Korean J. Math. **27** (2019), No. 2, pp. 505–514
https://doi.org/10.11568/kjm.2019.27.2.505

ON ESTIMATION OF UNIFORM CONVERGENCE OF ANALYTIC FUNCTIONS BY (p,q)-BERNSTEIN OPERATORS

M. Mursaleen*, Faisal Khan, Mohd Saif, and Abdul Hakim Khan

Abstract. In this paper we study the approximation properties of a continuous function by the sequence of (p,q)-Bernstein operators for $q > p > 1$. We obtain bounds of (p,q)-Bernstein operators. Further we prove that if a continuous function admits an analytic continuation into the disk $\{z : |z| \leq \rho\}$, then $B^n_{p,q}(g;z) \to g(z)$ $(n \to \infty)$ uniformly on any compact set in the given disk $\{z : |z| \leq \rho\}$, $\rho > 0$.

In 1912, S.N. Bernstein [4] defined the famous polynomial known as Bernstein polynomial to prove the Weierstrasss approximation theorem. Due to the fine properties of approximation, convergence and shape preserving, Bernstein polynomials play an important role in approximation theory as well as in analysis, geometry and computer science etc. Lupas [11] was the first who introduced the q-analogue of well known Bernstein polynomial and investigated its approximation and later various properties of q-Bernstein operators were handled by Phillip [22,23]. The approximation properties of q-generalization of other operators were studied by several authors. Recently, Mursaleen et al. [15] applied (p,q)-calculus in approximation theory and introduced the (p,q)-analogue of

Received February 17, 2019. Revised May 28, 2019. Accepted June 7, 2019.
2010 Mathematics Subject Classification: 41A65, 47A58, 30H05.
Key words and phrases: (p,q)-integers, (p,q)-Bernstein operators, divided difference, analytic function, uniform convergence.

∗ Corresponding author.

Bernstein operator. Later on many important papers have been appeared on (p,q)-approximations (e.g. [1]- [3], [5], [6]- [10], [12]- [21]).

Here we call up certain definitions and notations of (p,q)-calculus:
The (p,q)-integer $[n]_{p,q}$ is defined as

$$[n]_{p,q} := \frac{p^n - q^n}{p-q}, \quad (n \in \mathbb{N} \cup \{0\}).$$

The (p,q)-binomial expansion is defined as

$$(y+w)^n_{p,q} := (y+w)\,(py+qw)\,(p^2y+q^2w)\cdots(p^{n-1}y+q^{n-1}w),$$
$$= \sum_{k=0}^{n} p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} y^k,$$

and the (p,q)-binomial coefficients are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} := \frac{[n]_{p,q}!}{[k]_{p,q}![n-k]_{p,q}!}.$$

By induction, it can be easily seen that

$$(1+y)\,(p+qy)\,(p^2+q^2y)\cdots(p^{n-1}+q^{n-1}y) = \sum_{k=0}^{n} p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} y^k.$$

Let $g \in C\,[0,1]$ be such that $g : [0,1] \longrightarrow \mathbb{R}$ with $q > p > 1$. Then the (p,q)-Bernstein operators [16] of g are defined as

$$B^n_{p,q}(g;y) := \sum_{k=0}^{n} g\left(p^{n-k}\frac{[k]_{p,q}}{[n]_{p,q}}\right) p_{n,k}(p,q;y), \quad (n \in \mathbb{N})$$

where polynomial $p_{n,k}(p,q;y)$ is given by
(1)

$$p_{n,k}\,(p,q;y) = \frac{1}{p^{\frac{n(n-1)}{2}}} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\frac{k(k-1)}{2}}\, y^k \prod_{s=0}^{n-k-1} (p^s - q^s y), \quad x \in [0,1].$$

Note that, for $p = 1$, $B^n_{p,q}\,(g;y)$ yields q-Bernstein operators and upon agreement, the q-Bernstein operators are used only for the case $q \neq 1$.
The (p,q)-difference form of Bernstein operators [17] is as follows:

$$B^n_{p,q}(g;y) := \sum_{k=0}^{n} \lambda^n_{p,q}\, g\left[0, p^{n-1}\frac{[1]_{p,q}}{[n]_{p,q}}, \cdots, p^{n-k}\frac{[k]_{p,q}}{[n]_{p,q}}\right] y^k, \tag{2}$$

where $g[y_0, y_1, \dots, y_k]$ denotes the k-th order divided difference of g with pairwise distinct nodes, that is

