilde{\bigcup}_{i=1}^{i=n} cl_s(H_i, K) = \widetilde{\bigcup}_{i=1}^{i=n}[cl_s(H_i, K)\widetilde{\bigcap} (A, K)] = \widetilde{\bigcup}_{i=1}^{i=n} cl_{sU}(H_i, K)$. Thus a soft open subspace $((A, K), \tau_{(A,K)}, K)$ is almost soft semi-compact.

*Sufficiency:* Let $\{(G_i, K) : i \in I\}$ be a soft semi-open cover of $(A, K)$ in $(X, \tau, K)$. Now, $(A, K)\widetilde{\bigcap}(G_i, K)$ is a soft semi-open subset of $(X, \tau, K)$. By Corollary (3.15), we find that $(A, K)\widetilde{\bigcap}(G_i, K)$ is a soft semi-open subset of $((A, K), \tau_{(A,K)}, K)$. Since a soft open subspace $((A, K), \tau_{(A,K)}, K)$

is almost soft semi-compact, then $(A,K)\widetilde{\subseteq}\widetilde{\bigcup}_{i=1}^{i=n} cl_{sU}[(A,K)\widetilde{\bigcap}(G_i,K)]\widetilde{\subseteq}\widetilde{\bigcup}_{i=1}^{i=n} cl_{sU}(G_i,K)$. So $(A,K)\widetilde{\subseteq}\widetilde{\bigcup}_{i=1}^{i=n} cl_s(G_i,K)$. Thus $(A,K)$ is an almost soft semi-compact subset of $(X,\tau,K)$.
The case between parentheses can be proven similarly. □

Proposition 4.24. *If $(A,K)$ is an almost soft semi-compact subset of a soft hyperconnected soft semi $T_2'$-space $(X,\tau,K)$, then $(A,K)$ is soft semi-closed.*

*Proof.* Let the given conditions be satisfied and let $P_k^x \in (A,K)^c$. Then for each $P_k^y \in (A,K)$, there are two disjoint soft semi-open sets $(G_i,K)$ and $(W_i,K)$ such that $P_k^x \in (G_i,K)$ and $P_k^y \in (W_i,K)$. It follows that $\{(W_i,K) : i \in I\}$ forms a soft semi-open cover of $(A,K)$. Consequently, $(A,K)\widetilde{\subseteq}\widetilde{\bigcup}_{i=1}^{i=n} cl_s(W_i,K)$. Since $(X,\tau,K)$ is soft hyperconnected, then $\widetilde{\bigcap}_{i=1}^{i=n}(G_i,K) = (H,K)$ is a soft semi-open set and since $(H,K)\widetilde{\bigcap}(W_i,K) = \widetilde{\emptyset}$, then $(H,K)\widetilde{\bigcap}[\widetilde{\bigcup}_{i=1}^{i=n} cl_s(W_i,K)] = \widetilde{\emptyset}$. So $(H,K)\widetilde{\subseteq}(A,K)^c$. Thus $(A,K)^c$ is a soft semi-open set. Hence $(A,K)$ is soft semi-closed. □

Corollary 4.25. *If $(A,K)$ is a stable almost soft semi-compact subset of a soft hyperconnected soft semi $T_2$-space $(X,\tau,K)$, then $(A,K)$ is soft semi-closed.*

## 5. Approximately soft semi-compact spaces

Definition 5.1. An STS $(X,\tau,K)$ is called approximately soft semi-compact (resp. approximately soft semi-Lindelöf) space if every soft semi-open cover of $\widetilde{X}$ has a finite (resp. countable) sub-cover in which its soft semi-closure cover $\widetilde{X}$.

