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A BANACH ALGEBRA OF SERIES OF FUNCTIONS OVER PATHS

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Abstract. Let $C[0,T]$ denote the space of continuous real-valued functions on $[0,T]$. On the space $C[0,T]$, we introduce a Banach algebra of series of functions which are generalized Fourier-Stieltjes transforms of measures of finite variation on the product of simplex and Euclidean space. We evaluate analytic Feynman integrals of the functions in the Banach algebra which play significant roles in the Feynman integration theory and quantum mechanics.

1. Introduction

Let $C_0[0,T]$ denote the classical Wiener space, that is, the space of continuous real-valued functions on the interval $[0,T]$ with $x(0)=0$. On $C_0[0,T]$, Cameron and Storvick [2] introduced a Banach algebra $\mathcal{S}''$ which is the space of series of generalized Fourier-Stieltjes transforms of the $\mathbb{C}$-valued finite Borel measures over $\Delta_n\times\mathbb{R}^n$, where Δ_n is a simplex, that is, $\Delta_n=\{(t_1,\dots,t_n):0=t_0<t_1<\cdots<t_n<T\}$. They also showed that $\mathcal{S}''$ is isometrically embedded in the Banach algebra $\mathcal{S}'$, the

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space of generalized Fourier-Stieltjes transforms of the complex Borel measures on the space of functions of bounded variation on $[0,T]$.

On the other hand, let $C[0,T]$ denote an analogue of a generalized Wiener space, the space of continuous real-valued functions on the interval $[0,T]$. On $C[0,T]$, Ryu [9,10] introduced a finite measure $w_{\alpha,\beta;\varphi}$ and investigated its properties, where $\alpha,\beta:[0,T]\to\mathbb{R}$ are continuous functions such that β is strictly increasing, and φ is an arbitrary finite measure on the Borel class $\mathcal{B}(\mathbb{R})$ of $\mathbb{R}$. On this space $(C[0,T],w_{\alpha,\beta;\varphi})$, the author [3] introduced an Itô type integral $I_{\alpha,\beta}$ which generalizes the Paley-Wiener-Zygmund integrals on $C[0,T]$, and in [4,5] he derived two Banach algebras $\bar{\mathcal{S}}_{\alpha,\beta;\varphi}$ and $\bar{\mathcal{S}}'_{\alpha,\beta;\varphi}$ by using $I_{\alpha,\beta}$, which generalize the Cameron-Storvick's Banach algebra $\mathcal{S}$ and $\mathcal{S}'$, respectively, with the mean function and the variance function determined by α and β.

In this paper, we introduce a Banach algebra $\bar{\mathcal{S}}''_{\alpha,\beta;\varphi}$ which is defined over paths in $C[0,T]$ and consists of series of generalized Fourier-Stieltjes transforms of measures on $\Delta_n\times\mathbb{R}^n$. Then, we will prove that $\bar{\mathcal{S}}''_{\alpha,\beta;\varphi}$ is continuously embedded in $\bar{\mathcal{S}}'_{\alpha,\beta;\varphi}$. As an application, we derive evaluation formulas for the analytic Feynman integrals of functions in $\bar{\mathcal{S}}''_{\alpha,\beta;\varphi}$ which play significant roles in Feynman integration theory and quantum mechanics. In particular, if $\alpha(t)=0$, $\beta(t)=t$ for $t\in[0,T]$, and $\varphi=\delta_0$ which is the Dirac measure concentrated at 0, then $\bar{\mathcal{S}}''_{\alpha,\beta;\varphi}$ is reduced to $\mathcal{S}''$ so that the results of this paper generalize those in [2]. We also note that every path in $C[0,T]$ starts at an arbitrary point in $\mathbb{R}$ so that $C[0,T]$ generalizes $C_0[0,T]$.

2. An analogue of a generalized Wiener space

In this section we introduce an analogue of a generalized Wiener space with preliminaries which will be used in the next sections.

Let m_L denote the Lebesgue measure on $\mathcal{B}(\mathbb{R})$. Let α and β be absolutely continuous real-valued functions on $[0,T]$ such that β is strictly increasing and $|\alpha|'(t)+\beta'(t)>0$ for $t\in[0,T]$, where $|\alpha|$ denotes the total variation of α. For $\vec{t}_n=(t_0,t_1,\dots,t_n)$ with $0=t_0<t_1<\cdots<t_n\le T$, let $J_{\vec{t}_n}:C[0,T]\to\mathbb{R}^{n+1}$ be the function given by

$$J_{\vec{t}_n}(x)=(x(t_0),x(t_1),\dots,x(t_n)).$$

For $\prod_{j=0}^n B_j$ in $\mathcal{B}(\mathbb{R}^{n+1})$, the subset $J_{\vec{t}_n}^{-1}(\prod_{j=0}^n B_j)$ of $C[0,T]$ is called an interval I and let $\mathcal{C}$ be the set of all such intervals I. Define a premeasure

$m_{\alpha,\beta;\varphi}$ on $\mathcal{C}$ by

$$m_{\alpha,\beta;\varphi}(I) = \int_{B_0}\int_{\prod_{j=1}^n B_j} W(\vec{t}_n,\vec{u}_n,u_0)dm_L^n(\vec{u}_n)d\varphi(u_0),$$

where

$$\begin{aligned} W(\vec{t}_n,\vec{u}_n,u_0) &= \left[\frac{1}{\prod_{j=1}^n 2\pi[\beta(t_j)-\beta(t_{j-1})]}\right]^{\frac{1}{2}} \\ &\quad\times \exp\left\{-\frac{1}{2}\sum_{j=1}^n \frac{[u_j-\alpha(t_j)-u_{j-1}+\alpha(t_{j-1})]^2}{\beta(t_j)-\beta(t_{j-1})}\right\} \end{aligned}$$

for $\vec{u}_n = (u_1,\dots,u_n)$. The Borel σ-algebra $\mathcal{B}(C[0,T])$ of $C[0,T]$ with the supremum norm, coincides with the smallest σ-algebra generated by $\mathcal{C}$ and there exists a unique positive finite measure $w_{\alpha,\beta;\varphi}$ on $\mathcal{B}(C[0,T])$ with $w_{\alpha,\beta;\varphi}(I) = m_{\alpha,\beta;\varphi}(I)$ for $I\in\mathcal{C}$. This measure $w_{\alpha,\beta;\varphi}$ is called an analogue of a generalized Wiener measure on $(C[0,T],\mathcal{B}(C[0,T]))$ according to φ [9,10]. We now have the following theorem [9].

THEOREM 2.1. *If $f:\mathbb{R}^{n+1}\to\mathbb{C}$ is a Borel measurable function, then the following equality holds:*

$$\begin{aligned} &\int_{C[0,T]} f(x(t_0),x(t_1),\dots,x(t_n))dw_{\alpha,\beta;\varphi}(x) \\ \stackrel{*}{=}\ &\int_{\mathbb{R}^{n+1}} f(u_0,u_1,\dots,u_n)W(\vec{t}_n,\vec{u}_n,u_0)dm_L^n(\vec{u}_n)d\varphi(u_0), \end{aligned}$$

where $\stackrel{*}{=}$ means that if either side exists, then both sides exist and they are equal.

