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A NOTE ON MULTIPLIERS IN ALMOST DISTRIBUTIVE LATTICES

Kyung Ho Kim

Abstract. The notion of multiplier for an almost distributive lattice is introduced, and some related properties are investigated. Moreover, we introduce a congruence relation ϕ_a induced by $a \in L$ on an almost distributive lattice and derive some useful properties of ϕ_a.

1. Introduction

The notion of derivation, introduced from the analytic theory, is helpful to the research of structure and property in algebraic system. Recently, analytic and algebraic properties of lattice were widely researched ([4, 5]). Several authors ([1, 6]) have studied derivations in rings and near-rings after Posner ([7]) have given the definition of the derivation in ring theory. Bresar ([3]) introduced the generalized derivation in rings and many mathematicians studied on this concept. K. L. Xin, T. Y. Li and J. H. Lu applied the notion of the derivation in ring theory to lattices([9]). In ([7]), a partial multiplier on a commutative semigroup $(A, \cdot)$ has been introduced as a function F from a nonvoid subset D_F of A into A such that $F(x) \cdot y = x \cdot F(y)$ for all $x, y \in D_F$. In 1980, the concept of an almost distributive lattice was introduced by U. M. Swamy and G. C. Rao ([9]). This class of Almost distributive lattices include most of the existing ring theoretic generalizations of a Boolean

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algebra on one hand and the class of distributive lattices on the other. The notion of multiplier for an almost distributive lattice is introduced, and some related properties are investigated. Moreover, we introduce a congruence relation ϕ_a induced by $a \in L$ on an almost distributive lattice and derive some useful properties of ϕ_a.

2. Preliminaries

Throughout this paper, L stands for an almost distributive lattice $(L, \vee, \wedge)$ unless otherwise specified.

Definition 2.1. ([9]) An algebra $(L, \wedge, \vee)$ of type $(2, 2)$ is called an *Almost Distributive Lattice* if it satisfies the following axioms, for any $a, b, c \in L$.

L_1 : $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$.
L_2 : $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$.
L_3 : $(a \vee b) \wedge b = b$.
L_4 : $(a \vee b) \wedge a = a$.
L_5 : $a \vee (a \wedge b) = a$.

Definition 2.2. ([9]) Let L be any non-empty set. Define, for any $x, y \in L$, $x \vee y = x$ and $x \wedge y = y$. Then $(L, \vee, \wedge)$ is an almost distributive lattice on L and it is called a *discrete almost distributive lattice*

Lemma 2.3. *Let L be an almost distributive lattice. For any $a, b \in L$, we have*

(1) : $a \wedge a = a$.
(2) : $a \vee a = a$.
(3) : $(a \wedge b) \vee b = b$.
(4) : $a \wedge (a \vee b) = a$.
(5) : $a \vee (b \wedge a) = a$.
(6) : $a \vee b = a$ *if and only if* $a \wedge b = b$.
(7) : $a \vee b = b$ *if and only if* $a \wedge b = a$ (see[9]).

Definition 2.4. ([9]) For any $a, b \in L$, we say that a is less than or equal to b and write $a \leq b$, if $a \wedge b = a$, or, equivalently, $a \vee b = b$.

Theorem 2.5. *Let L be an almost distributive lattice. For any $a, b, c \in L$, we have*

(1) : *The relation $\leq$ is a partial ordering on L.*

(2) : *$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$.*
(3) : *$(a \vee b) \vee a = a \vee b = a \vee (b \vee a)$.*
(4) : *$(a \vee b) \wedge c = (b \vee a) \wedge c$.*
(5) : *The operation $\wedge$ is associative on L.*
(6) : *$a \wedge b \wedge c = b \wedge a \wedge c$* (see[9]).

LEMMA 2.6. *Let L be an almost distributive lattice. For any $a, b, c, d \in L$, the following identities hold.*

(1) : *$a \wedge b \leq b$ and $a \leq a \vee b$.*
(2) : *$a \wedge b = b \wedge a$ whenever $a \leq b$.*
(3) : *$[a \vee (b \vee c)] \wedge d = [(a \vee b) \vee c] \wedge d$.*
(4) : *$a \leq b$ implies $a \wedge c \leq b \wedge c, c \wedge a \leq c \wedge b$ and $c \vee a \leq c \vee b$* (see[9]).

DEFINITION 2.7. ([9]) An element 0 is called a *zero element* of L if $0 \wedge a = 0$ for all $a \in L$.