Proposition 5.2. *Every approximately soft semi-compact space is approximately soft semi-Lindelöf.*

*Proof.* The proof is straightforward. □

Proposition 5.3. *Every almost soft semi-compact (resp. almost soft semi-Lindelöf) space is approximately soft semi-compact (resp. approximately soft semi-Lindelöf).*

*Proof.* Since $\widetilde{\bigcup}_{i\in I} cl_s(G_i, K) \widetilde{\subseteq} cl_s(\widetilde{\bigcup}_{i\in I}(G_i, K))$, then the proposition is satisfied. □

COROLLARY 5.4. *Every soft semi-hyperconnected space is approximately soft semi-Lindelöf.*

The following example shows that the converse of the above proposition is not true.

EXAMPLE 5.5. Consider $(\mathcal{R}, \tau, K)$ is a soft topological space such that $K$ is any set of parameters and $\tau = \{\widetilde{\emptyset}, \widetilde{\mathcal{R}}, (G_1, K), (G_2, K), (G_3, K)\}$, where the soft sets in $\tau$ defined as follows:

$(G_1, K) = \{(k_i, \{1\}) : k_i \in K\}$
$(G_2, K) = \{(k_i, \{2\}) : k_i \in K\}$ and
$(G_3, K) = \{(k_i, \{1, 2\}) : k_i \in K\}$.

Then any soft set $(G, K)$ is soft semi-open if and only if $1 \in (G, K)$ or $2 \in (G, K)$. Now, we define a soft semi-open cover $\Lambda$ of $\widetilde{X}$ as follows: $\Lambda = \{(G, K) = \{(k_i, \{1, x\}) : k_i \in K$ and $x \neq 2$; and $(H, E) : H(k) = \{2\}\}$. This soft semi-open cover has not a countable sub-cover in which its soft semi-closure of whose members cover $\widetilde{X}$, hence $(\mathcal{R}, \tau, K)$ is not almost soft semi-Lindelöf. On the other hand, for any soft semi-open cover, we can choose a finite number of soft semi-open sets which contains a soft semi-open set $(G_3, E)$. Because $(G_3, E)$ is soft semi-dense, then $(\mathcal{R}, \tau, K)$ is approximately soft semi-compact.

PROPOSITION 5.6. *A finite (resp. countable) union of approximately soft semi-compact (resp. approximately soft semi-Lindelöf) subsets of $(X, \tau, K)$ is approximately soft semi-compact (resp. approximately soft semi-Lindelöf).*

*Proof.* Let $\{(A_s, K) : s \in S\}$ be approximately soft semi-Lindelöf subsets of $(X, \tau, K)$ and let $\{(G_i, K) : i \in I\}$ be a soft semi-open cover of $\widetilde{\bigcup}_{s\in S}(A_s, K)$. Then there exist countable sets $M_s$ such that $(A_1, K) \widetilde{\subseteq} cl_s(\widetilde{\bigcup}_{i\in M_1}(G_i, K)), \dots, (A_n, K) \widetilde{\subseteq} cl_s(\widetilde{\bigcup}_{i\in M_n}(G_i, K)), \dots$. Therefore $\widetilde{\bigcup}_{s\in S}(A_s, K) \widetilde{\subseteq} cl_s(\widetilde{\bigcup}_{i\in M_1}(G_i, K)) \widetilde{\bigcup} \dots \widetilde{\bigcup} cl_s(\widetilde{\bigcup}_{i\in M_n}(G_i, K)) \widetilde{\bigcup} \dots \widetilde{\subseteq} cl_s$ $(\widetilde{\bigcup}_{i\in \bigcup_{s\in S} M_s}(G_i, K))$.

A similar proof is given in the case of an approximately soft semi-compact space. □

DEFINITION 5.7. The collection $\Lambda = \{(F_i, K) : i \in I\}$ of soft sets is said to have the second type of finite (resp. countable) semi-intersection property if $int_s[\widetilde{\bigcap}_{i\in M}(F_i, K)] \neq \widetilde{\emptyset}$ for any finite (resp. countable) set $M$.

It is clear that the collection which satisfies the second type of finite (resp. countable) semi-intersection property, it also satisfies the first type of finite (resp. countable) semi-intersection property.