Let $F: C[0,T]\to\mathbb{C}$ be a measurable function and suppose that the integral

$$J_F(\lambda) \equiv \int_{C[0,T]} F(\lambda^{-\frac{1}{2}}x)dw_{\alpha,\beta;\varphi}(x)$$

exists as a finite number for all $\lambda>0$. If there exists a function $J_F^*(\lambda)$ analytic in

$$\mathbb{C}_+ \equiv \{\lambda\in\mathbb{C} : \operatorname{Re}\lambda > 0\}$$

such that $J_F^*(\lambda) = J_F(\lambda)$ for all $\lambda>0$, then $J_F^*(\lambda)$ is defined to be a generalized analytic Wiener $w_{\alpha,\beta;\varphi}$-integral of F over $C[0,T]$ with parameter

λ and it is denoted by

$$\int_{C[0,T]}^{anw_\lambda} F(x)dw_{\alpha,\beta;\varphi}(x) = J_F^*(\lambda)$$

for $\lambda \in \mathbb{C}_+$. Let q be a nonzero real number. If $\int_{C[0,T]}^{anw_\lambda} F(x)dw_{\alpha,\beta;\varphi}(x)$ has a limit as λ approaches $-iq$ through $\mathbb{C}_+$, then we call it a generalized analytic Feynman $w_{\alpha,\beta;\varphi}$-integral of F over $C[0,T]$ with parameter q and it is denoted by

$$\int_{C[0,T]}^{anf_q} F(x)dw_{\alpha,\beta;\varphi}(x) = \lim_{\lambda\to -iq} \int_{C[0,T]}^{anw_\lambda} F(x)dw_{\alpha,\beta;\varphi}(x).$$

We emphasize that φ need not be a probability measure so that $w_{\alpha,\beta;\varphi}$ also need not be a probability measure.

We observe that the functions α and β induce the obvious Lebesgue-Stieltjes measure $\nu_{\alpha,\beta}$ on $[0,T]$ by $\nu_{\alpha,\beta}(E) = \int_E d(|\alpha| + \beta)(t)$ for a Lebesgue measurable subset E of $[0,T]$. Define $L^2_{\alpha,\beta}[0,T]$ to be the space of functions on $[0,T]$ that are square integrable with respect to $\nu_{\alpha,\beta}$, that is,

$$L^2_{\alpha,\beta}[0,T] = \left\{ f : [0,T] \to \mathbb{R} \middle| \int_0^T [f(t)]^2 d\nu_{\alpha,\beta}(t) < \infty \right\}.$$

The space $L^2_{\alpha,\beta}[0,T]$ is a Hilbert space and has the inner product [8]

$$\langle f, g\rangle_{\alpha,\beta} = \int_0^T f(t)g(t)d\nu_{\alpha,\beta}(t) \text{ for } f,g \in L^2_{\alpha,\beta}[0,T].$$

Note that the space $L^2_{\alpha,\beta}[0,T]$ is separable. Let $S[0,T]$ denote the collection of all step functions on $[0,T]$. For f in $L^2_{\alpha,\beta}[0,T]$, let $\{\phi_n\}_{n=1}^\infty$ be a sequence of the step functions in $S[0,T]$ with $\lim_{n\to\infty} \|\phi_n - f\|_{\alpha,\beta} = 0$. Define $I_{\alpha,\beta}(f)$ by the $L^2(C[0,T])$-limit

$$I_{\alpha,\beta}(f)(x) = \lim_{n\to\infty} \int_0^T \phi_n(t)dx(t)$$

for all $x \in C[0,T]$ for which this limit exists, where $\int_0^T \phi_n(t)dx(t)$ denotes the Riemann-Stieltjes integral of ϕ_n with respect to x. We note that $I_{\alpha,\beta}(f)(x)$ exists for $w_{\alpha,\beta;\varphi}$ a.e. $x \in C[0,T]$ and it is independent of choice of the sequence $\{\phi_n\}_{n=1}^\infty$ in $S[0,T]$ to define it [3].

Let $\mathcal{M}_{\alpha,\beta}$ be the class of complex measures of finite variation on $L^2_{\alpha,\beta}[0,\ T]$ with Borel σ-algebra $\mathcal{B}(L^2_{\alpha,\beta}[0,T])$ of $L^2_{\alpha,\beta}[0,T]$ as its class of measurable sets. If $\mu \in \mathcal{M}_{\alpha,\beta}$, then we set $\|\mu\| = \operatorname{var}\mu$, the total variation of μ over $L^2_{\alpha,\beta}[0,T]$. Let $\bar{\mathcal{S}}_{\alpha,\beta;\varphi}$ be the space of functions of the form

$$F(x) = \int_{L^2_{\alpha,\beta}[0,T]} \exp\{iI_{\alpha,\beta}(f)(x)\}d\mu(f) \tag{1}$$

for all $x \in C[0,T]$ for which the integral exists, where $\mu \in \mathcal{M}_{\alpha,\beta}$. Here we take

$$\|F\| = \inf\{\|\mu\|\},$$

where the infimum is taken over all μ's so that F and μ are related by (1). We note that F is well-defined for $w_{\alpha,\beta;\varphi}$ a.e. $x \in C[0,T]$ and it is an integrable function of x on $C[0,T]$. Moreover, $\bar{\mathcal{S}}_{\alpha,\beta;\varphi}$ is a Banach algebra with unit over $\mathbb{C}$ [5].

Let $B[0,T]$ be the space of real-valued, right-continuous functions of bounded variation on $[0,T]$ that vanish at T. Let $\mathcal{A}'$ be the σ-algebra of subsets of $B[0,T]$ generated by the class of sets of the form

$$\{v \in B[0,T] : \langle v, f\rangle_{\alpha,\beta} < \lambda\},$$

where f and λ range over all elements of $L^2_{\alpha,\beta}[0,T]$ and all positive real numbers, respectively. Let $\mathcal{M}(B[0,T])$ be the class of complex measures of finite variation defined on subsets of $B[0,T]$ with $\mathcal{A}'$ as its class of measurable sets. If $\mu \in \mathcal{M}(B[0,T])$, we set $\|\mu\| = \operatorname{var}\mu$ over $B[0,T]$. Let $\bar{\mathcal{S}}'_{\alpha,\beta;\varphi}$ be the space of functions of the form

$$F(x) = \int_{B[0,T]} \exp\left\{i\int_0^T v(t)dx(t)\right\}d\mu(v) \tag{2}$$

for $w_{\alpha,\beta;\varphi}$ a.e. $x \in C[0,T]$, where $\mu \in \mathcal{M}(B[0,T])$. Here we take

$$\|F\|' = \inf\{\|\mu\|\},$$

where the infimum is taken over all μ's so that F and μ are related by (2). The space $\bar{\mathcal{S}}'_{\alpha,\beta;\varphi}$ is a Banach algebra with unit over $\mathbb{C}$. Moreover, $\bar{\mathcal{S}}'_{\alpha,\beta;\varphi}$ is continuously embedded in $\bar{\mathcal{S}}_{\alpha,\beta;\varphi}$, but need not be isometrically [4].

3. The Banach algebra $\bar{\mathcal{S}}''_{\alpha,\beta;\varphi}$

3.1. The Banach space $\bar{\mathcal{S}}''_{n,\alpha,\beta;\varphi}$. Let $\mathcal{M}''_n$ be the class of bounded complex Borel measures on $\Delta_n \times \mathbb{R}^n$. By Lemma 4.3 in [2], the space

$\mathcal{M}_n''$ is complete under the norm $\|\mu\| = \int_{\Delta_n\times\mathbb{R}^n} d|\mu|(\vec{t},\vec{v})$ for $\mu \in \mathcal{M}_n''$, that is, $\mathcal{M}_n''$ is a Banach space.