$$g[y_0] = g(y_0),\ g[y_0, y_1] = \frac{g(y_1) - g(y_0)}{y_1 - y_0},$$
$$g[y_0, y_1, \dots, y_k] = \frac{g[y_1, \dots, y_k] - g[y_0, \dots, y_{k-1}]}{[y_k - y_0]}$$

and $\lambda^n_{p,q}$ is given by

$$\lambda^n_{p,q} = \left[\begin{array}{c} n \\ k \end{array}\right]_{p,q} \frac{[k]_{p,q}!}{[n]^k_{p,q}} p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}}$$

$$= \left(1 - \frac{p^{n-1}[1]_{p,q}}{[n]_{p,q}}\right)\left(1 - \frac{p^{n-2}[2]_{p,q}}{[n]_{p,q}}\right) \cdots \left(1 - \frac{p^{n-k+1}[k-1]_{p,q}}{[n]_{p,q}}\right), \tag{3}$$

where, $\lambda^0_{p,q} = \lambda^1_{p,q} = 1, \quad 0 \le \lambda^n_{p,q} \le 1, \quad k = 0, 1, \dots, n.$
The k-th order divided difference [17] of an analytic function g can be expressed as

$$g[y_0, y_1, \dots, y_k] = \frac{1}{2\pi i} \oint_{\mathcal{L}} \frac{g(\eta)\, d\eta}{(\eta - y_0)(\eta - y_1) \cdots (\eta - y_k)}, \tag{4}$$

where $\mathcal{L}$ is contour encircling $y_0, \dots, y_k$ and g is assumed to be analytic on and within $\mathcal{L}$. Hence, when the nodes $0, \frac{[1]_{p,q}}{[n]_{p,q}}, \frac{[2]_{p,q}}{[n]_{p,q}}, \cdots, \frac{[k]_{p,q}}{[n]_{p,q}}$ are inside $\mathcal{L}$ and the pole $\alpha = p^m q^{-m}$ is outside, one has

$$g\left[0, \frac{p^{n-1}[1]_{p,q}}{[n]_{p,q}}, \cdots, \frac{p^{n-k}[k]_{p,q}}{[n]_{p,q}}\right] = \frac{1}{2\pi i} \oint_{\mathcal{L}} \frac{g_m(\eta)\, d\eta}{\left\{\eta \left(\eta - \frac{p^{n-1}[1]_{p,q}}{[n]_{p,q}}\right) \cdots \left(\eta - \frac{p^{n-k}[k]_{p,q}}{[n]_{p,q}}\right)\right\}}. \tag{5}$$

For a function $g(z)$ analytic in $\{z : |z| \le \rho\}$, we use the standard notation

$$M(r; g) = max_{|z| \le r} |g(z)|. \tag{6}$$

1. Auxiliary results

In this section, we prove two lemmas related to approximation of analytic functions on compact disk $\{z : |z| \le \rho\}$ in the complex plane.

Lemma 1.1. *Let $g(x)$ admit an analytic continuation as an entire function $g(z)$. Then*

$$B^n_{p,q}(g; z) \to g(z) \text{ as } n \to \infty \tag{7}$$

uniformly on any compact set in $\mathbb{C}$.

LEMMA 1.2. *If* $j > m,\ q > p > 1$, *then the following holds:*

$$\lim_{n\to\infty}\left\{\prod_{k=1}^{n-j}\left(1-p^{n-m-j}\,q^m\frac{[k]_{p,q}}{[n]_{p,q}}\right)\right\}=\left(\frac{p^{2j-m}}{q^{j-m}};\frac{p}{q}\right)_\infty.$$

Proof. It is clear that

$$\begin{aligned}\log\prod_{k=1}^{n-j}\left(1-p^{n-m-j}\,q^m\frac{[k]_{p,q}}{[n]_{p,q}}\right)&=\sum_{k=j}^{n-1}\left\{\log\left(1-p^{n-m-j}\,q^m\frac{[n-k]_{p,q}}{[n]_{p,q}}\right)\right\}\\&=\sum_{k=j}^{\infty}D_{k,n}^{p,q},\end{aligned}$$

where

$$D_{k,n}^{p,q}=\begin{cases}\log\left(1-p^{n-m-j}\,q^m\frac{[n-k]_{p,q}}{[n]_{p,q}}\right) & k<n\\ 0 \quad k\geq n\end{cases}.$$

