Korean J. Math. **20** (2012), No. 1, pp. 47–60

ANALYTIC CALCULATION OF EUROPEAN OPTION PRICING IN STOCHASTIC VOLATILITY ASSET MODEL

Jae-pill Oh

Abstract. We deal some analytic calculations for European option pricing by using the theory of elementary solution of generalized diffusion equation mainly.

1. Introduction

In this paper, we introduce a method of option pricing for several asset models which are special forms of a general class of volatility asset models.

In preprint [6], we can meet the general class of volatility asset models of the form

$$dS_t/S_t = \mu dt + f(\sigma_t)[\sqrt{1-\rho^2}dW_t^{(1)} + \rho dW_t^{(2)}] \tag{1}$$

$$d\sigma_t/\sigma_t = \beta(\sigma_t)dt + g(\sigma_t)dW_t^{(2)}, \tag{2}$$

where $W_t^{(1)}$ and $W_t^{(2)}$ are standard Brownian motions on a same probability space $(\Omega, \mathbf{F}, P)$. S_t denotes the price of the (traded) asset and σ_t is the (non-traded) stochastic local return variance at time t. The authors in [6] classified many stochastic volatility models by using some specifications and studied hedge strategies from an experimental as well as from an empirical perspective. Because the solutions of above stochastic differential equations can be represented by closed forms, perhaps we can get option pricing of asset models which are defined from above equations by same method. But, in this paper, we introduce some analytic

Received December 28, 2011. Revised February 27, 2012. Accepted March 6, 2012.

2010 Mathematics Subject Classification: 60H10, 60J60, 62P20.

Key words and phrases: SDE, volatility asset model, diffusion process, elementary solution of diffusion equation, European option pricing.

calculations of option pricing by using the theory of elementary solution of diffusion equation.

There are many asset models which are modified by stochastic volatilities. As we know, Black-Scholes volatility asset model is defined by two stochastic differential equations of the forms

$$dS_t = S_t(\mu dt + \sigma_t dW_t), \tag{3}$$

$$d\sigma_t = b(\sigma_t)dt + a(\sigma_t)d\bar{W}_t, \tag{4}$$

where μ is a constant, W_t and $\bar{W}_t$ are independent Brownian motions, and σ_t defined by (4) is volatility of asset price S_t at time t. As we know, some stochastic volatility models $\sigma_t, t \geq 0$ defined by the solutions of stochastic differential equations are diffusion processes.

Stein and Stein([8]) introduced a mean reverting diffusion process

$$d\sigma_t = -\delta(\sigma_t - \theta)dt + kd\bar{W}_t, \tag{5}$$

and used a Fourier inverse transformations to integrate the resulting partial differential equation for the price of a European option. Also, this model is a special form of above general class represented by (1) and (2). But this volatility model is a diffusion process, more in full, a Ornstein-Uhlenbeck process. Thus, we can use the elementary solution theory to get option prices.

We denote the density function f_{W_T} of random variable W_T of (3). Then we can get the European option pricing at time $t = 0$ with maturity T, if it is call and $S_T \geq K$,

$$\begin{aligned} u(t,0) &= e^{-rT} S_0 e^{\mu T} \int_{R^+} \int_0^\infty h(z) p(T, s(x), s(z)) dm(s(z)) f_{W_T}(y) dy \\ &\quad - e^{-rT} K. \end{aligned}$$

for the function $h(z) = \exp\{-(1/2)z^2 T + zy\}$.

Under some special case, we can get the distribution function

$$p(T, s(x), s(z)).$$

If we assume $b(x) = -b(constant)$ and in (4), this corresponding diffusion equation is periodic and we get

$$p(t,x,y) = (1/2\sqrt{\pi}t)\exp\{-\lambda_0 t - (x-y)^2/4t + b(x+y)/2\}.$$

Under some reflecting boundary conditions, we get for large t,

$$p(t,x,y) \sim (2\pi t)^{-1/2} e^{-B(0)},$$

and

$$p(t,x,y) \sim 1/M, \quad \text{for} \quad \text{some} \quad M$$

Therefore, in these case, we get for large t,

$$E[h(\sigma_t)] \sim (2\pi t)^{-1/2} \int_0^\infty a(x)^{-1} h(x) e^{B(x)-B(0)} dx,$$

and

$$E[h(\sigma_t)] \sim M^{-1} 2 \int_0^\infty a(x)^{-1} h(x) e^{B(x)} dx,$$

respectively.