For $x \in C[0,T]$, $\vec{t} = (t_1,\ldots,t_n)$ and $\vec{v} = (v_1,\ldots,v_n)$, let

$$J_n(x,\vec{t},\vec{v}) = \exp\bigg\{i\sum_{j=1}^n v_j[x(t_j) - x(0)]\bigg\}.$$

Define a relation $\sim$ on $\mathcal{M}_n''$ such that $\mu_1 \sim \mu_2$ for $\mu_1,\mu_2 \in \mathcal{M}_n''$ if

$$\int_{\Delta_n\times\mathbb{R}^n} J_n(x,\vec{t},\vec{v})d\mu_1(\vec{t},\vec{v}) = \int_{\Delta_n\times\mathbb{R}^n} J_n(x,\vec{t},\vec{v})d\mu_2(\vec{t},\vec{v})$$

for $w_{\alpha,\beta;\varphi}$ a.e. $x \in C[0,T]$. It is obvious that $\sim$ is an equivalence relation on $\mathcal{M}_n''$. Let $\bar{\mathcal{M}}_n''$ be the set of equivalence classes by $\sim$. For $[\mu_1],[\mu_2] \in \bar{\mathcal{M}}_n''$ and $c \in \mathbb{C}$, define $[\mu_1]+[\mu_2] = [\mu_1+\mu_2]$ and $c[\mu_1] = [c\mu_1]$. It is obvious that the addition and scalar multiplication are well-defined, and $\bar{\mathcal{M}}_n''$ is a linear space.

LEMMA 3.1. *Define* $\|[\mu]\| = \inf\{\|\mu_1\| : \mu_1 \in [\mu]\}$ *for* $[\mu] \in \bar{\mathcal{M}}_n''$. *Then* $(\bar{\mathcal{M}}_n'', \|\cdot\|)$ *is a normed space over* $\mathbb{C}$.

Proof. It remains to prove that $\|\cdot\|$ is a norm on $\bar{\mathcal{M}}_n''$. It is clear that $\|[0]\| = 0$. Suppose that $\|[\mu]\| = 0$ for $[\mu] \in \bar{\mathcal{M}}_n''$. Then for $w_{\alpha,\beta;\varphi}$ a.e. $x \in C[0,T]$ we have

$$\left|\int_{\Delta_n\times\mathbb{R}^n} J_n(x,\vec{t},\vec{v})d\mu(\vec{t},\vec{v})\right| = \left|\int_{\Delta_n\times\mathbb{R}^n} J_n(x,\vec{t},\vec{v})d\mu_1(\vec{t},\vec{v})\right| \le \|\mu_1\|$$

for all $\mu_1 \in [\mu]$ so that we have

$$\left|\int_{\Delta_n\times\mathbb{R}^n} J_n(x,\vec{t},\vec{v})d\mu(\vec{t},\vec{v})\right| \le \inf\{\|\mu_1\| : \mu_1 \in [\mu]\} = \|[\mu]\| = 0$$

which implies

$$\int_{\Delta_n\times\mathbb{R}^n} J_n(x,\vec{t},\vec{v})d\mu(\vec{t},\vec{v}) = 0.$$

Now we have $[\mu] = [0]$. Let $c \in \mathbb{C}$ and $[\mu] \in \bar{\mathcal{M}}_n''$. If $c = 0$, then $\|c[\mu]\| = |c|\|[\mu]\|$. Suppose that $c \neq 0$. Then

$$\begin{aligned}\|c[\mu]\| &= \inf\{\|\sigma\| : \sigma \in [c\mu]\} \\ &= \inf\bigg\{|c|\bigg\|\frac{1}{c}\sigma\bigg\| : \frac{1}{c}\sigma \in [\mu]\bigg\} = |c|\inf\{\|\tau\| : \tau \in [\mu]\} = |c|\|[\mu]\|.\end{aligned}$$

Moreover let $[\mu_1], [\mu_2] \in \bar{\mathcal{M}}''_n$ and let $\epsilon > 0$ arbitrary. Take $\sigma_1 \in [\mu_1]$ and $\sigma_2 \in [\mu_2]$ such that

$$\|\sigma_1\| < \|[\mu_1]\| + \frac{\epsilon}{2} \text{ and } \|\sigma_2\| < \|[\mu_2]\| + \frac{\epsilon}{2}.$$

Then

$$\begin{aligned} \|[\mu_1] + [\mu_2]\| &= \|[\sigma_1 + \sigma_2]\| \le \|\sigma_1 + \sigma_2\| \\ &\le \|\sigma_1\| + \|\sigma_2\| < \|[\mu_1]\| + \|[\mu_2]\| + \epsilon. \end{aligned}$$

Since ϵ is arbitrary, we have

$$\|[\mu_1] + [\mu_2]\| \le \|[\mu_1]\| + \|[\mu_2]\|.$$

Now $\|\cdot\|$ is a norm on $\bar{\mathcal{M}}''_n$ which completes the proof. □

THEOREM 3.2. *$\bar{\mathcal{M}}''_n$ is a Banach space.*

Proof. It only remains to be shown that $\bar{\mathcal{M}}''_n$ is complete under the norm given by Lemma 3.1. Let $\{[\mu_n]\}_{n=1}^{\infty}$ be a Cauchy sequence of elements in $\bar{\mathcal{M}}''_n$ and take a subsequence $\{[\mu_{n_k}]\}_{k=1}^{\infty}$ of $\{[\mu_n]\}_{n=1}^{\infty}$ satisfying

$$\|[\mu_{n_k}] - [\mu_{n_{k-1}}]\| < \frac{1}{2^k} \text{ for } k = 2, 3, \ldots.$$

Take $\sigma_1 \in [\mu_{n_1}]$ with

$$\|\sigma_1\| < \|[\mu_{n_1}]\| + 1.$$

For each $k = 2, 3, \ldots$, take $\sigma_k \in [\mu_{n_k} - \mu_{n_{k-1}}]$ with

$$\|\sigma_k\| < \|[\mu_{n_k}] - [\mu_{n_{k-1}}]\| + \frac{1}{2^k}.$$

Then we have

$$\sum_{k=1}^{\infty} \|\sigma_k\| < \|\mu_{n_1}\| + \sum_{k=1}^{\infty} \frac{1}{2^{k-1}} < \infty.$$

Let $\mu = \sum_{k=1}^{\infty} \sigma_k \in \mathcal{M}''_n$. Then we also have

$$\begin{aligned} \|[\mu] - [\mu_{n_k}]\| &= \left\|[\mu] - \sum_{j=1}^{k} [\sigma_j]\right\| \le \left\|\sum_{j=k+1}^{\infty} \sigma_j\right\| \le \sum_{j=k+1}^{\infty} \|\sigma_j\| \\ &\le \sum_{j=k+1}^{\infty} \frac{1}{2^{j-1}} = \frac{1}{2^{k-1}} \end{aligned}$$

which converges to 0 as $k \to \infty$. Since $\{[\mu_n]\}_{n=1}^{\infty}$ is a Cauchy sequence it follows that

$$\lim_{n\to\infty} \|[\mu] - [\mu_n]\| = 0$$

so that $\bar{\mathcal{M}}_n''$ is complete as desired. □

For each positive integer n, let $\bar{\mathcal{S}}_{n,\alpha,\beta;\varphi}''$ be the set of functions of the form

$$F(x) = \int_{\Delta_n \times \mathbb{R}^n} J_n(x, \vec{t}, \vec{v}) d\mu(\vec{t}, \vec{v}) \tag{3}$$

for $w_{\alpha,\beta;\varphi}$ a.e. $x \in C[0,T]$, where $\mu \in \mathcal{M}_n''$. Here we take

$$\|F\|_n'' = \inf\{\|\mu\|\},$$

where the infimum is taken over all μ's so that F and μ are related by (3). By using the same method as the proof of Lemma 3.1, we can prove that $(\bar{\mathcal{S}}_{n,\alpha,\beta;\varphi}'', \|\cdot\|_n'')$ is a normed space over $\mathbb{C}$. Note that for $F \in \bar{\mathcal{S}}_{n,\alpha,\beta;\varphi}''$, we have $|F(x)| \le \|F\|_n''$ for $w_{\alpha,\beta;\varphi}$ a.e. $x \in C[0,T]$.

