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ON NUMERICAL RANGE AND NUMERICAL RADIUS OF CONVEX FUNCTION OPERATORS

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Abstract. In this paper we prove some interesting inclusions concerning the numerical range of some operators and the numerical range of theirs ranges with a convex function. Further, we prove some inequalities for the numerical radius. These inclusions and inequalities are based on some classical convexity inequalities for non-negative real numbers and some operator inequalities.

1. Introduction

Let $\mathbb{B}(H)$ be the algebra of all bounded linear operators on a complex Hilbert space H. For all $A \in \mathbb{B}(H)$, we define the numerical range $W(A)$ as the collection of all complex numbers of the form $\langle Ax, x\rangle$ where $x \in H$. More precisely

$$W(A) = \{\langle Ax, x\rangle : x \in H,\ \|x\| = 1\}.$$

Also, the numerical radius $w(A)$ of $A \in \mathbb{B}(H)$ is the radius of the smallest circle centered at the origin containing $W(A)$. More precisely

$$w(A) = \sup\{|\lambda| : \lambda \in W(A)\}.$$

The numerical range has been studied widely since the early years of the last century. Previous research has established that the numerical

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range is a convex set (Toeplitz-Hausdorf theorem) and, the spectrum contained in the closure of its numerical range. Moreover, we can extract an algebraic properties of an operator A by studying his numerical range and numerical radius. For example, A is self-adjoint operator if, and only if, its numerical range is a part of $\mathbb{R}$, A is positive if, and only if, its numerical range is a part of $\mathbb{R}^+$, and an invertible operator A is unitary if, and only if, numerical radius of A coincides with the numerical radius of its inverse, for more information we refer the reader to [6] and references therein.
Let f be a real valued function continuous defined on interval I ($I \subset \mathbb{R}$) and let $A \in \mathbb{B}(H)$ be self-adjoint have its spectrum in I. We define $f(A)$ by the familiar function calculus, see [14].
If $f(t) \geq 0$ for any $t \in \sigma(A)$, then $f(A) \geq 0$, i.e. $f(A)$ is a positive operator on H. Moreover, if both f and g are real valued functions on $\sigma(A)$, then the following important property holds:

$$f(t) \geq g(t) \text{ for any } t \in \sigma(A) \text{ implies that } f(A) \geq g(A),$$

in the operator order of $\mathbb{B}(H)$.
A real valued continuous function f on an interval I is said to be convex operator if

$$f((1-\lambda)A + \lambda B) \leq (1-\lambda)f(A) + \lambda f(B),$$

concave if

$$f((1-\lambda)A + \lambda B) \geq (1-\lambda)f(A) + \lambda f(B),$$

in the operator order, for all $\lambda \in [0; 1]$ and for every self-adjoint operator A and B on a Hilbert space H whose theirs spectra are contained in I. Notice that a function f is function concave if $-f$ is function convex.
A real valued continuous function f on an interval I is said to be monotone operator if it is monotone with respect to the operator order, i.e.,

$$A \leq B \text{ with } \sigma(A),\ \sigma(B) \subset I \text{ imply } f(A) \leq f(B).$$

For some fundamental results on convex operator (concave operator) and monotone function see [15] and the references therein. In this paper we obtain a few additional results about the numerical range and the numerical radius of convex function operator, we will give some basic properties of the numerical range and the numerical radius of the convex operator function.

2. Preliminaries

To prove the numerical range inclusions and numerical radius inequalities, we need several known lemmas. All these lemmas are concerning the convex function properties.

LEMMA 2.1. [9] *(i) Assume that $A \in \mathbb{B}(H)$ is normal and $U \in \mathbb{B}(H)$ is unitary, then for every real function on $\sigma(A)$*

$$f(U^*AU) = U^*f(A)U. \tag{2.1}$$

(ii) For every $X \in \mathbb{B}(H)$ and every real function on $\sigma(XX^*)$.

$$Xf(X^*X) = f(XX^*)X. \tag{2.2}$$

LEMMA 2.2. [13] *Let $A \in \mathbb{B}(H)$ be self-adjoint and assume that $\sigma(A) \subseteq [m, M]$, for some scalars m, M with $m \leq M$. If f is a convex function on $[m, M]$, then*

$$f\Big(\langle Ax, x\rangle\Big) \leq \langle f(A)x, x\rangle,$$

for each $x \in H$ with $\|x\| = 1$.

LEMMA 2.3. [11] *Let $A \in B(H)$ be positive and let $x \in H$ be any unit vector. Then (i) $\langle Ax, x\rangle^r \leq \langle A^r x, x\rangle$ for $r \geq 1$.*
(ii) $\langle A^r x, x\rangle \leq \langle Ax, x\rangle^r$ for $0 < r \leq 1$.
(iii) If A is invertible, then $\langle A^r x, x\rangle \geq \langle Ax, x\rangle^r$ for all $r < 0$ and $x \in H$, with $\|x\| = 1$.