In Section 2, we deal the elementary solution $p(t,x,y)$ of the generalized diffusion equation. In Section 3, we introduce the calculation of the price of European call option for, so-called, the Stein/Stein volatility model.

2. Distribution of diffusion process

Let $a \in C^1(R^+)$, $b \in C(R^+)$, $R^+ = (0,\infty)$ with $a(x) > 0$ and $(\bar{W}_t, P)$ be a standard Brownian motion. Let $\sigma_t, t > 0$ be the solution of stochastic differential equation (4);

$$\sigma_t = \sigma_0 + \int_R a(\sigma_t) d\bar{W}_t + \int_R b(\sigma_t) dt \tag{6}$$

with initial condition $\sigma_0 = x \in (0,\infty)$. Then we know that the solution $\sigma_t, t > 0$ is a diffusion process with the generator

$$\mathbf{L} = \frac{1}{2} a^2(x) \frac{d^2}{dx^2} + b(x) \frac{d}{dx}. \tag{7}$$

Thus, by the differential equations theory in diffusion process, we get the scale function $s(x)$ and the speed density $dM(y) := m(y)dy$ with the natural scale $y = s(x)$ as following;

$$s(x) = \int_1^x a(z)^{-1} e^{-B(z)} dz, \quad x \in (0,\infty), \tag{8}$$

$$m(y) = 2 \int_1^{s^{-1}(y)} a(z)^{-1} e^{B(z)} dz, \quad y = s(x) \in (s(0), s(\infty)), \tag{9}$$

where

$$B(z) := \int_1^z \frac{2b(\xi)}{a(\xi)^2} d\xi.$$

Then the value of functional $E[h(\sigma_t)]$ is the expectation of $h(\sigma_t)$ with respect to the probability P;

$$E[h(\sigma_t)] = \int_0^\infty h(z)p(t, s(x), s(z))dm(s(z)), \tag{10}$$

where h is a smooth function and σ_t is the solution of stochastic differential equation (6).

Let $S = (l_1, l_2)$ be an open interval in the sense of McKean(c.f. [4]) with $-\infty \leq l_1 < 0 < l_2 \leq \infty$ and $m(x)$ a real valued nontrivial right continuous nondecreasing function on it with $m(0) = 0$. For a functional

$$v(t, x) = E[h(\sigma_t)|\sigma_0 = x], \tag{11}$$

where h is a smooth function on R with polynomial growth order, we get the second order partial differential equation which is so-called the generalized diffusion equation;

$$\begin{aligned} \frac{\partial v(t,x)}{\partial t} &= \mathbf{A}v(t,x), \\ v(0,x) &= h(x), \end{aligned} \tag{12}$$

where

$$\mathbf{A} = \frac{1}{2}a(x)^2\frac{\partial^2}{\partial x^2} + b(x)\frac{\partial}{\partial x}$$

on the interval S. Further, from the elementary solution (the fundamental solution of (12)) $p(t, x, y)$ of the generalized diffusion equation, we get

$$v(t, x) = \int_R h(y)p(t, x, y)dy. \tag{13}$$

To get the elementary solution $p(t, x, y)$ of (12) which is in (13) on the interval S, so-called in the McKean sense, we need several a little bit complex steps. For each $\alpha \in \mathbf{C}$ and $x \in S$, let $\varphi_1(x, \alpha)$ and $\varphi_2(x, \alpha)$ be the solution of the integral equations

$$\begin{aligned} \varphi_1(x,\alpha) &= 1 + \alpha\int_{0+}^{x+}(x-y)\varphi_1(x,\alpha)dm(y), \\ \varphi_2(x,\alpha) &= x + \alpha\int_{0+}^{x+}(x-y)\varphi_2(x,\alpha)dm(y), \end{aligned}$$

respectively. Then for each $\alpha > 0$, there exist the limits

$$\begin{aligned} h_1(\alpha) &= -\lim_{x \downarrow l_1} \varphi_2(x,\alpha)/\varphi_1(x,\alpha), \\ h_2(\alpha) &= \lim_{x \uparrow l_2} \varphi_2(x,\alpha)/\varphi_1(x,\alpha). \end{aligned}$$