REMARK 3.3. In the space $\bar{\mathcal{S}}_{n,\alpha,\beta;\varphi}''$, the function $F \in \bar{\mathcal{S}}_{n,\alpha,\beta;\varphi}''$ does not determine the measure μ in (3) uniquely [2].

THEOREM 3.4. *$\bar{\mathcal{S}}_{n,\alpha,\beta;\varphi}''$ is a Banach space. Moreover, $\bar{\mathcal{S}}_{n,\alpha,\beta;\varphi}''$ is isometrically isomorphic to $\bar{\mathcal{M}}_n''$.*

Proof. Define $\phi : \bar{\mathcal{M}}_n'' \to \bar{\mathcal{S}}_{n,\alpha,\beta;\varphi}''$ by

$$\phi([\mu])(x) = \int_{\Delta_n \times \mathbb{R}^n} J_n(x, \vec{t}, \vec{v}) d\mu(\vec{t}, \vec{v})$$

for $w_{\alpha,\beta;\varphi}$ a.e. $x \in C[0,T]$. If $[\mu_1] = [\mu_2]$, then $\mu_1 \sim \mu_2$. By the definition of $\sim$, we have $\phi([\mu_1]) = \phi([\mu_2])$ which implies that ϕ is well-defined. Now we have for $w_{\alpha,\beta;\varphi}$ a.e. $x \in C[0,T]$

$$\begin{aligned} \phi(c_1[\mu_1] + c_2[\mu_2])(x) &= \phi([c_1\mu_1 + c_2\mu_2])(x) \\ &= \int_{\Delta_n \times \mathbb{R}^n} J_n(x, \vec{t}, \vec{v}) d(c_1\mu_1 + c_2\mu_2)(\vec{t}, \vec{v}) \\ &= c_1\phi([\mu_1])(x) + c_2\phi([\mu_2])(x). \end{aligned}$$

By the definition of $\bar{\mathcal{S}}''_{n,\alpha,\beta;\varphi}$, it is obvious that ϕ is onto. If $\phi([\mu_1]) = \phi([\mu_2])$, then

$$\begin{aligned}\int_{\Delta_n\times\mathbb{R}^n} J_n(x,\vec{t},\vec{v})d\mu_1(\vec{t},\vec{v}) &= \phi([\mu_1])(x)\\ &= \phi([\mu_2])(x) = \int_{\Delta_n\times\mathbb{R}^n} J_n(x,\vec{t},\vec{v})d\mu_2(\vec{t},\vec{v})\end{aligned}$$

for $w_{\alpha,\beta;\varphi}$-a.e. $x \in C[0,T]$ so that $\mu_1 \sim \mu_2$. Thus we have $[\mu_1] = [\mu_2]$ which implies that ϕ is one-to-one. Moreover, we have

$$\begin{aligned}\|\phi([\mu_1])\|''_n &= \left\|\int_{\Delta_n\times\mathbb{R}^n} J_n(\cdot,\vec{t},\vec{v})d\mu_1(\vec{t},\vec{v})\right\|''_n\\ &= \inf\{\|\mu\| : \mu \in \mathcal{M}''_n \text{ and } \mu\sim\mu_1\}\\ &= \inf\{\|\mu\| : \mu \in [\mu_1]\} = \|[\mu_1]\|\end{aligned}$$

so that ϕ is an isometric, bijective linear transformation. Since $\bar{\mathcal{M}}''_n$ is complete by Theorem 3.2, $\bar{\mathcal{S}}''_{n,\alpha,\beta;\varphi}$ is a Banach space. Now the proof is completed. □

Remark 3.5. It is not obvious whether $\bar{\mathcal{S}}''_{n,\alpha,\beta;\varphi}$ is a Banach algebra or not.

By Theorem 3.4, we have the following corollary.

Corollary 3.6. *For each positive integer j, let $F_j \in \bar{\mathcal{S}}''_{n,\alpha,\beta;\varphi}$ with*

$$\sum_{j=1}^{\infty} \|F_j\|''_n < \infty.$$

Then the function F defined by

$$F(x) = \sum_{j=1}^{\infty} F_j(x) \text{ for } w_{\alpha,\beta;\varphi} \text{ a.e. } x \in C[0,T],$$

converges absolutely and uniformly, and is an element of $\bar{\mathcal{S}}''_{n,\alpha,\beta;\varphi}$.

The following theorem is needed to prove our main results. Its proof is similar to that of Lemma 4.0 in [2].

Theorem 3.7. *If n and k are positive integers and $F \in \bar{\mathcal{S}}''_{n,\alpha,\beta;\varphi}$, then we have $\bar{\mathcal{S}}''_{n,\alpha,\beta;\varphi} \subseteq \bar{\mathcal{S}}''_{n+k,\alpha,\beta;\varphi}$ and $\|F\|''_{n+k} \le \|F\|''_n$.*

3.2. The Banach space $\bar{\mathcal{S}}^{\wedge}_{n,\alpha,\beta;\varphi}$. Let $\mathcal{M}^{\wedge}_n$ be the class of bounded complex Borel measures on $(0,T]^n \times \mathbb{R}^n$. The space $\mathcal{M}^{\wedge}_n$ is a Banach space under the norm $\|\mu\| = \int_{(0,T]^n\times\mathbb{R}^n} d|\mu|(\vec{t},\vec{v})$ for $\mu \in \mathcal{M}^{\wedge}_n$. Define a relation $\approx$ on $\mathcal{M}^{\wedge}_n$ satisfying that $\mu_1 \approx \mu_2$ for $\mu_1, \mu_2 \in \mathcal{M}^{\wedge}_n$ if

$$\int_{(0,T]^n\times\mathbb{R}^n} J_n(x,\vec{t},\vec{v})d\mu_1(\vec{t},\vec{v}) = \int_{(0,T]^n\times\mathbb{R}^n} J_n(x,\vec{t},\vec{v})d\mu_2(\vec{t},\vec{v})$$

for $w_{\alpha,\beta;\varphi}$ a.e. $x \in C[0,T]$. Then, $\approx$ is an equivalence relation on $\mathcal{M}^{\wedge}_n$. Let $\bar{\mathcal{M}}^{\wedge}_n$ be the set of equivalence classes by $\approx$. For $[\mu_1],[\mu_2] \in \bar{\mathcal{M}}^{\wedge}_n$ and $c \in \mathbb{C}$, define $[\mu_1]+[\mu_2] = [\mu_1+\mu_2]$ and $c[\mu_1] = [c\mu_1]$. It is obvious that the addition and scalar multiplication are well-defined, and $\bar{\mathcal{M}}^{\wedge}_n$ is a linear space. Define $\|[\mu]\| = \inf\{\|\mu_1\| : \mu_1 \in [\mu]\}$ for $[\mu] \in \bar{\mathcal{M}}^{\wedge}_n$. Then, one can show that $\bar{\mathcal{M}}^{\wedge}_n$ is a Banach space by using the same process as the proofs of Lemma 3.1 and Theorem 3.2.