LEMMA 2.4. [14] *Let $A_j \in B(H)$ be a self-adjoint operators with $\sigma(A_j) \subseteq [m, M]$, $i = 1, ..., n$. For some scalars $m < M$ and $x_j \in H, i = 1, ..., n$, with $\sum_{i=1}^{n} \|x_i\| = 1$. If f is convex function on $[m, M]$, then*

$$f\left(\sum_{i=1}^{n}\langle A_i x_i, x_i\rangle\right) \leq \sum_{i=1}^{n}\langle f(A_i)x_i, x_i\rangle.$$

LEMMA 2.5. [8] *For any continuous function f defined on an interval I the following conditions are equivalent:*

(i) *f is convex operator .*
(ii) *For each positive integer number n we have the inequality*

$$f\left(\sum_{i=1}^{n} A_i^* X_i A_i\right) \leq \sum_{i=1}^{n} A_i^* f\Big(X_i\Big) A_i, \tag{2.3}$$

for every $n-$tuple $(X_1, \cdots, X_n)$ of bounded self-adjoint operators on an arbitrary Hilbert space H with spectra contained in I and every n-tuple $(A_1, \cdots, A_n)$ of operators on H with $\sum_{i=1}^{n} A_i^* A_i = I$.

(iii) *$f(V^*XV) \leq V^* f(X) V$ for each isometry V on an infinite-dimensional Hilbert space H and every self-adjoint operator X with spectrum in I.*

(vi) *$Pf(PXP + s(1-P))P \leq Pf(X)P$ for each orthogonal projection P on an infinite-dimensional Hilbert space H, every self-adjoint operator X with spectrum in I and every s in I.*

We finish this part by a lemma given a property of convex and concave function.

Lemma 2.6. [1] *Let $A \in \mathbb{B}(H)$ be self-adjoint and positive, $\alpha \geq 1$ and f be non-negative function on $[0, \infty)$ with $f(0) = 0$.*
(i) If f is convex, then $f(\alpha A) \geq \alpha f(A)$.
(ii) If f is concave, then $\alpha f(A) \geq f(\alpha A)$.

3. Main results

This section contains two part, the first consist to study some proprieties of numerical range and radius of convex function for one or two operators where we applied some of this result to special convex function as exponential and power function , in the second part, we generalized some result of the first part to the case of n operators.

3.1. Numerical range and radius for convex function operators.

We define the numerical range of convex function operator by

$$W(f(A)) = \{\langle f(A)x, x\rangle : \|x\| = 1,\ x \in H\},$$

such that $A \in \mathbb{B}(H)$ be self-adjoint.
$W(f(A))$ verified all the properties of the classical numerical range also it is a convex set, as we show in the next theorem.

Theorem 3.1. *Let $A, B \in \mathbb{B}(H)$ be self-adjoint, $\alpha \in [0,1]$ and f be a non-negative convex function, then*
(i) $W(f(\alpha A + (\alpha - 1)B)) \subseteq \alpha W(f(A)) + (\alpha - 1) W(f(B))$.
(ii) Suppose that $X \in \mathbb{B}(H)$ be isometry satisfies $XX^* = 1$. Then

$$W(f(X^*AX)) \subseteq W(f(A)).$$

(iii) $W(f(A)) \subset \mathbb{R}$.

Proof. By definition we have

$$\begin{aligned} W(f(\alpha A+(\alpha-1)B)) &\subseteq W(\alpha f(A)+(\alpha-1)f(B)) \\ &\subseteq \alpha W(f(A))+(\alpha-1)W(f(B)). \end{aligned}$$

The proof of the second assertion, we have

$$f(X^*AX) \leq X^*f(A)X,$$

then

$$\begin{aligned} W(f(X^*AX)) &= \{\langle f(X^*AX)x,x\rangle \ \|x\|=1,\ x\in H\} \\ &\subseteq \{\langle X^*f(A)Xx,x\rangle \ \|x\|=1,\ x\in H\} \\ &\subseteq \{\langle f(A)x,x\rangle \ \|x\|=1,\ x\in H\}. \end{aligned}$$

It easy to check this assertion because A is self-adjoint and $W(A) \subseteq \mathbb{R}$. □

REMARK 3.1. (i) The first inclusion in the previous theorem be equality if $\alpha = 0$ or $\alpha = 1$.
(ii) The second inequality in the previous theorem equivalent to

$$W(Pf(PAP+s(1-P))P) \subset W(f(A)),$$

for each orthogonal projection P on an infinite-dimensional Hilbert space H, every self-adjoint operator A and $s \in I$.