We will use usual convention $1/\infty = 0$, $(\pm a)/0 = \pm\infty$, $\infty \pm a = \infty$, and $-\infty \pm a = -\infty$ for positive real number a. Define the function $h/(\alpha)$ by the equality

$$1/h(\alpha) = 1/h_1(\alpha) + 1/h_2(\alpha)$$

and $u_i(x,\alpha), i = 1,2, \alpha > 0, x \in S$ by

$$u_i(x,\alpha) = \varphi_1(x,\alpha) + (-1)^{i+1}\varphi_2(x,\alpha)/h_i(\alpha).$$

Then it is known that $u_1(x,\alpha)[u_2(x,\alpha)]$ is positive non-decreasing[resp. non-increasing] in $x \in S$ with $u_1(0,\alpha) = u_2(0,\alpha) = 1$. Let

$$\begin{aligned} h_{11}(\alpha) &= h(\alpha), \quad h_{22}(\alpha) = -(h_1(\alpha) + h_2(\alpha))^{-1}, \\ h_{12}(\alpha) &= h_{21}(\alpha) = -h(\alpha)/h_2(\alpha). \end{aligned}$$

Then it is seen that all these functions $h_{ij}(\alpha), i,j = 1,2$ can be analytically continued to the exterior of the half line $(-\infty, 0]$ in the complex plane. The spectral measure $\sigma_{ij}(d\lambda), i,j = 1,2$ are given by

$$\sigma_{ij}([\lambda_1,\lambda_2]) = \lim_{\epsilon \downarrow 0} \frac{1}{\pi} \int_{\lambda_1}^{\lambda_2} \mathrm{Im} h_{ij}(-\lambda - \sqrt{-1}\epsilon) d\lambda$$

for all continuity points $\lambda_1 < \lambda_2$. Then the matrix valued measure $[\sigma_{ij}(d\lambda)]_{i,j=1,2}$ is symmetric and nonnegative definite. Thus, we define the elementary solution $p(t,x,y)$ of the generalized diffusion equation (12) by

$$p(t,x,y) = \sum_{i,j=1}^{2} \int_{0-}^{\infty} e^{-\lambda t} \varphi_i(x,-\lambda)\varphi_j(y,-\lambda)\sigma_{ij}(d\lambda), \tag{14}$$

for $t > 0$ and $x, y \in S$.

We call the generalized diffusion equation (12) is periodic if $\mathbf{A}$ in (12) denoted by $\mathbf{A} = d(d/ds)/dm$ satisfy following (a), (b) and (c);

(a) $s(x)$ defined by (8) is continuous and increasing function,

(b) $m(y)$ defined by (9) is non-trivial, right continuous and nondecreasing,

(c) There is a positive ρ such that

$$s(x+1) - s(y+1) = \rho^{-1}(s(x) - s(y)), \tag{15}$$

$$m(x+1) - m(y+1) = \rho^{-1}(m(x) - m(y)). \tag{16}$$

Then we get that the elementary solution $p(t, x, y)$ of periodic diffusion equation (12) has some asymptotic behaviors to some more simple values; for large $t > 0$, and $x, y \in R$,

$$p(t, x, y) \sim \alpha(x, y, \lambda_0) t^{-1/2} \exp(-\lambda_0 t), \tag{17}$$

where λ_0 is the principle eigenvalue of $\mathbf{A}$, and $\alpha(x, y, \lambda_0)$ is a positive constant depending on x, y and λ_0.

3. Calculation of price in European options

As we know, the solution of stochastic differential equation (3) is a closed form as following;

$$S_t = S_0 \exp\{\mu t - \frac{1}{2}\sigma_t^2 t + \sigma_t W_t\}. \tag{18}$$

The price of European option is defined by

$$u(t, x) = E[e^{-r(T-t)} g(S_T) | S_t = s_t].$$

where the function g is the pay-off function and T is the maturity. The option is call if $g(x) = (x - K)^+$, and is put if $g(x) = (K - x)^+$ where $(x)^+ = \max\{0, x\}$ and K is the strike price.