Since

$$\left|D_{k,n}^{p,q}\right|\leq\left|\log\left(1-p^{n-m-j}\,q^m\frac{[n-k]_{p,q}}{[n]_{p,q}}\right)\right|$$

$$\leq\frac{q}{q-p}\frac{p^{n-m-k}\,q^m\,[n-k]_{p,q}}{[n]_{p,q}}\ \leq\frac{q}{q-p}p^{n-k-m}\,q^m,$$

it follows that $\sum_{k=j}^{\infty}\left|D_{k,n}^{p,q}\right|<\infty$, and the Lebesgue Dominated Convergence Theorem yields:

$$\begin{aligned}&\lim_{n\to\infty}\left\{\sum_{k=j}^{n-1}\log\left(1-p^{n-m-j}q^m\frac{[n-k]_{p,q}}{[n]_{p,q}}\right)\right\}\\&=\sum_{k=j}^{\infty}\left\{\lim_{n\to\infty}\log\left(1-p^{n-m-j}q^m\frac{[n-k]_{p,q}}{[n]_{p,q}}\right)\right\}\\&=\sum_{k=j}^{\infty}\lim_{n\to\infty}\left(1-p^{n-k-m}\,\frac{q^m}{q^k}\right).\end{aligned}$$

Hence

$$\lim_{n\to\infty}\log\prod_{k=1}^{n-j}\left(1-p^{n-m-j}\,q^m\,\frac{[n-k]_{p,q}}{[n]_{p,q}}\right)=\log\prod_{k=j}^{\infty}\left(1-p^{n-k-m}\,\frac{q^m}{p^k}\right).$$

The above derived Lemmas will be used to obtain main results in this paper. $\square$

2. Main Results

Our first theorem illustrates bounds of Bernstein operators $B^n_{p,q}(g; y)$ and the second theorem states that for every uniformly continuous function g which admits an analytic continuation on a closed disk, there is a sequence of Bernstein operators $B^n_{p,q}(g; y)$ which is uniformly convergent to g.

THEOREM 2.1. *Let $g(y)$ be bounded on $[0, 1]$ such that it has property of analytic continuation into disk $\{z : |z| \leq \rho\}, \rho > 0$. If*

$$B^n_{p,q}(g; z) := \sum_{k=0}^{n} D^{k,n}_{p,q} z^k \quad (n \in \mathbb{N}), \tag{8}$$

then the following relation holds

$$\left|D^{k,n}_{p,q}\right| \leq \frac{\mathcal{D}}{\rho^k}$$

where $\mathcal{D} = \mathcal{D}^n_{g,k}$ is independent of both k and n.

Proof. **Case (i).** Let $0 < \rho < 1$.
Since the divided difference form of (p, q)-Bernstein operators is given by

(9)

$$g\left[0, \frac{p^{n-1}[1]_{p,q}}{[n]_{p,q}}, \cdots, \frac{p^{n-k}[k]_{p,q}}{[n]_{p,q}}\right] = \frac{1}{2\pi i}\oint_{|\eta|=\rho} \frac{g(\eta)\, d\eta}{\eta\left(\eta - \frac{p^{n-1}[1]_{p,q}}{[n]_{p,q}}\right)\cdots\left(\eta - \frac{p^{n-k}[k]_{p,q}}{[n]_{p,q}}\right)},$$

then, for $|\eta| = \rho$ and $k < n - j$, taking modulus of denominator of (9), we get

$$\begin{aligned}
&\left|\eta\left(\eta - \frac{p^{n-1}[1]_{p,q}}{[n]_{p,q}}\right)\cdots\left(\eta - \frac{p^{n-k}[k]_{p,q}}{[n]_{p,q}}\right)\right| \\
&\geq \eta\left(\eta - \frac{p^{n-1}[1]_{p,q}}{[n]_{p,q}}\right)\cdots\left(\eta - \frac{p^{n-k}[k]_{p,q}}{[n]_{p,q}}\right) \\
&\qquad \geq \mathcal{D}_1 = \mathcal{D}_{p,q,\rho} > 0.
\end{aligned}$$

Now applying Lemma 1.2, we obtain

$$\left|D_{k,n}^{p,q}\right| \leq \left|g\left[0, \frac{p^{n-1}[1]_{p,q}}{[n]_{p,q}}, \cdots, \frac{p^{n-K}[k]_{p,q}}{[n]_{p,q}}\right]\right| \leq \frac{1}{2\pi}.\frac{2\pi\rho\, M(\rho,g)}{\mathcal{D}_1\rho^{k+1}} = \frac{\mathcal{D}_2}{\rho^k}$$

where $\mathcal{D}_2 = \mathcal{D}_{p,q,\rho,g}$.