For each positive integer n, let $\bar{\mathcal{S}}^{\wedge}_{n,\alpha,\beta;\varphi}$ be the set of functions of the form

$$F(x) = \int_{(0,T]^n\times\mathbb{R}^n} J_n(x,\vec{t},\vec{v})d\mu(\vec{t},\vec{v}) \tag{4}$$

for $w_{\alpha,\beta;\varphi}$ a.e. $x \in C[0,T]$, where $\mu \in \mathcal{M}^{\wedge}_n$. Here we take

$$\|F\|^{\wedge}_n = \inf\{\|\mu\|\},$$

where the infimum is taken over all μ's so that F and μ are related by (4). By using the same method as the proof of Theorem 3.4, we can prove that $(\bar{\mathcal{S}}^{\wedge}_{n,\alpha,\beta;\varphi}, \|\cdot\|^{\wedge}_n)$ is a Banach space over $\mathbb{C}$ and $\bar{\mathcal{S}}^{\wedge}_{n,\alpha,\beta;\varphi}$ is isometrically isomorphic to $\bar{\mathcal{M}}^{\wedge}_n$. Clearly we have $\bar{\mathcal{S}}''_{n,\alpha,\beta;\varphi} \subseteq \bar{\mathcal{S}}^{\wedge}_{n,\alpha,\beta;\varphi}$ and for $F \in \bar{\mathcal{S}}''_{n,\alpha,\beta;\varphi}$

$$\|F\|^{\wedge}_n \le \|F\|''_n$$

since every measure in $\mathcal{M}''_n$ can be extended to be in $\mathcal{M}^{\wedge}_n$ without increasing its variation. We note that if $F \in \bar{\mathcal{S}}^{\wedge}_{n,\alpha,\beta;\varphi}$, then $|F(x)| \le \|F\|^{\wedge}_n$ for $w_{\alpha,\beta;\varphi}$ a.e. $x \in C[0,T]$. Furthermore, we have similar forms of Corollary 3.6 and Theorem 3.7.

3.3. The Banach algebra $\bar{\mathcal{S}}''_{\alpha,\beta;\varphi}$. Define $\bar{\mathcal{S}}''_{0,\alpha,\beta;\varphi}$ and $\bar{\mathcal{S}}^{\wedge}_{0,\alpha,\beta;\varphi}$ to be the space of constant functions and define their norms to be their absolute values. Let $\bar{\mathcal{S}}''_{\alpha,\beta;\varphi}$ be the space of functions F on $C[0,T]$ defined by

$$F(x) = \sum_{n=0}^{\infty} F_n(x) \text{ for } w_{\alpha,\beta;\varphi} \text{ a.e. } x \in C[0,T], \tag{5}$$

where $F_n \in \bar{\mathcal{S}}''_{n,\alpha,\beta;\varphi}$ and $\sum_{n=0}^{\infty}\|F_n\|''_n < \infty$. Since $|F_n(x)| \le \|F_n\|''_n$ for $w_{\alpha,\beta;\varphi}$ a.e. $x \in C[0,T]$, the series in (5) converges absolutely and uniformly over $C[0,T]$. For $F \in \bar{\mathcal{S}}''_{\alpha,\beta;\varphi}$, define $\|F\|''$ by

$$\|F\|'' = \inf\left\{\sum_{n=0}^{\infty}\|F_n\|''_n\right\},$$

where the infimum is taken over all representations of F given by (5). Moreover, we have

$$|F(x)| \le \|F\|'' \text{ for } w_{\alpha,\beta;\varphi} \text{ a.e. } x \in C[0,T]. \tag{6}$$

Lemma 3.8. *$(\bar{\mathcal{S}}''_{\alpha,\beta;\varphi}, \|\cdot\|'')$ is a normed space over $\mathbb{C}$.*

Proof. It is obvious that $\bar{\mathcal{S}}''_{\alpha,\beta;\varphi}$ is a linear space over $\mathbb{C}$. If $F(x) = 0 \in \bar{\mathcal{S}}''_{\alpha,\beta;\varphi}$ for $w_{\alpha,\beta;\varphi}$ a.e. $x \in C[0,T]$, take $F_n(x) = 0 \in \bar{\mathcal{S}}''_{n,\alpha,\beta;\varphi}$ for $w_{\alpha,\beta;\varphi}$ a.e. $x \in C[0,T]$ so that F and $\{F_n\}$ are related by (5). Then $\|F\|'' = 0$ clearly. For $F \in \bar{\mathcal{S}}''_{\alpha,\beta;\varphi}$, suppose that $\|F\|'' = 0$. Then we have $F(x) = 0$ for $w_{\alpha,\beta;\varphi}$ a.e. $x \in C[0,T]$ by (6). Let $c \in \mathbb{C}$ and $F \in \bar{\mathcal{S}}''_{\alpha,\beta;\varphi}$. If $c = 0$, then $\|cF\|'' = \|0\|'' = 0 = |c|\|F\|''$. Suppose that $c \ne 0$. If F and $\{F_n\}$ are related by (5), then cF and $\{cF_n\}$ are also related by (5) so that $\|cF\|'' \le |c|\sum_{n=0}^{\infty}\|F_n\|''_n$ which implies $\|cF\|'' \le |c|\|F\|''$, since $\{F_n\}$ is arbitrary. Let $\epsilon > 0$ arbitrary. Take a sequence $\{G_n\}$ such that cF and $\{G_n\}$ are related by (5) with $\sum_{n=0}^{\infty}\|G_n\|''_n < \|cF\|'' + \epsilon$. Then F and $\{\frac{1}{c}G_n\}$ are related by (5) so that $\|F\|'' \le \frac{1}{|c|}\sum_{n=0}^{\infty}\|G_n\|''_n < \frac{1}{|c|}(\|cF\|'' + \epsilon)$. Hence $|c|\|F\|'' < \|cF\|'' + \epsilon$. Since ϵ is arbitrary, we have $|c|\|F\|'' \le \|cF\|''$ so that $|c|\|F\|'' = \|cF\|''$. Let $G \in \bar{\mathcal{S}}''_{\alpha,\beta;\varphi}$. Let F, G and $\{H_n\}, \{I_n\}$ be related by (5), respectively, with $\sum_{n=0}^{\infty}\|H_n\|''_n < \|F\|'' + \frac{\epsilon}{2}$ and $\sum_{n=0}^{\infty}\|I_n\|''_n < \|G\|'' + \frac{\epsilon}{2}$. Then $\|F+G\|'' \le \sum_{n=0}^{\infty}\|H_n + I_n\|''_n \le \sum_{n=0}^{\infty}\|H_n\|''_n + \sum_{n=0}^{\infty}\|I_n\|''_n < \|F\|'' + \|G\|'' + \epsilon$. Since ϵ is arbitrary, we have $\|F+G\|'' \le \|F\|'' + \|G\|''$ which completes the proof. □

The following lemma is needed to prove one of our main results. Its proof is motivated from Lemma 4.1 in [2].