THEOREM 5.8. *An STS $(X, \tau, K)$ is approximately soft semi-compact (resp. approximately soft semi-Lindelöf) if and only if every collection of soft semi-closed subsets of $(X, \tau, K)$, satisfying the second type of finite (resp. countable) semi-intersection property, has, itself, a non-null soft intersection.*

*Proof.* We only prove the theorem when $(X, \tau, K)$ is approximately soft semi-compact, the other case can be made similarly.
Let $\Lambda = \{(F_i, K) : i \in I\}$ be a soft semi-closed subsets of $\widetilde{X}$. Suppose that $\widetilde{\bigcap}_{i\in I}(F_i, K) = \widetilde{\emptyset}$. Then $\widetilde{X} = \widetilde{\bigcup}_{i\in I}(F_i^c, K)$. As $(X, \tau, K)$ is approximately soft semi-compact, then $\widetilde{X} = cl_s(\widetilde{\bigcup}_{i=1}^{i=n}(F_i^c, K))$. Therefore $\widetilde{\emptyset} = (cl_s(\widetilde{\bigcup}_{i=1}^{i=n}(F_i^c, K)))^c = int_s(\widetilde{\bigcap}_{i=1}^{i=n}(F_i, K))$. Hence the *necessary* condition holds.
Conversely, Let $\Lambda$ be a soft semi-closed subsets of $\widetilde{X}$ which satisfies the second type of finite semi-intersection property. Then it also satisfies the first type of finite semi-intersection property. Since $\Lambda$ has a non-null soft intersection, then $(X, \tau, K)$ is an almost soft semi-compact space. It follows, by Proposition (5.3), that $(X, \tau, K)$ is approximately soft semi-compact. □

DEFINITION 5.9. A topological space $(X, \tau)$ is called approximately semi-compact (resp. approximately semi-Lindelöf) space if every semi-open cover of $X$ has a finite (resp. countable) sub-cover in which its semi-closure cover $X$.

THEOREM 5.10. *A soft open subset $(A, K)$ of $(X, \tau, K)$ is approximately soft semi-compact (resp. approximately soft semi-Lindelöf) if and only if a soft subspace $((A, K), \tau_{(A,K)}, K)$ is approximately soft semi-compact (resp. approximately soft semi-Lindelöf).*

*Proof.* The proof is similar of that Theorem (4.23). □

DEFINITION 5.11. A soft set $(F, K)$ is called soft semi-dense set provided that $cl_s(F, K) = \widetilde{X}$.

Proposition 5.12. *If there exists a finite (resp. countable) soft semi-dense subset of* $(X, \tau, K)$ *such that* $K$ *is finite (resp. countable), then* $(X, \tau, K)$ *is approximately soft semi-compact (resp. approximately soft semi-Lindelöf).*

*Proof.* Let $\Lambda = \{(G_i, K) : i \in I\}$ be a soft semi-open cover of $(X, \tau, K)$ and let $(B, K)$ be a finite (resp. countable) soft semi-dense subset of $(X, \tau, K)$. Then for each $P_{k_s}^{x_s} \in (B, K)$, there exists $(G_{x_s}, K) \in \Lambda$ containing $P_{k_s}^{x_s}$. This implies that $\widetilde{X} = cl_s[\widetilde{\bigcup}(G_{x_s}, K)]$. Since $K$ is finite (resp. countable), then the collection $\{(G_{x_s}, K)\}$ is finite (resp. countable). Hence the proof is complete. □

Proposition 5.13. *The soft semi-irresolute image of an approximately soft semi-compact (resp. approximately soft semi-Lindelöf) set is approximately soft semi-compact (resp. approximately soft semi-Lindelöf).*

*Proof.* By using a similar technique of the proof of Proposition (4.13) and employing item **(iii)** of Theorem (4.12), this theorem holds. □