Now, we estimate the coefficients $\left|D_{k,n}^{p,q}\right|$ for $k > n-j$, that is, to consider the case $\frac{[k]_{p,q}}{[n]_{p,q}} > \rho$. As we know that

$$g[y_0, y_1, \ldots, y_k] = \frac{1}{2\pi i}\sum_{s=0}^{k}\frac{g(\eta)\, d\eta}{(\eta-y_0)(\eta-y_1)\cdots(\eta-y_k)},$$

therefore

$$g\left[0, \frac{p^{n-1}[1]_{p,q}}{[n]_{p,q}}, \cdots, \frac{p^{n-k}[k]_{p,q}}{[n]_{p,q}}\right] = \sum_{s=0}^{k}\frac{g(\eta)\, d\eta}{\eta\left(\eta-\frac{p^{n-1}[1]_{p,q}}{[n]_{p,q}}\right)\cdots\left(\eta-\frac{p^{n-k}[k]_{p,q}}{[n]_{p,q}}\right)}$$

$$= \sum_{s=0}^{n-j} + \sum_{s=n-j+1}^{k}.$$

With the help of Residue theorem and equation (6), we have

$$\sum_{s=0}^{n-j} = \frac{1}{2\pi i}\int_{|\eta|=\rho}\frac{g(\eta)\, d\eta}{\eta\left(\eta-\frac{p^{n-1}[1]_{p,q}}{[n]_{p,q}}\right)\cdots\left(\eta-\frac{p^{n-k}[k]_{p,q}}{[n]_{p,q}}\right)}$$

$$= \frac{1}{2\pi i}\int_{|\eta|=\rho}\frac{g(\eta)\, d\eta}{\eta^{k+1}\left(1-\frac{p^{n-1}[1]_{p,q}}{\eta[n]_{p,q}}\right)\cdots\left(1-\frac{p^{n-k}[k]_{p,q}}{\eta[n]_{p,q}}\right)}.$$

$$\left|\sum_{s=0}^{n-j}\right| = \frac{1}{2\pi}\left|\int_{|\eta|=\rho}\frac{g(\eta)\, d\eta}{\eta\left(\eta-\frac{p^{n-1}[1]_{p,q}}{[n]_{p,q}}\right)\cdots\left(\eta-\frac{p^{n-k}[k]_{p,q}}{[n]_{p,q}}\right)}\right|$$

$$\leq \frac{1}{2\pi}\frac{M(\rho;g)}{\rho^k\mathcal{C}_3} = \frac{\mathcal{C}_4}{\rho^k}. \tag{10}$$

To estimate $\sum_{s=n-j+1}^{n-t}$, we consider

$$\left|\frac{p^{n-s}[s]_{p,q}}{[n]_{p,q}}.\frac{p^{n-s}[s]_{p,q}-p^n}{[n]_{p,q}}\cdots\frac{p^{n-s}[s]_{p,q}-p^{n-(s-1)}}{[n]_{p,q}}\frac{p^{n-s}[s]_{p,q}-p^{n-(s+1)}}{[n]_{p,q}}\cdots\frac{p^{n-s}[s]_{p,q}-p^{n-(n-t)}}{[n]_{p,q}}\right|^{-1}$$

$$= \frac{[n]_{p,q}^{n-t}(q-p)^{n-t}}{\left((\frac{q}{p})^s-1\right)\left((\frac{q}{p})^s-\frac{q}{p}\right)\cdots\left((\frac{q}{p})^s-(\frac{q}{p})^{s-1}\right)\left((\frac{q}{p})^{s+1}-(\frac{q}{p})^s\right)\cdots\left((\frac{q}{p})^{n-t}-(\frac{q}{p})^s\right)}$$

$$\leq \prod_{j=1}^{\infty}\left(1-\frac{p^j}{q^j}\right)\left(\frac{p}{q}\right)^{\frac{t(t-1)}{2}}\left(\frac{q}{p}\right)^{\frac{n^2-s^2}{2}-\frac{n-s}{2}}$$

$$\leq D_5\left(\frac{q}{p}\right)^{n(n-s)},$$

where $D_5 = D_{p,q,\rho}$. Since $s \geq n - j + 1$, it gives

$$\left(\frac{q}{p}\right)^{n(n-s)} \leq \left(\frac{q}{p}\right)^{n(j-1)} \leq \frac{1}{\rho^n}.$$

Since $M = max_{y\in[0,1]}\left|g(y)\right|$, we have

$$\left|\sum_{s=n-j+1}^{k}\right| \leq M \sum_{s=n-j+1}^{n-t} \frac{D_5}{\rho^n} \leq \frac{D_6(j-t)}{\rho^n} = \frac{D_7}{\rho^k}, \tag{11}$$

where $D_7 = \mathcal{D}_{p,q,g}$.
Finally, by equations (10) and (11), we get our required result.