The numerical radius has similar properties.

THEOREM 3.2. *Let f be a convex function on I. Then*

$$w(f(\alpha A+(1-\alpha)B)) \leq \alpha w(f(A))+(\alpha-1)w(f(B)),$$

for all $A, B \in \mathbb{B}(H)$ be self-adjoint and $0 \leq \alpha \leq 1$. Further $0 \in I$ and $f(0) \leq 0$. Then

$$w(f(X^*AX)) \leq w(f(A)),$$

for all $A \in \mathbb{B}(H)$ be self-adjoint, $X \in \mathbb{B}(H)$ such that $XX^* = I$ and X commute with A.

Proof. For every unit vector $x \in H$ and $\alpha \in [0,1]$

$$\begin{aligned} |\langle f(\alpha A+(1-\alpha)B)x,x\rangle| &\leq |\langle [\alpha f(A)+(1-\alpha)f(B)]\,x,x\rangle| \\ &\qquad \text{(since f is convex)} \\ &= |\alpha\langle f(A)x,x\rangle+(1-\alpha)\langle f(B)x,x\rangle| \\ &\leq |\alpha\langle f(A)x,x\rangle|+|(1-\alpha)\langle f(B)x,x\rangle| \\ &\leq \alpha|\langle f(A)x,x\rangle|+(1-\alpha)|\langle f(B)x,x\rangle|. \end{aligned}$$

So that,

$$\begin{aligned} &w(f(\alpha A+(1-\alpha)B)) \\ &\qquad = \sup\{|\langle f(\alpha A+(1-\alpha)B)x,x\rangle| : x\in H, \|x\|=1\} \\ &\qquad \le \sup\{|\alpha\langle f(A)x,x\rangle| + |(1-\alpha)\langle f(B)x,x\rangle| : x\in H, \|x\|=1\} \\ &\qquad \le \sup\{\alpha|\langle f(A)x,x\rangle| : x\in H, \|x\|=1\} \\ &\qquad\quad + \sup\{(1-\alpha)|\langle f(B)x,x\rangle| : x\in H, \|x\|=1\} \\ &\qquad = \alpha w(f(A)) + (1-\alpha)w(f(B)). \end{aligned}$$

To prove the second assertion, for any unit vector $x\in H$. Then, we have

$$|\langle f(X^*AX)x,x\rangle| \le |\langle X^*f(A)Xx,x\rangle| \qquad \text{(by Lemma 2.5(iii))}.$$

So that

$$\begin{aligned} w(f(X^*AX)) &= \sup\{|\langle f(X^*AX)x,x\rangle| : x\in H,\ \|x\|=1\} \\ &\le \sup\{|\langle X^*f(A)Xx,x\rangle| : x\in H,\ \|x\|=1\} \\ &\le \sup\{|\langle f(A)x,x\rangle| : x\in H,\ \|x\|=1\} \\ &= w(f(A)). \end{aligned}$$

Also, we have these inequalities for numerical radius. □

REMARK 3.2. Let f be a convex function on I, $0\in I$ and $f(0)\le 0$. Then

$$w(f(P^*AP)) \le w(f(A)),$$

for all $A\in\mathbb{B}(H)$ be self-adjoint and $P\in\mathbb{B}(H)$ orthogonal projection. And

$$w(Pf(PXP+s(1-P))P) \le w(f(X)),$$

for each orthogonal projection P, every self-adjoint operator A with spectrum in I and every s in I.
Indeed. For every unit vector $x\in H$

$$\begin{aligned} w(f(PAP)) &= \sup\{|\langle f(PAP)x,x\rangle| : x\in H, \|x\|=1\} \\ &\le \sup\{|\langle Pf(A)Px,x\rangle| : x\in H, \|x\|=1\} \\ &\le \sup\{|\langle f(A)Px,Px\rangle| : x\in H, \|x\|=1\} \\ &\le \sup\{|\langle f(A)x,x\rangle| : x\in H, \|x\|=1\} \\ &= w(f(A)). \end{aligned}$$

The second inequality

$$\begin{aligned}
&w(Pf(PXP+S(1-P))P)\\
&\qquad = \sup\{|\langle Pf(PXP+S(1-P))Px, x\rangle| : x \in H, \|x\| = 1\}\\
&\qquad \leq \sup\{|\langle Pf(A)Px, x\rangle| : x \in H, \|x\| = 1\}\\
&\qquad \leq \sup\{|\langle f(A)Px, Px\rangle| : x \in H, \|x\| = 1\}\\
&\qquad \leq \sup\{|\langle f(A)x, x\rangle| : x \in H, \|x\| = 1\}\\
&\qquad = w(f(A)).
\end{aligned}$$

The follows from Theorem 3.2, taking $f(t) = t^r : r \geq 0$.