If we use the general class of stochastic volatility model (1) and (2) for the Stein/Stein volatility model in [6], we get of the form

$$\frac{d\sigma_t}{\sigma_t} = \frac{\delta(\theta - \sigma_t)}{\sigma_t} dt + \frac{k}{\sigma_t} d\bar{W}_t.$$

Thus, we can get the closed form of solution;

$$\sigma_t^2 = \sigma_0^2 \exp\{\int \frac{\delta(\theta - \sigma_t)}{\sigma_t} dt + \int (\frac{k}{\sigma_t}) d\bar{W}_T\}. \tag{19}$$

Thus, from this random variable σ_T^2, the asset price S_T at time T is of the form;

$$S_T = S_0 \exp\{\mu T - \frac{1}{2}\sigma_T^2 T + \sigma_T W_T\}. \tag{20}$$

From the definition of option price, the price of European call option at time $t = 0$ is calculated as

$$\begin{aligned} (21) \quad u(t,0) &= E[e^{-rT} g(S_T)|S_0 = s_0] \\ &= E[e^{-rT}(S_T - K)^+|S_0 = s_0] \\ &= e^{-rT}E[S_T|S_0 = s_0] - e^{-rT}K, \quad if \quad S_T \geq K \\ &= e^{-rT}S_0 E[\exp\{\mu T - \frac{1}{2}\sigma_T^2 T + \sigma_T W_T\}] - e^{-rT}K. \end{aligned}$$

If we assume Brownian motion W_t has mean μ_t^* and variance σ_t^*, we get density function f_{W_T} of random variable W_T. For the solution σ_t of stochastic differential equation (5), we assume Brownian motion $\bar{W}_t$ which is independent with Brownian motion W_t, has mean $\bar{\mu}_t^*$ and variance $\bar{\sigma}_t^*$. Then, we get density function $f_{\bar{W}_T}$ of random variable $\bar{W}_T$. Thus, from equations (19) and (21), we get

$$\begin{aligned} (22) \qquad u(t,0) = e^{-rT}s_0 e^{\mu T} E[\exp\{-\frac{1}{2}\exp\{\sigma_0^2 \exp\{\int \frac{\delta(\theta - \sigma_t)}{\sigma_t} dt \\ + \int (\frac{k}{\sigma_t}) d\bar{W}_T\}\}T + \exp\{\sigma_0^2 \exp\{\int \frac{\delta(\theta - \sigma_t)}{\sigma_t} dt \\ + \int (\frac{k}{\sigma_t}) d\bar{W}_T\}\}W_t\}] - e^{-rT}K. \end{aligned}$$

Thus, we can get the European call option price at $t = T$; if we know the density functions f_{W_T} and $f_{\bar{W}_T}$.

Stein and Stein volatility model is a diffusion process of the form of solution of stochastic differential equation (5) having a form of (reflected) Ornstein-Uhlenbeck diffusion process. The solution of this stochastic differential equation is represented as an integral form;

$$(23) \qquad \sigma_t = \sigma_0 + (-\delta)\int_0^t (\sigma_s - \theta) ds + \int_0^t k d\bar{W}_s.$$

To get the price of European option, we can use the expectation (10) of the random variable $h(\sigma_T)$ for the solution σ_t of (4);

$$(24) \qquad E[h(\sigma_T)|\sigma_0 = x] = \int_0^\infty p(T, s(x), s(z))h(z)dm(s(z)).$$

In here,

$$\begin{aligned} s(x) &= \int_1^x a(z)^{-1} e^{-B(z)} dz, \quad x \in (0,\infty) \\ &= \int_1^x \frac{1}{k} e^{-B(z)} dz, \end{aligned}$$

and

$$\begin{aligned} m(y) &= 2\int_1^{s_1^{-1}(y)} a(z)^{-1} e^{B(z)} dz \\ &= 2\int_1^{s_1^{-1}(y)} \frac{1}{k} e^{B(z)} dz, \quad y = s(x) \in (s(0), s(\infty)), \end{aligned}$$

where

$$\begin{aligned} B(z) &:= \int_1^z \frac{2b(\xi)}{a(\xi)^2} d\xi = \int_1^z \frac{-2\delta(\xi-\theta)}{k^2} d\xi \\ &= \frac{\delta}{k}(2\theta z + 1 - \delta z^2 - 2\delta\theta), \end{aligned}$$

because of $a(x) = k$, $b(x) = -\delta(x-\theta)$. From the fact (21), if $S_T \geq K$, we get

$$u(t,0) = e^{-rT} s_0 \exp\{\mu T\} E[\exp\{-\frac{1}{2}\sigma_T^2 T + \sigma_T W_T\}] - e^{-rT} K,$$

where the random variable σ_T is the solution of (4) at $t = T$. Thus, we get the European option price for the function $h(z) = \exp\{-(1/2)z^2 T + zy\}$,