Case (ii). Let $\rho > 1$.
By using (4) and Lemma 1.2, we get
(12)

$$D_{k,n}^{p,q} \leq \left|g\left[0, \frac{p^{n-1}[1]_{p,q}}{[n]_{p,q}}, \cdots, \frac{p^{n-k}[k]_{p,q}}{[n]_{p,q}}\right]\right| \leq \frac{1}{2\pi}.\frac{2\pi\rho\, M(\rho, g)}{\mathcal{D}_1\, \rho^{k+1}} = \frac{\mathcal{D}_2}{\rho^k}.$$

Now, by combining (10) and (11) we get our desired result. □

THEOREM 2.2. *Let $g(y)$ be bounded on $[0, 1]$ and admit an analytic continuation into disk $\{z : |z| \leq \rho\}, \rho > 0$. Then*

$$B_{p,q}^n(g; z) \to g(z) \text{ as } n \to \infty$$

on any compact set $S \subset \{z : |z| < a\}$, where $\rho < a \leq 1$.

Proof. Let $q > p > 1$ and $g : [0, 1] \to \mathbb{C}$ be a bounded function such that $g \in C\,[0, a], a \leq 1$. Then

$$\lim_{n\to\infty} B_{p,q}^n(g; p^m q^{-m}) = g(p^m q^{-m}) \ for\ all\ p^m q^{-m} \in [0, a].$$

Therefore,

$$B_{p,q}^n(g; z) = \sum_{k=0}^{n} g\left(\frac{p^k[n-k]_{p,q}}{[n]_{p,q}}\right) p_{n,n-k}\,(p, q; z).$$

Since we know that

$$\lim_{n\to\infty} p_{n,n-k}\left(p,q;p^m q^{-m}\right) = \begin{cases} 0 \ k>m \\ 0 \quad k<m \\ 1 \ k=m \end{cases}$$

and g is continuous at $p^m q^{-m}$, the statement follows. Now let $S \subset \{z : |z| \leq \rho\}$ be any compact set. Choose $0 < \theta < \rho < a$ such that $S \subset \{z : |z| \leq \rho\}$. Theorem 2.1 implies that for $|z| \leq \rho$, we have

$$\left|B^n_{p,q}(g;z)\right| \leq \sum_{k=0}^{n} \left|D\theta^k/\rho^k\right| \leq \frac{D}{\left(1-\frac{\theta}{\rho}\right)}.$$

Therefore, the sequence $\{B^n_{p,q}(g,z)\}$ of operators is uniformly bounded in the disk $\{z : |z| \leq \rho\}$. Also, it converges on the sequence $\{p^m q^{-m}\}$ and has limit point 0 to the function $g(z)$ analytic in this disc. By Vitali's Convergence Theorem, $B^n_{p,q}(g;z) \to g(z)$ as $n \to \infty$ uniformly on any compact set. This completes the proof. □

References

[1] T. Acar, *(p,q)-generalization of Szá sz-Mirakyan operators*, Math. Methods Appl. Sci. **9** (10) (2016), 2685–2695.

[2] T. Acar, A. Aral and S.A. Mohiuddine, *Approximation by bivariate (p,q)-Bernstein Kantorovich operators*, Iranian Jour. Sci. Techn., Trans A: Science, **42** (2) (2016), 655–662.

[3] T. Acar and A. Mohiuddine, *On Kantorovich modification of (p,q)-Baskakov operators,* J. Inequal. Appl. (2016), 2016: 98.

[4] S.N. Bernstein, *Démonstration du théorème de Weierstrass fondée sur la calcul des probabilités*, Commun. Soc. Math. Charkow Sér. **13** (1912), 1–2.

[5] H. Bin Jebreen, M. Mursaleen and Ambreen Naaz, *Approximation by quaternion (p,q)-Bernstein polynomials and Voronovskaja type result on compact disk*, Adv. Difference Equ. **2018** (2018), 448.