LEMMA 2.8. *Let L be an almost distributive lattice. If L has 0, then for any $a, b \in L$, we have the following identities.*

(1) : *$a \vee 0 = a$ and $0 \vee a = a$.*
(2) : *$a \wedge 0 = 0$.*
(3) : *$a \wedge b = 0$ if and only if $b \wedge a = 0$* (see[9]).

DEFINITION 2.9. ([9]) A non-empty subset I of L is called an *ideal* of L if $a \vee b \in I$ and $a \wedge x \in I$ whenever $a, b \in I$ and $x \in L$.

If I is an ideal of L and $a, b \in L$, then $a \wedge b \in I$ if and only if $b \wedge a \in I$.

3. Multipliers in almost distributive lattices

In what follows, let L denote an almost distributive lattice unless otherwise specified.

DEFINITION 3.1. Let L be an almost distributive lattice. A function $f : L \to L$ is called a *multiplier* if

$$f(x \wedge y) = f(x) \wedge y$$

for all $x, y \in L$.

LEMMA 3.2. *The identity map on L is a multiplier on L. This is called an identity multiplier on L.*

EXAMPLE 3.3. Let L be an almost distributive lattice and $0 \in L$. A function f defined by $f(x) = 0$ for all $x \in L$ is called a *zero-multiplier* on L.

EXAMPLE 3.4. In a discrete almost distributive lattice $L = \{0, a, b\}$, if we define a function f by $f(0) = 0, f(a) = b, f(b) = a$, then f is a multiplier on L.

EXAMPLE 3.5. Let $L = \{0, a, b, c\}$ be a set in which "$\wedge$" and "$\vee$" is defined by

$\vee$	0	a	b	c
0	0	a	b	c
a	a	a	b	b
b	b	b	b	b
c	c	b	b	c

$\wedge$	0	a	b	c
0	0	0	0	0
a	0	a	a	0
b	0	a	b	c
c	0	0	c	c

Then it is easy to check that $(L, \wedge, \vee, 0)$ is an almost distributive lattice. Define a map $f : L \to L$ by

$$f(x) = \begin{cases} 0 & \text{if } x = 0, c \\ a & \text{if } x = a, b \end{cases}$$

Then it is easy to check that f is a multiplier on L.

LEMMA 3.6. *Let f be a multiplier of L. Then the following conditions hold.*

(1) $f(x) \leq x$, *for every* $x \in L$.
(2) $f(x) \wedge f(y) \leq f(x \wedge y)$, *for any* $x, y \in L$.
(3) *If I is an ideal of L, then* $f(I) \subseteq I$.
(4) *If L has* 0, *then* $f(0) = 0$.

Proof. (1) Let $x \in L$. Then $f(x) = f(x \wedge x) = f(x) \wedge x$, which implies that $f(x) \leq x$.

(2) Let $x, y \in L$. Then $f(x \wedge y) = f(x) \wedge y$. Since $f(y) \leq y$ for any $y \in L$, we get $f(x) \wedge f(y) \leq f(x) \wedge y = f(x \wedge y)$. Hence $f(x) \wedge f(y) \leq f(x \wedge y)$ for any $x, y \in L$.

(3) Let $a \in I$. Then by (1) above, we have $f(a) \leq a$, and hence $f(a) \in I$. Thus, $f(I) \subseteq I$.

(4) If L has 0, then by (1) above, $f(0) \leq 0$. Thus $0 \leq f(0) \leq 0$, and hence $f(0) = 0$. $\square$

LEMMA 3.7. *Let L be an almost distributive lattice. Define a function f_a by $f_a(x) = a \wedge x$ for all $x \in L$. Then f_a is a multiplier of L. Such a multiplier of L are called a principal multiplier of L.*

Proof. Let $x, y \in L$. Then

$$f_a(x \wedge y) = a \wedge (x \wedge y) = (a \wedge x) \wedge y = f_a(x) \wedge y$$

for all $x, y \in L$. □

PROPOSITION 3.8. *Let L be an almost distributive lattice. Then $f_a(x) = a \wedge x$ is an isotone multiplier of L.*

Proof. Let $x, y \in L$ be such that $x \leq y$. Then

$$f_a(x) = f_a(x \wedge y) = a \wedge x \wedge a \wedge y = f_a(x) \wedge f_a(y),$$

which implies that $f_a(x) \leq f_a(y)$. Hence f_a is an isotone multiplier of L. □