(25)

$$\begin{aligned} u(t,0) &= e^{-rT} S_0 e^{\mu T} \int_{R^+} \int_0^\infty h(z) p(T, s(x), s(z)) f_{W_T}(y) dy dm(s(z)) - e^{-rT} K \\ &= e^{-rT} S_0 e^{\mu T} \int_{R^+} \int_0^\infty [\exp\{-\frac{1}{2} z^2 T + zy\}] f_{W_T}(y) \\ &\qquad p(T, s(x), s(z)) dm(s(z)) dy - e^{-rT} K, \end{aligned}$$

where $f_{W_T}(y)$ is the density function of W_T. To get calculation of $p(T, s(x), s(z))$, we will introduce several cases.

[I]. The following formulation of this subsection is found in [5] mainly.

Let us assume $b(\cdot) = 0$ in (4). From the stochastic differential equation (4), if we use $s(x)$ and $m(y)$ of (8) and (9), we get

$$\begin{aligned} s(x) &= \int^x 1 d\xi = x, \\ m(x) &= 1/\sigma^2(x)s'(x) = 1/\sigma^2(x). \end{aligned}$$

Then, for the generator $\mathbf{L}$ in (7) and the function h in (10), we get

$$\begin{aligned} \mathbf{L}h(x) &= \frac{1}{2}a^2(x)\frac{d^2}{dx^2}h(x) + b(x)\frac{d}{dx}h(x) \\ &= \frac{1}{2}\frac{d}{dM}\frac{d}{ds}h(x) = \frac{1}{2}\sigma^2(x)h''(x). \end{aligned}$$

Then we get that $\rho = 1$ of (15) and (16). Therefore, in this case, we can use the theory of periodic diffusion equation easily.

As a more general case of periodic diffusion operator $\mathbf{A}$ in (12), if we think a generator

$$\mathbf{L} = a^2(x)\frac{d^2}{dx^2} - b\frac{d}{dx},$$

where b is a real number, then we can get following;

$$\rho = e^b, \quad ds(x) = e^{bx}dx, \quad dm(x) = e^{-bx}dx,$$

satisfying (15) and (16). Therefore, as we see in [4] and [5], we get $S = [\lambda_0, \infty)$, $\lambda_0 = b^2/4$,

$$\alpha(x, y, \lambda_0) = (1/2\sqrt{\pi})e^{b(x+y)/2},$$

and for $t > 0$, $x, y \in R$,

$$p(t, x, y) = (1/2\sqrt{\pi}t)\exp\{-\lambda_0 t - (x-y)^2/4t + b(x+y)/2\}. \tag{26}$$

From this elementary solution, we get option price by using (25). Further, if we know some real data, we can get the period of the movement of prices of asset and volatility.

The following formulation of this subsection **[II]** and **[III]** can be found in [4].

[II]. From the stochastic differential equation (4), if we assume

$$\int_0^1 |\frac{2b(x)}{a(x)^2}|dx = \int_0^1 |\frac{-2\delta(x-\theta)}{k^2}|dx < \infty, \tag{27}$$

$$\int_0^1 a(x)^{-1}dx = \int_0^1 \frac{1}{k}dx = \infty, \tag{28}$$

$$\int_1^\infty a(x)^{-1}e^{B(x)}\int_1^x a(z)^{-1}e^{-B(z)}dz = \int_1^\infty \frac{1}{k}e^{B(x)}\int_1^x \frac{1}{k}e^{-B(z)}dz < \infty, \tag{29}$$

and we impose the reflecting boundary condition at upper boundary B_1 in case of

$$B_1 := \int_1^\infty a(x)^{-1} e^{-B(x)} dx < \infty,$$

then we get $l_1 = s(0+) = -\infty$, $l_2 = \infty$, and

$$m(y) \sim 2e^{2B(0)} y \quad as \quad y \to -\infty,$$
$$\int_0^\infty y dm(y) < \infty.$$