[6] S.G. Gal, *Approximation by complex q-Lorentz polynomial, $q > 1$*, Mathematica (Cluj) **54(77)** (1) (2012).

[7] U. Kadak, *On weighted statistical convergence based on (p,q)-integers and related approximation theorems for functions of two variables*, J. Math. Anal. Appl. **443** (2) (2016), 752–764.

[8] U. Kadak, *Weighted statistical convergence based on generalized difference operator involving (p,q)-gamma function and its applications to approximation theorems*, J. Math. Anal. Appl., **448** (2) (2017), 1633–1650.

[9] U. Kadak, V.N. Mishra and S. Pandey, *Chlodowsky type generalization of* (p,q)*-Szasz operators involving Brenke type polynomials*, Revista de la Real Academia de Ciencias Exactas, FiÌsicas y Naturales. Serie A, MatemaÌticas (2017), DOI: 10.1007/s13398-017-0439-y.

[10] P.P. Korovkin, *Convergence of linear positive operator in the space of contineous function.* Dokl. Akad. Nauk. Russian. SSSR (N.S.) **90** (1953), 961–964.

[11] A. Lupas, *A q-analogue of the Bernstein operators,* seminar on numerical and statistical calculus, University of Cluj-Napoca (1987), 85–92.

[12] V.N. Mishra and S. Pandey, *On Chlodowsky variant of* (p,q)*-Kantorovich–Stancu–Schurer operators*, Int. J. Anal. Appl. **11** (1) (2016), 28–39.

[13] V.N. Mishra and S. Pandey, *On* (p,q) *Baskakov–Durrmeyer–Stancu operators,* Adv.Appl.Clifford Algebra **27** (2) (2017), 1633–1646.

[14] M. Mursaleen and M. Ahasan, *The Dunkl generalization of Stancu type q-Szász-Mirakjan-Kantorovich operators and some approximation results*, Carpathian J. Math., **34(3)** (2018), 363-370.

[15] M. Mursaleen, K.J. Ansari and A. Khan, *On* (p,q)*-analogue of Bernstein operators*, Appl. Math. Comput. **266** (2015), 874–882.
Erratum: Appl. Math. Comput. **278** (2016), 70–71.

[16] M. Mursaleen, F. Khan and A. Khan, *Approximation by* (p,q)*-Lorentz polynomial on a compact disk,* Complex. Anal. Oper. Theory, **10** (8) (2016), 1725–1740.

[17] M. Mursaleen, Md. Nasiruzzaman, F. Khan and A. Khan, (p,q)*-analogue of divided difference and Bernstein operators*, J. Nonlinear. Funct. Anal. **2017** (2017), Article ID 25.

[18] M. Mursaleen, Ambreen Naaz and Asif Khan, *Improved approximation and error estimations by King type* (p,q)*-Szasz-Mirakjan-Kantorovich operators*, Appl. Math. Comput. **348** (2019), 175–185.

[19] M. Mursaleen, S. Rahman and A.H. Alkhaldi, *Convergence of iterates of q-Bernstein and* (p,q)*-Bernstein operators and the Kelisky-Rivlin type theorem*, Filomat **32** (12) (2018), (to appear).

[20] S. Ostrovska, *q-Bernstein polynomial of the Cauchy kernel,* Appl. Math. Comput., **198** (1) (2008), 261–270.

[21] S. Ostrovska, *q-Bernstein polynomial and their iterates,* Jour. Approx. Theory **123** (2003), 232–255.

[22] G.M. Phillips, *Bernstein polynomials based on the q-integers,* Ann. Numer. Math. **4** (1997), 511–518.

[23] G.M. Phillips, *A generalization of the Bernstein polynomial based on the q-integers.* ANZIAMZ, **42** (2000), 79–86 .

M. Mursaleen
Department of Mathematics
Aligarh Muslim University, Aligarh, India
E-mail: mursaleem@gmail.com

Faisal Khan
Department of Mathematics
Aligarh Muslim University, Aligarh, India
E-mail: faisalamu2011@gmail.com

Mohd Saif
Department of Applied Mathematics
Aligarh Muslim University, Aligarh, India
E-mail: usmanisaif153@gmail.com

Abdul Hakim Khan
Department of Applied Mathematics
Aligarh Muslim University, Aligarh, India
E-mail: ahkhan.amu@gmail.com