LEMMA 3.9. *Let L be an almost distributive lattice and let f be a multiplier of L. If $x \leq y$ and $f(y) = y$, then $f(x) = x$.*

Proof. Let $x \leq y$ and $f(y) = y$. Then by Lemma 2.6(2), we have

$$f(x) = f(x \wedge y) = f(y) \wedge x = y \wedge x = x \wedge y = x.$$

□

THEOREM 3.10. *Let L be an almost distributive lattice and let f be a multiplier of L. Then f is an isotone multiplier of L.*

Proof. Let $x, y \in L$ be such that $x \leq y$. Then by Lemma 2.9(2) and $f(y) \leq y$, we have

$$f(x) = f(x \wedge y) = f(y \wedge x) = f(y) \wedge x \leq f(y) \wedge y = f(y).$$

This implies that $f(x) \leq f(y)$, that is, f is isotone. □

PROPOSITION 3.11. *Let L be an almost distributive lattice and let f be a multiplier of L. Then $f(x \vee y) = f(x) \vee f(y)$ for any $x, y \in L$.*

Proof. Let $x, y \in L$. Then we get $f(x) = f((x \vee y) \wedge x)$ and $f(y) = f((x \vee y) \wedge y)$. Hence

$$f(x) \vee f(y) = (f(x \vee y) \wedge x) \vee (f(x \vee y) \wedge y)) = f(x \vee y) \wedge (x \vee y),$$

which implies that $f(x \vee y)$. □

Theorem 3.12. *Let L be an almost distributive lattice and let f be a multiplier of L. Then the following conditions are equivalent.*

(1) *f is an identity function on L.*
(2) *$f(x \vee y) = f(x) \vee y$ for any $x, y \in L$.*
1.

Proof. (1) $\Rightarrow$ (2) Let f be an identity function on L. Then $f(x \vee y) = x \vee y = f(x) \vee y$ for all $x, y \in L$.

(2) $\Rightarrow$ (1) Let $f(x \vee y) = f(x) \vee y$ for any $x, y \in L$. Putting $y = x$ in this relation, we have $f(x) = f(x) \vee x = x$ for all $x \in L$, which implies that f is an identity map on L. This completes the proof. □

Proposition 3.13. *Let L be an almost distributive lattice with* 0 *and f be a multiplier of L. Then $f : L \to L$ is an identity map if it satisfies $x \vee f(y) = f(x) \vee y$ for all $x, y \in L$.*

Proof. Let $x, y \in L$ be such that $x \vee f(y) = f(x) \vee y$. Now $f(x) = 0 \vee f(x) = f(0) \vee x = 0 \vee x = x$. Thus f is an identity map of L. □

In general, every multiplier of L need not be identity. However, in the following theorem, we give a set of conditions which are equivalent to be an identity multiplier of L.

Theorem 3.14. *Let L be an almost distributive lattice with* 0. *A multiplier f of L is an identity map if and only if the following conditions are satisfied for all $x, y \in L$.*

(1) *f is idempotent, i.e., $f^2(x) = f(x)$.*
(2) *$f^2(x) \vee y = f(x) \vee f(y)$.*

Proof. The condition for necessary is trivial. For sufficiency, assume that (1) and (2) hold. Then we get $f(x) \vee y = f^2(x) \vee y = f(x) \vee f(y) = f(x \vee y)$ for $x, y \in L$ by Proposition 3.11. Hence by Theorem 3.12, f is an identity multiplier of L. □

Let L be an almost distributive lattice and let f_1 and f_2 be two self-maps. We define $f_1 \circ f_2 : L \to L$ by

$$(f_1 \circ f_2)(x) = f_1(f_2(x))$$

for all $x \in L$.

Proposition 3.15. *Let L be an almost distributive lattice and f_1, f_2 two multipliers of L. Then $f_1 \circ f_2$ is also a multiplier of L.*

Proof. Let L be an almost distributive lattice and let f_1, f_2 be two multipliers of L. Then we have

$$\begin{aligned}(f_1 \circ f_2)(a \wedge b) &= f_1(f_2(a \wedge b)) = f_1(f_2(a) \wedge b)\\ &= f_1(f_2(a)) \wedge b = (f_1 \circ f_2)(a) \wedge b\end{aligned}$$

for any $a, b \in L$. This completes the proof. □

Let L be an almost distributive lattice and f_1, f_2 two self-maps. We define $f_1 \vee f_2 : L \to L$ by

$$(f_1 \vee f_2)(x) = f_1(x) \vee f_2(x)$$

for all $x \in L$.