Proposition 5.14. *If* $(A, K)$ *is an approximately soft semi-compact subset of a soft hyperconnected soft semi* $T_2'$*-space* $(X, \tau, K)$*, then* $(A, K)$ *is soft semi-closed.*

*Proof.* The proof is similar of that Proposition (4.24). □

Corollary 5.15. *If* $(A, K)$ *is a stable approximately soft semi-compact subset of a soft hyperconnected soft semi* $T_2$*-space* $(X, \tau, K)$*, then* $(A, K)$ *is soft semi-closed.*

## 6. Mildly soft semi-compact spaces

Definition 6.1. An STS $(X, \tau, K)$ is called mildly soft semi-compact (resp. mildly soft semi-Lindelöf) if every soft semi-clopen cover of $\widetilde{X}$ has a finite (resp. countable) soft subcover.

The proofs of the next two propositions are easy and so will be omitted.

Proposition 6.2. *A finite (resp. countable) union of mildly soft semi-compact (resp. mildly soft semi-Lindelöf) subsets of* $(X, \tau, K)$ *is mildly soft semi-compact (resp. mildly soft semi-Lindelöf).*

Proposition 6.3. *Every mildly soft semi-compact space is mildly soft semi-Lindelöf.*

If we replace a word "finite" by "countable" in Example (4.16). then $(\mathcal{R}, \tau, K)$ is mildly soft semi-Lindelöf. However, it is not mildly soft semi-compact. Hence the converse of Proposition (6.3) fails.

Proposition 6.4. *Every almost soft semi-compact (resp. almost soft semi-Lindelöf) space $(X, \tau, K)$ is mildly soft semi-compact (resp. mildly soft semi-Lindelöf).*

*Proof.* When $(X, \tau, K)$ is almost soft semi-Lindelöf.
Let $\Lambda = \{(H_i, K) : i \in I\}$ be a soft semi-clopen coverof $(X, \tau, K)$. Then $\widetilde{X} = \widetilde{\bigcup}_{s \in S} cl_s(H_i, K)$. Now, $cl_s(H_i, K) = (H_i, K)$. Therefore $(X, \tau, K)$ is mildly soft semi-Lindelöf.
A similar proof is given when $(X, \tau, K)$ is mildly soft semi-compact. □

Corollary 6.5. *Every soft semi-compact (resp. soft semi-Lindelöf) space is mildly soft semi-compact (resp. mildly soft semi-Lindelöf).*

Corollary 6.6. *If $(X, \tau, K)$ is soft semi-hyperconnected, then the following six properties are equivalent:*
**(i):** *Almost soft semi-compact;*
**(ii):** *Almost soft semi-Lindelöf;*
**(iii):** *Approximately soft semi-compact;*
**(iv):** *Approximately soft semi-Lindelöf;*
**(v):** *Mildly soft semi-compact;*
**(vi):** *Mildly soft semi-Lindelöf.*

Proposition 6.7. *Every soft semi-connected space $(X, \tau, K)$ is mildly soft semi-compact.*

*Proof.* In view of $(X, \tau, K)$ is soft semi-connected, then $\widetilde{X}$ and $\widetilde{\emptyset}$ the only soft semi-clopen subsets of $(X, \tau, K)$. Therefore it is mildly soft semi-compact. □

The next example illustrates that the converse of the above proposition fails.

Example 6.8. Let $(X, \tau, K)$ be the same as in Example (4.16). On the one hand, $(X, \tau, K)$ is mildly soft semi-compact. On the other hand, the following soft sets:

$(G, K) = \{(k_1, \mathcal{R} \setminus \{5\}), (k_2, \mathcal{R} \setminus \{5\})\}$ and

$$(H, K) = \{(k_1, \{5\}), (k_2, \{5\})\}$$

are two disjoint soft open sets. Since their union is $\widetilde{\mathcal{R}}$, then $(\widetilde{\mathcal{R}}, \tau, K)$ is soft semi-disconnected.

In the next example, we illuminate that an approximately soft semi-compact space need not be mildly soft semi-Lindelöf.