Lemma 3.9. *For each $n \ge 0$,*

$$\bar{\mathcal{S}}''_{n,\alpha,\beta;\varphi} \subseteq \bar{\mathcal{S}}'_{\alpha,\beta;\varphi}$$

and if $F \in \bar{\mathcal{S}}''_{n,\alpha,\beta;\varphi}$, then

$$\|F\|' \le \|F\|''_n.$$

Proof. The case $n = 0$ is trivial so let $n \geq 1$ be given and let F given by (3) be an element of $\bar{\mathcal{S}}''_{n,\alpha,\beta;\varphi}$, where $\mu \in \mathcal{M}''_n$. Let $B_n[0,T]$ be a subset of $B[0,T]$ consisting of right continuous step functions which have no more than n points of discontinuities. Define a function $\Phi : \Delta_n \times \mathbb{R}^n \to B_n[0,T]$ by

$$\Phi(\vec{t}, \vec{v}) = u_{\vec{t},\vec{v}},$$

where

$$u_{\vec{t},\vec{v}}(s) = \begin{cases} \sum_{j=p}^{n} v_j & \text{if} \quad t_{p-1} \leq s < t_p,\ p = 1, \dots, n \\ 0 & \text{if} \quad t_n \leq s \leq T \end{cases}$$

for $(\vec{t}, \vec{v}) = ((t_1, \dots, t_n), (v_1, \dots, v_n)) \in \Delta_n \times \mathbb{R}^n$. Let $\mathcal{A}'_n$ be the class of subsets $E \subseteq B_n[0,T]$ such that $\Phi^{-1}(E)$ is a Borel subset of $\Delta_n \times \mathbb{R}^n$. It is clear that $\mathcal{A}'_n$ is a σ-algebra. Define a measure σ on $\mathcal{A}'_n$ as follows: If E is an element of $\mathcal{A}'_n$, let

$$\sigma(E) = \mu(\Phi^{-1}(E)).$$

It is clear that σ is a bounded complex measure on $\mathcal{A}'_n$ and $\|\sigma\| \leq \|\mu\|$. We now define a complex measure $\hat{\sigma}$ on $B[0,T]$ as follows: For $E \subseteq B[0,T]$,

$$\hat{\sigma}(E) = \sigma(E \cap B_n[0,T]),$$

whenever the latter exists. Then $\hat{\sigma}$ is defined on $\mathcal{A}'$ by the following assertion: Let

$$E = \{u \in B[0,T] : \langle u, f \rangle_{\alpha,\beta} < \lambda\},$$

where $f \in L^2_{\alpha,\beta}[0,T]$ and $\lambda \in \mathbb{R}$. We have

$$\begin{aligned} \langle \Phi(\vec{t}, \vec{v}), f \rangle_{\alpha,\beta} &= \langle u_{\vec{t},\vec{v}}, f \rangle_{\alpha,\beta} = \sum_{j=1}^{n} \sum_{l=j}^{n} v_l \int_{[t_{j-1}, t_j)} f(t)\, d\nu_{\alpha,\beta}(t) \\ &= \sum_{l=1}^{n} v_l \sum_{j=1}^{l} \int_{[t_{j-1}, t_j)} f(t)\, d\nu_{\alpha,\beta}(t) \\ &= \sum_{l=1}^{n} v_l \int_{[0, t_l)} f(t)\, d\nu_{\alpha,\beta}(t) \end{aligned}$$

so that it follows from the definition of E and Φ that

$$\Phi^{-1}(E \cap B_n[0,T]) = \left\{ (\vec{t}, \vec{v}) : (\vec{t}, \vec{v}) \in \Delta_n \times \mathbb{R}^n, \sum_{j=1}^{n} v_j \theta(t_j) < \lambda \right\},$$

where $\theta(t) = \int_0^t f(s)d\nu_{\alpha,\beta}(s)$. Since θ is continuous on $[0,T]$, $\Phi^{-1}(E \cap B_n[0,T])$ is a Borel set, and $E \cap B_n[0,T] \in \mathcal{A}'_n$. Since $\mathcal{A}'$ generated by E of this form, it follows that $\hat{\sigma}(E)$ is at least defined on $\mathcal{A}'$. Clearly, $\hat{\sigma} \in \mathcal{M}(B[0,T])$ and $\|\hat{\sigma}\| \le \|\sigma\|$. We define for each positive integer m,

$$J_m(w) = \exp\left\{\frac{q}{m}2\pi i\right\}, \text{ where } \frac{2\pi(q-1)}{m} < w \le \frac{2\pi q}{m} \text{ for } q = 1, 2, \dots, m$$

and note that $\lim_{m\to\infty} J_m(w) = \exp\{iw\}$ boundedly. Then for $w_{\alpha,\beta;\varphi}$ a.e. $x \in C[0,T]$,

$$\lim_{m\to\infty} F_m(x) = F(x),$$

where

$$F_m(x) = \int_{\Delta_n\times\mathbb{R}^n} J_m\left(\sum_{j=1}^n v_j[x(t_j) - x(0)]\right)d\mu(\vec{t},\vec{v}).$$

For $w_{\alpha,\beta;\varphi}$ a.e. $x \in C[0,T]$, let

$$E_{m,q} = \left\{u \in B[0,T] \,\middle|\, \frac{2\pi(q-1)}{m} < \int_0^T u(t)dx(t) \le \frac{2\pi q}{m}\right\}.$$

Since

$$\begin{aligned}\int_0^T \Phi(\vec{t},\vec{v})(t)dx(t) &= \sum_{j=1}^n\sum_{l=j}^n v_l \int_{[t_{j-1},t_j)} dx(t)\\ &= \sum_{l=1}^n v_l \sum_{j=1}^l \int_{[t_{j-1},t_j)} dx(t) = \sum_{l=1}^n v_l[x(t_l) - x(0)]\end{aligned}$$

we have

$$\begin{aligned}&\Phi^{-1}(E_{m,q} \cap B_n[0,T])\\ =\ &\left\{(\vec{t},\vec{v}) : (\vec{t},\vec{v}) \in \Delta_n \times \mathbb{R}^n, \Phi(\vec{t},\vec{v}) \in E_{m,q}\right\}\\ =\ &\left\{(\vec{t},\vec{v}) \in \Delta_n \times \mathbb{R}^n \,\middle|\, \frac{2\pi(q-1)}{m} < \sum_{j=1}^n v_j[x(t_j) - x(0)] \le \frac{2\pi q}{m}\right\}\\ \equiv\ &\delta_{m,q}.\end{aligned}$$

Now we have

$$\begin{aligned} F_m(x) &= \sum_{q=1}^{m} \exp\left\{\frac{q}{m}2\pi i\right\}\mu(\delta_{m,q}) = \sum_{q=1}^{m} \exp\left\{\frac{q}{m}2\pi i\right\}\hat{\sigma}(E_{m,q}) \\ &= \int_{B[0,T]} J_m\left(\int_0^T u(t)dx(t)\right)d\hat{\sigma}(u). \end{aligned}$$

By letting $m \to \infty$, we have for $w_{\alpha,\beta;\varphi}$ a.e. $x \in C[0,T]$

$$F(x) = \lim_{m\to\infty} F_m(x) = \int_{B[0,T]} \exp\left\{i\int_0^T u(t)dx(t)\right\}d\hat{\sigma}(u) \in \bar{\mathcal{S}}'_{\alpha,\beta;\varphi}$$

so that $\bar{\mathcal{S}}''_{n,\alpha,\beta;\varphi} \subseteq \bar{\mathcal{S}}'_{\alpha,\beta;\varphi}$ and $\|F\|' \le \|\hat{\sigma}\| \le \|\sigma\| \le \|\mu\|$. Since μ is arbitrary measure such that F and μ are related by (3), we have $\|F\|' \le \|F\|''_n$ and the lemma is proved. □

The following theorem is one of our main results. By using Lemma 3.9, we can prove this theorem.