Proposition 3.16. *Let L be an almost distributive lattice and f_1, f_2 two multipliers of L. Then $f_1 \vee f_2$ is also a multiplier of L.*

Proof. Let L be an almost distributive lattice and f_1, f_2 two multipliers of L. Then we have

$$\begin{aligned}(f_1 \vee f_2)(a \wedge b) &= f_1(a \wedge b) \vee f_2(a \wedge b) = (f_1(a) \wedge b) \vee (f_2(a) \wedge b)\\ &= (f_1(a) \vee f_2(a)) \wedge b = (f_1 \vee f_2)(a) \wedge b\end{aligned}$$

for any $a, b \in L$. This completes the proof. □

Let L_1 and L_2 be two almost distributive lattices. Then $L_1 \times L_2$ is also an almost distributive lattice with respect to the point-wise operation given by

$$(a, b) \wedge (c, d) = (a \wedge c, b \wedge d) \text{ and } (a, b) \vee (c, d) = (a \vee c, b \vee d)$$

for all $a, c \in L_1$ and $b, d \in L_2$.

Proposition 3.17. *Let L_1 and L_2 be two almost distributive lattices with 0. Define a map $f : L_1 \times L_2 \to L_1 \times L_2$ by $f(x, y) = (0, y)$ for all $(x, y) \in L_1 \times L_2$. Then f is a multiplier of $L_1 \times L_2$ with respect to the point-wise operation.*

Proof. Let $(x_1, y_1), (x_2, y_2) \in L_1 \times L_2$. The we have

$$\begin{aligned}f((x_1, y_1) \wedge (x_2, y_2)) &= f(x_1 \wedge x_2, y_1 \wedge y_2)\\ &= (0, y_1 \wedge y_2) = (0 \wedge x_2, y_1 \wedge y_2)\\ &= (0, y_1) \wedge (x_2, y_2) = f(x_1, y_1) \wedge (x_2, y_2).\end{aligned}$$

Therefore f is a multiplier of the direct product $L_1 \times L_2$. □

Definition 3.18. Let L be an almost distributive lattice and let f be a multiplier of L. Define a set $Fix_f(L)$ by

$$Fix_f(L) := \{x \in L \mid f(x) = x\}.$$

Lemma 3.19. *Let f be a multiplier of L. If $x \in Fix_f(L)$ and $y \in L$, then $x \wedge y \in Fix_f(L)$.*

Proof. Let $y \in Fix_f(L)$ and $x \in L$. Then we obtain

$$f(x \wedge y) = f(x) \wedge y = x \wedge y,$$

which implies that $x \wedge y \in Fix_f(L)$. This completes the proof. □

Proposition 3.20. *Let L be an almost distributive lattice and let f_1 and f_2 be two multipliers of L. Then $f_1 = f_2$ if and only if $Fix_{f_1} = Fix_{f_2}$.*

Proof. If $f_1 = f_2$, then clearly $Fix_{f_1}(L) = Fix_{f_2}(L)$. Suppose that $Fix_{f_1}(L) = Fix_{f_2}(L)$. For any $x \in L$, $f_1(f_1(x)) = f_1(x)$, thus $f_1(x) \in Fix_{f_1}(L)$. Hence $f_1(x) \in Fix_{f_2}(L)$. Therefore, $f_2(f_1(x)) = f_1(x)$ and hence $f_2 f_1 = f_1$. Similarly, we obtain $f_1 f_2 = f_2$. Since f_1 and f_2 are isotone by Proposition 3.10 and $f_1(x) \leq x$, we have $f_2(f_1(x)) \leq f_2(x)$ and so, $f_2 f_1 \leq f_2$. That is, $f_1 \leq f_2$. By symmetry, we get $f_2 = f_1$. □

Theorem 3.21. *Let L be an almost distributive lattice and let $\mathcal{M}(L)$ be the set of all multipliers on L. Then $(\mathcal{M}(L), \vee, \wedge)$ is an almost distributive lattice, where for any $f_1, f_2 \in \mathcal{M}(L)$, $(f_1 \wedge f_2)(x) = f_1(x) \wedge f_2(x)$ and $(f_1 \vee f_2)(x) = f_1(x) \vee f_2(x)$ for all $x \in L$.*