EXAMPLE 6.9. Assume that $(\mathcal{R}, \tau, K)$ is the same as in Example (5.5). We illustrated that $(\mathcal{R}, \tau, K)$ is an approximately soft semi-Lindelöf space. It can be noted that the given collection $\Lambda$ forms a soft semi-open cover of $\mathcal{R}$. It also forms a soft semi-closed cover of $\mathcal{R}$. Since that collection has not a countable sub-cover, then $(\mathcal{R}, \tau, K)$ is not a mildly soft semi-Lindelöf space.

THEOREM 6.10. *An STS $(X, \tau, K)$ is mildly soft semi-compact (resp. mildly soft semi-Lindelöf) if and only if every collection of soft semi-clopen subsets of $(X, \tau, K)$, satisfying the finite (resp. countable) intersection property, has, itself, a non-null soft intersection.*

*Proof.* We only prove the theorem when $(X, \tau, K)$ is mildly soft semi-compact. The other case can be made similarly.
Let $\Lambda = \{(F_i, K) : i \in I\}$ be a soft semi-clopen subsets of $\widetilde{X}$. Suppose that $\widetilde{\bigcap}_{i\in I}(F_i, K) = \widetilde{\emptyset}$. Then $\widetilde{X} = \widetilde{\bigcup}_{i\in I}(F_i^c, K)$. As $(X, \tau, K)$ is mildly soft semi-compact, then $\widetilde{\bigcup}_{i=1}^{i=n}(F_i^c, K) = \widetilde{X}$. Therefore $\widetilde{\bigcap}_{i=1}^{i=n}(F_i, K) = \widetilde{\emptyset}$. Hence the *necessary* condition holds.
Conversely, let $\Lambda = \{(H_i, K) : i \in I\}$ be a soft semi-clopen cover of $\widetilde{X}$. Suppose $\Lambda$ has no finite sub-collection which cover $\widetilde{X}$. Then $\widetilde{X} \setminus \bigcup_{i=1}^{i=n}(H_i, K) \neq \widetilde{\emptyset}$, for any $n \in \mathcal{N}$. Now, $\widetilde{\bigcap}_{i=1}^{i=n}(H_i^c, K) \neq \widetilde{\emptyset}$ implies that $\{(H_i^c, K) : i \in I\}$ is the collection of soft semi-clopen subsets of $\widetilde{X}$ which has the finite intersection property. Thus $\widetilde{\bigcap}_{i\in I}(H_i^c, K) \neq \widetilde{\emptyset}$. This implies that $\widetilde{X} \neq \widetilde{\bigcup}_{i\in I}(H_i, K)$. But this contradicts that $\Lambda$ is a soft semi-clopen cover of $\widetilde{X}$. Hence $(X, \tau, K)$ is mildly soft semi-compact. □

For the sake of economy, the proofs of the following two propositions will be omitted.

PROPOSITION 6.11. *If $(D, K)$ is a soft semi-clopen subset of a mildly soft semi-compact (resp. mildly soft semi-Lindelöf) space $(X, \tau, K)$, then $(D, K)$ is mildly soft semi-compact (resp. mildly soft semi-Lindelöf).*

PROPOSITION 6.12. *If $(G, K)$ is a mildly soft semi-compact (resp. mildly soft semi-Lindelöf) set and $(D, K)$ is a soft semi-clopen set in $(X, \tau, K)$, then $(G, K)\widetilde{\bigcap}(D, K)$ is mildly soft semi-compact (resp. mildly soft semi-Lindelöf).*

PROPOSITION 6.13. *The soft semi-irresolute image of a mildly soft semi-compact (resp. mildly soft semi-Lindelöf) set is mildly soft semi-compact (resp. mildly soft semi-Lindelöf).*

*Proof.* By using a similar technique of the proof of Proposition (3.10), the proposition holds. □

DEFINITION 6.14. An STS $(X, \tau, K)$ is said to be soft semi-partition provided that a soft set is soft semi-open if and only if it is soft semi-closed.