THEOREM 3.10. *The space $\bar{\mathcal{S}}''_{\alpha,\beta;\varphi}$ is contained in $\bar{\mathcal{S}}'_{\alpha,\beta;\varphi}$ and if $F \in \bar{\mathcal{S}}''_{\alpha,\beta;\varphi}$, then $\|F\|' \le \|F\|''$.*

Note that $\bar{\mathcal{S}}''_{n,\alpha,\beta;\varphi}$ is a Banach space by Theorem 3.4. The following theorem is needed to prove one of our main results. Its proof is similar to the proof of Lemma 4.5 in [2].

THEOREM 3.11. *The space $\bar{\mathcal{S}}''_{\alpha,\beta;\varphi}$ is complete under the norm $\|F\|''$ for $F \in \bar{\mathcal{S}}''_{\alpha,\beta;\varphi}$.*

The following lemma is needed to prove our main results. Its proof is similar to the proof of Lemma 4.6 in [2].

LEMMA 3.12. *If $G \in \bar{\mathcal{S}}^{\wedge}_{n,\alpha,\beta;\varphi}$, we can express G in the form*

$$G = F + H,$$

where $F \in \bar{\mathcal{S}}''_{n,\alpha,\beta;\varphi}$ and $H \in \bar{\mathcal{S}}^{\wedge}_{n-1,\alpha,\beta;\varphi}$ if $n > 1$ and $H = 0$ if $n = 1$, and where

$$\|G\|^{\wedge}_n \ge \|F\|''_n + \|H\|^{\wedge}_{n-1}.$$

The following lemma follows from repeated applications of Lemma 3.12.

Lemma 3.13. *If $G \in \bar{\mathcal{S}}^{\wedge}_{n,\alpha,\beta;\varphi}$, we can express G in the form*

$$G = \sum_{q=1}^{n} F_q,$$

where $F_q \in \bar{\mathcal{S}}''_{q,\alpha,\beta;\varphi}$ for $q = 1, \ldots, n$. Moreover, $\|G\|^{\wedge}_n \geq \sum_{q=1}^{n} \|F_q\|''_q$.

By Lemma 3.13, we have the following theorem.

Theorem 3.14. *For each $n = 0, 1, 2, \ldots$, we have $\bar{\mathcal{S}}^{\wedge}_{n,\alpha,\beta;\varphi} = \bar{\mathcal{S}}''_{n,\alpha,\beta;\varphi}$, and if $F \in \mathcal{S}^{\wedge}_{n,\alpha,\beta;\varphi}$, then $\|F\|^{\wedge}_n = \|F\|''_n$.*

We now have the following lemma and its proof is similar to that of Lemma 4.9 in [2].

Lemma 3.15. *If $F \in \bar{\mathcal{S}}^{\wedge}_{m,\alpha,\beta;\varphi}$, $G \in \bar{\mathcal{S}}^{\wedge}_{n,\alpha,\beta;\varphi}$ and $H(x) = F(x)G(x)$ for $w_{\alpha,\beta;\varphi}$ a.e. $x \in C[0,T]$, where $m, n = 0, 1, 2, \ldots$, then $H \in \bar{\mathcal{S}}^{\wedge}_{m+n,\alpha,\beta;\varphi}$ and $\|H\|^{\wedge}_{m+n} \leq \|F\|^{\wedge}_m \|G\|^{\wedge}_n$.*

The following corollary immediately follows from Theorem 3.14 and Lemma 3.15.

Corollary 3.16.

1. *For $n = 1, 2, \ldots$, $\bar{\mathcal{M}}''_n$ is isometrically isomorphic to $\bar{\mathcal{M}}^{\wedge}_n$.*
2. *If we replace $\bar{\mathcal{S}}^{\wedge}_{n,\alpha,\beta;\varphi}$ by $\bar{\mathcal{S}}''_{n,\alpha,\beta;\varphi}$ in Lemma 3.15, the results hold still.*

By Theorem 3.11 and Corollary 3.16, one can prove the following theorem which is one of our main results. For a detailed proof, see that of Theorem 4.2 in [2].

Theorem 3.17. *The space $\bar{\mathcal{S}}''_{\alpha,\beta;\varphi}$ is a Banach algebra with the norm $\|\cdot\|''$.*

4. Evaluations of the analytic Feynman integrals

Feynman integrals were introduced by Feynman in his formulation of quantum mechanics, but they are inadequate mathematically [6]. One of approaches to define rigorously them, is to use an analytic continuation from real to imaginary time, which is now called the analytic Feynman integral [7].

In this section we evaluate analytic Feynman integrals of the functions in $\bar{\mathcal{S}}''_{\alpha,\beta;\varphi}$, which play important roles in treating the heat equation and the Schrödinger equation by integration over the Wiener space [1].

THEOREM 4.1. *Let $n>0$. Let $F_n(\in \bar{\mathcal{S}}''_{n,\alpha,\beta;\varphi})$ and $\mu_n(\in \mathcal{M}''_n)$ be related by (3). Then for $\lambda>0$*

$$(7)\ J_{F_n}(\lambda) = \varphi(\mathbb{R})\int_{\Delta_n\times\mathbb{R}^n}\exp\bigg\{-\frac{1}{2\lambda}\sum_{l=1}^n[\beta(t_l)-\beta(t_{l-1})]\bigg(\sum_{j=l}^n v_j\bigg)^2 +i\lambda^{-\frac12}\sum_{l=1}^n\sum_{j=l}^n[\alpha(t_l)-\alpha(t_{l-1})]v_j\bigg\}d\mu_n(\vec{t},\vec{v}),$$

where $\vec{t}=(t_1,\dots,t_n)$ and $\vec{u}=(u_1,\dots,u_n)$. In addition, if there exists $M_n>0$ satisfying

$$(8)\qquad \int_{\Delta_n\times\mathbb{R}^n}\exp\bigg\{\mathrm{Re}(i\lambda^{-\frac12})\sum_{l=1}^n\sum_{j=l}^n[\alpha(t_l)-\alpha(t_{l-1})]v_j\bigg\}d|\mu_n|(\vec{t},\vec{v}) \le M_n$$

for any $\lambda\in\mathbb{C}_+$, then $\int_{C[0,T]}^{anw_\lambda}F_n(x)dw_{\alpha,\beta;\varphi}(x)$ is given by the right-hand side of (7). Moreover, if (8) holds for all $\lambda\in\{z\in\mathbb{C}:\mathrm{Re}\,z\ge 0, z\neq 0\}$, then for any nonzero real q, $\int_{C[0,T]}^{anf_q}F_n(x)dw_{\alpha,\beta;\varphi}(x)$ is given by the right-hand side of (7) with replacing λ by $-iq$.