Proof. Let $f_1, f_2 \in \mathcal{M}(L)$ and $x, y \in L$. Then

$$\begin{aligned}
(f_1 \wedge f_2)(x \wedge y) &= f_1(x \wedge y) \wedge f_2(x \wedge y) \\
&= f_1(x) \wedge y \wedge f_2(x) \wedge y \\
&= f_1(x) \wedge f_2(x) \wedge y \\
&= (f_1 \wedge f_2)(x) \wedge y.
\end{aligned}$$

This implies that $f_1 \wedge f_2$ is a multiplier on L. Also, we have

$$\begin{aligned}
(f_1 \vee f_2)(x \wedge y) &= f_1(x \wedge y) \vee f_2(x \wedge y) \\
&= (f_1(x) \wedge y) \vee (f_2(x) \wedge y) \\
&= (f_1(x) \vee f_2(x)) \wedge y \\
&= (f_1 \vee f_2)(x) \wedge y.
\end{aligned}$$

This implies that $f_1 \vee f_2$ is a multiplier on L. Therefore $\mathcal{M}(L)$ is closed under $\wedge$ and $\vee$, and clearly, it satisfies the properties of an almost distributive lattice. $\square$

THEOREM 3.22. *Let L be an almost distributive lattice and let $\mathcal{F} = \{Fix_f(L) \mid f \in \mathcal{M}(L)\}$. For any $f_1, f_2 \in \mathcal{M}(L)$, if we define $Fix_{f_1}(L) \vee Fix_{f_2}(L) = Fix_{f_1 \vee f_2}(L)$ and $Fix_{f_1}(L) \wedge Fix_{f_2}(L) = Fix_{f_1 \wedge f_2}(L)$, then $(\mathcal{F}, \vee, \wedge)$ is an almost distributive lattice and it is isomorphic to $\mathcal{M}(L)$.*

Proof. Let $\mathcal{F} = \{Fix_f(L) \mid f \in \mathcal{M}(L)\}$. Define $Fix_{f_1}(L) \vee Fix_{f_2}(L) = Fix_{f_1 \vee f_2}(L)$ and $Fix_{f_1}(L) \wedge Fix_{f_2}(L) = Fix_{f_1 \wedge f_2}(L)$ for any $f_1, f_2 \in \mathcal{M}$. Then by Theorem 3.21, $\mathcal{F}$ is closed under $\wedge$ and $\vee$. Since $(\mathcal{M}, \vee, \wedge)$ is an almost distributive lattice, we can very that $(\mathcal{F}, \vee, \wedge)$ is an almost distributive lattice. Now define $\phi : \mathcal{M}(L) \to \mathcal{F}$ by $\phi(f) = Fix_f(L)$. By Theorem 3.20, ϕ is well-defined and injective. Clearly, ϕ is surjective. Also, for any $f_1, f_2 \in \mathcal{M}$, we have $\phi(f_1 \wedge f_2) = Fix_{f_1 \wedge f_2}(L) = Fix_{f_1}(L) \wedge Fix_{f_2}(L) = \phi(f_1) \wedge \phi(f_2)$ and $\phi(f_1 \vee f_2) = Fix_{f_1 \vee f_2}(L) = Fix_{f_1}(L) \vee Fix_{f_2}(L) = \phi(f_1) \vee \phi(f_2)$. Hence ϕ is an isomorphism. $\square$

Let us recall from Proposition 3.20 that the composition of two multipliers f and g of an almost distributive lattice L is a multiplier of L where $(f \circ g)(x) = f(g(x))$ for all $x \in L$.

THEOREM 3.23. *Let f and g be two idempotent multipliers of L such that $f \circ g = g \circ f$. Then the following conditions are equivalent.*

(1) $f = g$.
(2) $f(L) = g(L)$.
(3) $Fix_f(L) = Fix_g(L)$.

Proof. (1)$\Rightarrow$ (2): It is obvious.

(2) $\Rightarrow$ (3): Assume that $f(L) = g(L)$. Let $x \in Fix_f(L)$. Then $x = f(x) \in f(L) = g(L)$. Hence $x = g(y)$ for some $y \in L$. Now $g(x) = g(g(y)) = g^2(y) = g(y) = x$. Thus $x \in Fix_g(L)$. Therefore, $Fix_f(L) \subseteq Fix_g(L)$. Similarly, we can obtain $Fix_g(L) \subseteq Fix_f(L)$. Thus $Fix_f(L) = Fix_g(L)$.