THEOREM 6.15. *Let $(X, \tau, K)$ be a soft semi-partition topological space. Then the following four statements are equivalent.*

- **(i):** *$(X, \tau, K)$ is soft semi-Lindelöf (resp. soft semi-compact);*
- **(ii):** *$(X, \tau, K)$ is almost soft semi-Lindelöf (resp. almost soft semi-compact);*
- **(iii):** *$(X, \tau, K)$ is approximately soft semi-Lindelöf (resp. approximately soft semi-compact);*
- **(iv):** *$(X, \tau, K)$ is mildly soft semi-Lindelöf (resp. mildly soft semi-compact).*

*Proof.* **(i)** $\rightarrow$ **(ii)**: It follows from Proposition (4.3).
**(ii)** $\rightarrow$ **(iii)**: It follows from Proposition (5.3).
**(iii)** $\rightarrow$ **(iv)**: Let $\{(G_i, K) : i \in I\}$ be a semi-clopen cover of $\widetilde{X}$. As $(X, \tau, K)$ is approximately soft semi-Lindelöf, then $\widetilde{X}\widetilde{\subseteq}cl_s(\widetilde{\bigcup}_{s\in S}(G_i, K))$ and as $(X, \tau, K)$ is soft semi-partition, then $cl_s(\widetilde{\bigcup}_{s\in S}(G_i, K)) = \widetilde{\bigcup}_{s\in S}(G_i, K)$. Therefore $(X, \tau, K)$ is mildly soft semi-Lindelöf.
**(iv)** $\rightarrow$ **(i)**: Let $\{(G_i, K) : i \in I\}$ be a soft semi-open cover of $\widetilde{X}$. As $(X, \tau, K)$ is soft semi-partition, then $\{(G_i, K) : i \in I\}$ is a semi-clopen cover of $\widetilde{X}$ and as $(X, \tau, K)$ is mildly soft semi-Lindelöf, then $\widetilde{X} = \widetilde{\bigcup}_{s\in S}(G_i, K)$.
A similar proof can be given for the case between parentheses. □

DEFINITION 6.16. Let $(F, K)$ be a soft subset of $(X, \tau, K)$. Then $(int_s(F), K)$ is defined as $int_s(F)(k) = int_s(F(k))$, where $int_s(F(k))$ is the semi-interior of $F(k)$ in $(X, \tau_k)$ for each $k \in K$.

Proposition 6.17. *Let $(L, K)$ be a soft subset of an enriched STS $(X, \tau, K)$. Then:*

- **(i):** $int_s(L, K) \widetilde{\subseteq} (int_s(L), K)$.
- **(ii):** $int_s(L, K) = (int_s(L), K)$ *if and only if* $(int_s(L), K)$ *is soft semi-open.*

*Proof.* **(i):** For any $k \in K$, $int_s(L(k))$ is the largest semi-open subset of $(X, \tau_k)$ contained in $L(k)$. Putting $int_s(L, K) = (F, K)$ then $F(k)$ is a semi-open subset of $(X, \tau_k)$ is contained in $L(k)$ as well. This means that $F(k) \subseteq int_s(L(k)) \subseteq ((int_s(L))(k)$. Hence $int_s(L, K) \widetilde{\subseteq} (int_s(L), K)$.

**(ii):** If $(int_s(L), K) = int_s(L, K)$, then $(int_s(L), K)$ is a soft semi-open set. Conversely, let $(int_s(L), K)$ be a soft semi-open set. Obviously, $(int_s(L), K)$ contained in $(L, K)$. So from the definition of soft semi-interior of $(L, K)$, we infer that $(int_s(L), K) \widetilde{\subseteq} int_s(L, K)$ and from **(i)** above, we obtain that $int_s(L, K) \widetilde{\subseteq} (int_s(L), K)$. Hence $(int_s(L), K) = int_s(L, K)$.