Thus, we get long time asymptotic behavior of the elementary solution $p(t, x, y)$ of the generalized diffusion equation as following

$$p(t, x, y) \sim (2\pi t)^{-1/2} e^{-B(0)} \quad as \quad t \to \infty. \tag{30}$$

For the option pricing of asset price process $S_t, t \geq 0$, we write the expectation of $h(\sigma_t)$ with respect to probability measure P as following

$$\begin{aligned} E[h(\sigma_t)] &= \int_0^\infty p(t, s(x), s(z)) h(z) dm(s(z)) \\ &\sim (2\pi t)^{-1/2} \int_0^\infty a(x)^{-1} h(x) e^{B(x)-B(0)} dx \quad as \quad t \to \infty. \end{aligned} \tag{31}$$

for all h such that the integral in the right-hand side converges absolutely.

[III]. Instead of (27), (28) and (29), if we assume

$$2 \int_0^\infty a(x)^{-1} e^{B(x)} dx < \infty$$

and impose the reflecting boundary condition at lower boundary B_2 in case of

$$B_2 := -\int_0^1 a(x)^{-1} e^{-B(x)} dx > -\infty,$$

then we get $l_1 = -\infty$, $l_2 = \infty$ and

$$M := m(\infty) - m(-\infty) = 2 \int_0^\infty a(x)^{-1} e^{B(x)} dx < \infty.$$

Then, we get

$$p(t, x, y) \sim 1/M \quad as \quad t \to \infty. \tag{32}$$

and

$$E[h(\sigma_t)] \sim M^{-1} 2 \int_0^\infty a(x)^{-1} h(x) e^{B(x)} dx \quad as \quad t \to \infty \tag{33}$$

for all h such that the integral in the right-hand side converges absolutely.

4. Some other volatility asset models

I. Hull and White volatility model.

From [3], Hull and White volatility model is defined by the solution σ_t^2 of the stochastic differential equation

$$d\sigma_t^2 = \sigma_t^2(pdt + qd\bar{W}_t). \tag{34}$$

Thus, from the solution, we can get a non-negative random variable of the form;

$$\sigma_T^2 = \sigma_0^2 \exp\{pT - \frac{1}{2}(q^2T) + q\bar{W}_T\}. \tag{35}$$

From this random variable σ_T^2 and the asset price S_T at time T, the price of European call option at time $t = 0$ is calculated as (21).

If we assume the Brownian motion W_t has the density function f_{W_T} of random variable W_T. For the solution σ_t of stochastic differential equation (34), we assume the Brownian motion $\bar{W}_t$ has the density function $f_{\bar{W}_T}$ of random variable $\bar{W}_T$. Then, we can calculate option price by following;

$$\begin{aligned} u(t,0) &= e^{-rT}s_0e^{\mu T}E[\exp\{-\frac{1}{2}\exp\{pT - \frac{1}{2}(q^2T) + q\bar{W}_T\}T \\ &\quad + \exp\{pT - \frac{1}{2}(q^2T) + q\bar{W}_T\}^{\frac{1}{2}}W_T\}\}] - e^{-rT}K, \quad if \quad S_T \geq K. \end{aligned} \tag{36}$$

To calculate option price by using elementary solution of diffusion process, we rewrite (34) as

$$\sigma_t = \sigma_0 + \int_0^\infty q\sigma_t d\bar{W}_t + \int_0^\infty p\sigma_t dt, \quad \sigma_t > 0$$

with condition $\sigma_0 = x > 0$. Then we get generator

$$\mathbf{L} = \frac{1}{2}(qx)^2\frac{d^2}{dx^2} + px\frac{d}{dx},$$

scale function $s(x)$, and speed density $m(x)$ for $x \in (0,\infty)$;

$$\begin{aligned} s(x) &= \int_0^x e^{-B(z)}dz = \int_0^x z^{-\frac{2p}{q^2}}dz = \int_1^x z^{\frac{q^2}{2p}}dz = \frac{2p}{q^2+2p}x^{\frac{q^2+2p}{2p}}, \\ m(x) &= 1/(qx)^2s'(x) = 1/(qx)^2x^{\frac{q^2}{2p}}, \quad x \in (0,\infty). \end{aligned}$$

because

$$B(z) := \int_0^z \frac{2p\xi}{(q\xi)^2}d\xi = \int_1^z \frac{2p}{q^2}\frac{1}{\xi}d\xi = \frac{2p}{q^2}\ln z.$$

To get the elementary solution $p(T,x,y)$, we can use the theory of diffusion equation. But, as we know, the solution of (34) is a geometric Brownian motion which is more simple than diffusion process in general to get option prices. Thus, it is more simple if we use (36) by using density functions of W_T and $\bar{W}_T$.