(3) $\Rightarrow$ (1): Assume that $Fix_f(L) = Fix_g(L)$. Let $x \in L$. Since $f(x) \in Fix_f(L) = Fix_g(L)$, we have $g(f(x)) = f(x)$. Also, we obtain $g(x) \in Fix_g(L) = Fix_f(L)$. Hence we get $f(g(x)) = g(x)$. Thus we have

$$f(x) = g(f(x)) = (g \circ f)(x) = (f \circ g)(x) = f(g(x)) = g(x).$$

Therefore, f and g are equal in the sense of mappings. $\square$

Definition 3.24. Let $(L, \vee, \wedge, 0)$ be an almost distributive lattice. For any $a \in L$, define $\phi_a = \{(x, y) \in L \times L \mid f_a(x) = f_a(y)\}$ where f_a is a principal multiplier induced by $a \in L$.

Proposition 3.25. *Let L be an almost distributive lattice. Then for any $a \in L$, ϕ_a is a congruence relation on L.*

Proof. Clearly, ϕ_a is an equivalence relation on L. Now, let $(x, y), (p, q) \in \phi_a$. Then $a \wedge x = a \wedge y$ and $a \wedge p = a \wedge q$. Now $a \wedge x \wedge p = a \wedge x \wedge a \wedge p = a \wedge y \wedge a \wedge q = a \wedge y \wedge q$ and $a \wedge (x \vee p) = (a \wedge x) \vee (a \wedge p) = (a \wedge y) \vee (a \wedge q) = a \wedge (y \vee q)$. Therefore, $(x \wedge p, y \wedge q), (x \vee p, y \vee q) \in \phi_a$. Hence ϕ_a is a congruence relation on L. $\square$

Proposition 3.26. *Let L be an almost distributive lattice. Then the following identities hold for any $a, b \in L$.*
(1) $\phi_{a \wedge b} = \phi_{b \wedge a}$.
(2) $\phi_{a \vee b} = \phi_{b \vee a}$.
(3) $\phi_a \cap \phi_b = \phi_{a \vee b}$.

Proof. (1) and (2) Since $a \wedge b \wedge x = b \wedge a \wedge x$ and $(a \vee b) \wedge x = (b \vee a) \wedge x$, we obtain $\phi_{a \wedge b} = \phi_{b \wedge a}$ and $\phi_{a \vee b} = \phi_{b \vee a}$.
(3) Again, we obtain

$$\begin{aligned}(x, y) \in \phi_a \cap \phi_b &\Leftrightarrow a \wedge x = a \wedge y \text{ and } b \wedge x = b \wedge y \\ &\Leftrightarrow (a \vee b) \wedge x = (a \vee b) \wedge y \Leftrightarrow (x, y) \in \phi_{a \vee b},\end{aligned}$$

which implies that $\phi_{a \vee b} = \phi_a \cap \phi_b$. $\square$

Theorem 3.27. *Let L be an almost distributive lattice and let $\mathcal{M}(L)$ be the set of all multipliers on L. Then the set of all principal multipliers $\mathcal{P}(L) = \{f_a \mid a \in L\}$ is a distributive lattice with the following operations*

$$f_a \vee f_b = f_{a \vee b} \text{ and } f_a \wedge f_b = f_{a \wedge b}$$

for all $a, b \in L$.

Proof. Let $a, b \in L$. Then

$$(f_a \vee f_b))(x) = f_a(x) \vee f_b(x) = (a \wedge x) \vee (b \wedge x) = (a \vee b) \wedge x = f_{a \vee b}(x)$$

for any $x \in L$, which implies that $f_a \wedge f_b = f_{a \vee b} \in \mathcal{P}(L)$. Also,

$$(f_a \wedge f_b))(x) = f_a(x) \wedge f_b(x) = (a \wedge x) \wedge (b \wedge x) = (a \wedge b) \wedge x = f_{a \wedge b}(x)$$

for any $x \in L$, which implies that $f_a \wedge f_b = f_{a \wedge b} \in \mathcal{P}(L)$. Hence $\mathcal{P}(L)$ is closed under $\vee$ and $\wedge$, and so $\mathcal{P}(L)$ is a sub-almost distributive lattice of L. Next, for any $x \in L$, $f_{a\wedge b}(x) = a \wedge b \wedge x = b \wedge a \wedge x = f_{b\wedge a}(x)$. Thus $f_{a\wedge b} = f_{b\wedge a}$. That is, $f_a \wedge f_b = f_b \wedge f_a$. Hence $\mathcal{P}(L)$ is a distributive lattice. □

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Kyung Ho Kim
Department of Mathematics
Korea National University of Transportation
Chungju 27469, Korea
E-mail: ghkim@ut.ac.kr