□

Proposition 6.18. *If $(X, \tau, K)$ is an enriched mildly soft semi-compact (resp. enriched mildly soft semi-Lindelöf) space, then $K$ is finite (resp. countable).*

*Proof.* Let $(X, \tau, K)$ be mildly soft semi-compact (resp. mildly soft semi-Lindelöf). Since $(X, \tau, K)$ is enriched, then the collection $\{(G, K) : G(k) = X$ and $G(\alpha) = \emptyset$ for each $\alpha \neq k\}$ forms a soft semi-clopen cover of $(X, \tau, K)$. Hence it must be that $K$ is finite (resp. countable). □

Definition 6.19. A collection $\beta$ of soft semi-open sets is called soft semi-base of $(X, \tau, K)$ if every soft semi-open subset of $\widetilde{X}$ can be written as a soft union of members of $\beta$

Theorem 6.20. *Consider $(X, \tau, K)$ has a soft semi-base consists of soft semi-clopen sets. Then $(X, \tau, K)$ is soft semi-compact (resp. soft semi-Lindelöf) if and only if it is mildly soft semi-compact (resp. mildly soft semi-Lindelöf).*

*Proof.* The *necessary* condition is obvious.
To verify the *sufficient* condition, assume that $\Lambda$ is a soft semi-open cover of a mildly soft semi-compact space $(X, \tau, K)$. Since $\widetilde{X}$ is a soft union of members of the soft semi-base and $\widetilde{X}$ is mildly soft semi-compact, then

we can find a finite number $(H_s, K)$ of the soft semi-base which satisfies that $\widetilde{X} = \widetilde{\bigcup}_{s=1}^{s=n}(H_s, K)$. So for each member $(G_s, K)$ of $\Lambda$, there exists a member $(H_s, K)$ of the soft semi-base such that $(H_s, K)\widetilde{\subseteq}(G_s, K)$. Thus $\widetilde{X} = \widetilde{\bigcup}_{s=1}^{s=n}(G_s, K)$. Hence $(X, \tau, K)$ is soft semi-compact.
The proof in the case of a mildly soft semi-Lindelöf space is similar. □

## Conclusion

In this study, we establish and investigate the concepts of soft semi-Lindelöf spaces, almost (approximately, mildly) soft semi-compact spaces and almost (approximately, mildly) soft semi-Lindelöf spaces. We show the relationships among these concepts with the help of illustrative examples and we discuss the image of these spaces under soft semi-irresolute maps. Also, we point out the relationships of some of these spaces with soft semi $T_2$-spaces and soft semi $T_2'$-spaces. We present two new types of of the finite semi-intersection property and utilize them to give the equivalent conditions for almost (approximately) soft semi-compact and almost (approximately) soft semi-Lindelöf spaces. Furthermore, we illustrate under what conditions the four sorts of soft semi-compact (the four sorts of soft semi-Lindelöf) spaces are equivalent. We study the relationships between enriched soft topological spaces and the initiated spaces in different cases and obtain interesting results. Finally, the introduced concepts are compared in relation with many soft topological notions such as soft semi-connectedness, soft semi-irresolute maps and soft subspaces. The presented concepts in this study are elementary and fundamental for further researches and will open a way to improve more applications on soft topology.

## Conflict of interests

The authors declare that there is no conflict of interests regarding the publication of this article.

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**Tareq Mohammed Al-shami**
Department of Mathematics
Sana'a University
Sana'a, Yemen

Department of Mathematics
Mansoura University
Mansoura, Egypt
*E-mail*: tareqalshami83@gmail.com

**Mohammed E. El-Shafei**
Department of Mathematics
Mansoura University
Mansoura, Egypt
*E-mail*: meshafei@hotmail.com

**Mohammed Abo-Elhamayel**
Department of Mathematics
Mansoura University
Mansoura, Egypt
*E-mail*: mohamedaboelhamayel@yahoo.com