II. Heston volatility model.

The Heston volatility model(c.f. [2]) is denoted by a stochastic differential equation having a solution of the form;

$$\sigma_t^2 = \sigma_0^2 + \int_0^t \delta(\theta - \sigma_s^2)ds + \int_0^t k\sigma_s d\bar{W}_s. \tag{37}$$

As a similar method as the Stein/Stein model, if W_t and $\bar{W}_t$ are independent, to get the price of European option we need to get the expectation of random variable $h(\sigma_t^2)$, where σ_t^2 is the solution of stochastic differential equation (37);

$$E[h(\sigma_t^2)|\sigma_0 = x] = \int_0^\infty p(t, s(x), s(z))h(z)dm(s(z)). \tag{38}$$

In here,

$$\begin{aligned} s(x) &= \int_1^x a(z)^{-1}e^{-B(z)}dz, \quad x \in (0,\infty) \\ &= \int_1^x \frac{1}{kz^{1/2}}e^{-B(z)}dz, \quad x \in (0,\infty), \end{aligned}$$

and

$$m(y) = 2\int_1^{s^{-1}(y)} \frac{1}{kz^{1/2}}e^{B(z)}dz, \quad y = s(x) \in (s(0), s(\infty)),$$

where

$$\begin{aligned} B(z) &:= \int_1^z \frac{2b(\xi)}{a(\xi)^2}d\xi = \int_1^z \frac{2\delta(\theta-\xi)}{k^2\xi}d\xi \\ &= \frac{2\delta\theta}{k^2}(\ln 2) - \frac{2\delta}{k^2}z + \frac{2\delta}{k^2}, \end{aligned}$$

because of $a(x) = kz^{1/2}$, $b(x) = \delta(\theta - x)$. From the solution of (3), we get

$$S_T = S_0 \exp\{\mu T - \frac{1}{2}\sigma_T^4 T + \sigma_T^2 W_T\},$$

and, if $S_T \geq K$, we get

$$u(t,0) = e^{-rT}s_0 \exp\{\mu T\}E[\exp\{-\frac{1}{2}\sigma_T^4 T + \sigma_T W_T\}] - e^{-rT}K,$$

where the random variable σ_T^2 is the solution of (37) at $t = T$. Thus, we can get the European option price by using

$$\begin{aligned} u(t,0) &= e^{-rT}S_0\exp\{\mu T\}\int_{R^+}\int_{-1}^\infty [\exp\{-\frac{1}{2}z^4T + z^2y\}] \\ &\qquad f_{W_T}(y)p(T,s(x),s(z))dm(s(z))dy - e^{-rT}K, \end{aligned} \tag{39}$$

where $f_{W_T}(y)$ is the density function of W_T in (3).

But, Heston volatility model has a big meaning when W_t and $\bar{W}_t$ are not independent. In this case, we can get $\mathrm{Corr}(W_t, \bar{W}_t) \neq 0$ which discharge for the leverage works. Thus, to study leverage works, our diffusion method is needed not.

5. Summary

As we know from [6] and above, for many types of volatility asset models which are defined by the solutions of stochastic differential equations and are represented by closed forms, we can calculate option prices if we know the distribution of volatility σ_t basically. For the types of volatility models represented by stochastic differential equations, if the solutions(volatility models) are diffusion processes, we can calculate the option prices by using the theory of diffusion equations, i.e., using the theory of elementary solutions of diffusion equations derived from some functionals. Further, if we impose some conditions to define various types of diffusion equations (c.f. [4] and [5]), we can get option prices for various types of volatility models which are defined by some diffusion processes.

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Department of Mathematics
Kangweon National University
Chuncheon 200-701, Korea
E-mail: jpoh@kangwon.ac.kr