Proof. By Theorem 2.1, we have for $\lambda>0$

$$\begin{aligned}
&J_{F_n}(\lambda)\\
&= \int_{C[0,T]}\int_{\Delta_n\times\mathbb{R}^n}J_n(\lambda^{-\frac12}x,\vec{t},\vec{v})d\mu_n(\vec{t},\vec{v})dw_{\alpha,\beta;\varphi}(x)\\
&= \int_{\Delta_n\times\mathbb{R}^n}\int_{C[0,T]}\exp\bigg\{i\lambda^{-\frac12}\sum_{j=1}^n v_j[x(t_j)-x(0)]\bigg\}dw_{\alpha,\beta;\varphi}(x)d\mu_n(\vec{t},\vec{v})\\
&= \int_{\Delta_n\times\mathbb{R}^n}\bigg[\prod_{j=1}^n\frac{1}{2\pi[\beta(t_j)-\beta(t_{j-1})]}\bigg]^{\frac12}\int_{\mathbb{R}^{n+1}}\exp\bigg\{i\lambda^{-\frac12}\sum_{j=1}^n v_j[u_j-u_0]\\
&\quad -\frac12\sum_{j=1}^n\frac{[u_j-\alpha(t_j)-u_{j-1}+\alpha(t_{j-1})]^2}{\beta(t_j)-\beta(t_{j-1})}\bigg\}dm_L^n(\vec{u})d\varphi(u_0)d\mu_n(\vec{t},\vec{v}).
\end{aligned}$$

For $j=1,\dots,n$, let $z_j=u_j-u_{j-1}$. Then $u_j=\sum_{l=1}^{j} z_l+u_0$ so that

$$\begin{aligned}
&J_{F_n}(\lambda)\\
&= \int_{\Delta_n\times\mathbb{R}^n}\left[\prod_{j=1}^{n}\frac{1}{2\pi[\beta(t_j)-\beta(t_{j-1})]}\right]^{\frac{1}{2}}\int_{\mathbb{R}^{n+1}}\exp\bigg\{i\lambda^{-\frac{1}{2}}\sum_{l=1}^{n}\sum_{j=l}^{n}v_jz_l\\
&\quad -\frac{1}{2}\sum_{j=1}^{n}\frac{[z_j-\alpha(t_j)+\alpha(t_{j-1})]^2}{\beta(t_j)-\beta(t_{j-1})}\bigg\}dm_L^n(\vec{z})d\varphi(u_0)d\mu_n(\vec{t},\vec{v}),
\end{aligned}$$

where $\vec{z}=(z_1\dots,z_n)$. By a simple calculation, we have

$$\begin{aligned}
J_{F_n}(\lambda) = \varphi(\mathbb{R})\int_{\Delta_n\times\mathbb{R}^n}&\exp\bigg\{-\frac{1}{2\lambda}\sum_{l=1}^{n}[\beta(t_l)-\beta(t_{l-1})]\bigg(\sum_{j=l}^{n}v_j\bigg)^2\\
&+i\lambda^{-\frac{1}{2}}\sum_{l=1}^{n}\sum_{j=l}^{n}[\alpha(t_l)-\alpha(t_{l-1})]v_j\bigg\}d\mu_n(\vec{t},\vec{v})
\end{aligned}$$

which implies (7). If (8) holds, then we have the remainder part of this theorem by the analytic continuation and the dominated convergence theorem. □

By letting $M_n=\|\mu_n\|$ in (8) of Theorem 4.1, we now have the following corollary.

Corollary 4.2. *Let $n>0$. Let $F_n(\in\bar{\mathcal{S}}''_{n,\alpha,\beta;\varphi})$ and $\mu_n(\in\mathcal{M}''_n)$ be related by* (3). *If α is a constant function on $[0,T]$, then we have, for any $\lambda\in\mathbb{C}_+$,*

$$\begin{aligned}
(9)\qquad &\int_{C[0,T]}^{anw_\lambda}F_n(x)dw_{\alpha,\beta;\varphi}(x)\\
&= \varphi(\mathbb{R})\int_{\Delta_n\times\mathbb{R}^n}\exp\bigg\{-\frac{1}{2\lambda}\sum_{l=1}^{n}[\beta(t_l)-\beta(t_{l-1})]\bigg(\sum_{j=l}^{n}v_j\bigg)^2\bigg\}d\mu_n(\vec{t},\vec{v}).
\end{aligned}$$

Moreover, for any nonzero real q, $\int_{C[0,T]}^{anf_q}F_n(x)dw_{\alpha,\beta;\varphi}(x)$ is given by the right-hand side of (9) *with replacing λ by $-iq$.*

Theorem 4.3. *Let F be given by* (5). *Then for $\lambda>0$*

$$J_F(\lambda)=\sum_{n=0}^{\infty}J_{F_n}(\lambda) \tag{10}$$

where $J_{F_0}(\lambda) = F_0\varphi(\mathbb{R})$ *and* $J_{F_n}(\lambda)$ *is given by* (7) *if* $n \geq 1$. *In addition, if for each positive integer* n, *there exists* $M_n > 0$ *satisfying* (8) *for any* $\lambda \in \mathbb{C}_+$ *such that*

$$\sum_{n=1}^{\infty} M_n < \infty, \tag{11}$$

then $\int_{C[0,T]}^{anw_\lambda} F(x)dw_{\alpha,\beta;\varphi}(x)$ *is given by*

$$\int_{C[0,T]}^{anw_\lambda} F(x)dw_{\alpha,\beta;\varphi}(x) = \sum_{n=0}^{\infty} \int_{C[0,T]}^{anw_\lambda} F_n(x)dw_{\alpha,\beta;\varphi}(x), \tag{12}$$

where $\int_{C[0,T]}^{anw_\lambda} F_0(x)dw_{\alpha,\beta;\varphi}(x) = F_0\varphi(\mathbb{R})$ *and* $\int_{C[0,T]}^{anw_\lambda} F_n(x)dw_{\alpha,\beta;\varphi}(x)$ *is given by the right-hand side of* (7) *if* $n \geq 1$. *Moreover, if* (8) *and* (11) *hold for all* $\lambda \in \{z \in \mathbb{C} : \operatorname{Re} z \geq 0, z \neq 0\}$, *then for any nonzero real* q, $\int_{C[0,T]}^{anf_q} F(x)dw_{\alpha,\beta;\varphi}(x)$ *is given by*

$$\int_{C[0,T]}^{anf_q} F(x)dw_{\alpha,\beta;\varphi}(x) = \sum_{n=0}^{\infty} \int_{C[0,T]}^{anf_q} F_n(x)dw_{\alpha,\beta;\varphi}(x), \tag{13}$$

where $\int_{C[0,T]}^{anf_q} F_0(x)dw_{\alpha,\beta;\varphi}(x) = F_0\varphi(\mathbb{R})$ *and* $\int_{C[0,T]}^{anf_q} F_n(x)dw_{\alpha,\beta;\varphi}(x)$ *is given by the right-hand side of* (7) *with replacing* λ *by* $-iq$ *if* $n \geq 1$.

Corollary 4.4. *Let* F *be given by* (5). *If* α *is a constant function on* $[0,T]$, *then* $\int_{C[0,T]}^{anw_\lambda} F(x)dw_{\alpha,\beta;\varphi}(x)$ *exists for any* $\lambda \in \mathbb{C}_+$ *and it is given by* (12), *where* $\int_{C[0,T]}^{anw_\lambda} F_n(x)dw_{\alpha,\beta;\varphi}(x)$ *is expressed by* (9) *if* $n \geq 1$. *Moreover, for any nonzero real* q, $\int_{C[0,T]}^{anf_q} F(x)dw_{\alpha,\beta;\varphi}(x)$ *is given by* (13), *where* $\int_{C[0,T]}^{anf_q} F_n(x)dw_{\alpha,\beta;\varphi}(x)$ *is expressed by* (9) *with replacing* λ *by* $-iq$ *if* $n \geq 1$.

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