Corollary 3.1. *Let $A, B \in \mathbb{B}(H)$ be self-adjoint. Then*

$$w((A+B)^r) \leq 2^{r-1}(w(A^r) + w(B^r))$$

for all $r \leq 0$.

The equivalence between the numerical radius and the norm have been given this corollary.

Corollary 3.2. *Let f be a convex function on I. Then*

$$\|f(\alpha A + (1-\alpha)B)\| \leq \alpha\|f(A)\| + (\alpha - 1)\|f(B)\|,$$

for all $A, B \in \mathbb{B}(H)$ be self-adjoint and $0 \leq \alpha \leq 1$. Further $0 \in I$ and $f(0) \leq 0$. Then

$$\|f(X^*AX\| \leq \|f(A)\|, .$$

where $X \in \mathbb{B}(H)$ such that $XX^* = I$ and X commute with A.

A particular case of Corollary 3.2, choosing the non-negative convex function $f(t) = |t|^r, r \geq 1$ on $(-\infty, +\infty)$, we find the inequality above

$$\||A+B|^r\| \leq 2^{r-1}\||A|^r + |B|^r\|.$$

Another result for numerical range and numerical radius.

Proposition 3.1. *Let $A \in \mathbb{B}(H)$ be self-adjoint, $\alpha \geq 1$ and let f be a non-negative function on $(0, \infty)$, with $f(0) = 0$.*
(i) If f is convex, then $\alpha W(f(A)) \subseteq W(f(\alpha A))$.
(ii) If f is concave, then $W(f(\alpha A)) \subseteq \alpha W(f(A))$.

Proof. The result follows from Lemma 2.6. □

We have similar result for the numerical radius.

THEOREM 3.3. *Let* $A \in \mathbb{B}(H)$ *be self-adjoint,* $\alpha \geq 1$ *and let* f *be non-negative function on* $(0, \infty)$*, with* $f(0) = 0$.
(i) *If* f *is convex, then* $w(\alpha f(A)) \leq w(f(\alpha A))$.
(ii) *If* f *is concave, then* $w(f(\alpha A)) \leq \alpha w(f(A))$.

Proof. For any unit vector $x \in H$. Then

$$\begin{aligned} w(\alpha f(A)) &= \sup\{|\langle \alpha f(A)x, x\rangle| : x \in H, \|x\| = 1\} \\ &\leq \sup\{|\langle f(\alpha A)x, x\rangle| : x \in H, \|x\|\} \\ &\qquad (\text{ by Lemma 2.6}) \\ &= w(f(\alpha A). \end{aligned}$$

The proof of (ii) is similar to (i). $\square$

COROLLARY 3.3. *Let* $A \in \mathbb{B}(H)$ *be self-adjoint,* $\alpha \geq 1$ *and let* f *be a non-negative function on* $(0, \infty)$*, with* $f(0) = 0$.
(i) *If* f *is convex, then* $\|\alpha f(A)\| \leq \|f(\alpha A)\|$.
(ii) *If* f *is concave, then* $\|f(\alpha A)\| \leq \alpha \|f(A)\|$.

REMARK 3.3.
- The Propositions 3.1, 3.3 and their corollaries may not be true if f not non-negative function. To see this one may take $f(t) = -\log t$.
- If, in addition, in Theorem 3.2 we assume that f is increasing (or decreasing) we have the same previous result.

Every non-negative decreasing function f on $[0, \infty)$ satisfies $f(2t) \leq 2f(t), t \in [0, \infty)$. The next theorems give a new inequality for the numerical range and numerical radius of convex function.

THEOREM 3.4. *Let* f *be a convex decreasing function on* $(0, \infty)$ *and let* $A, B \in \mathbb{B}(H)$ *be self-adjoint. Then*

$$W(f(A + B)) \subset W(f(A)) + W(f(B)).$$

Proof. Since f is decreasing convex function, we have

$$f(\frac{A + B}{2}) \leq \frac{1}{2}(f(A) + f(B))$$

for all $A, B \in \mathbb{B}(H)$ be self-adjoint.
And, we have $f(2t) \leq 2f(t)$, then

$$\begin{aligned} f\left(\frac{2A + 2B}{2}\right) &\leq \frac{1}{2} f(2A + 2B) \\ &\leq f(A) + f(B). \end{aligned}$$

So, we have

$$
\begin{aligned}
&W(f(A+B))\\
&\quad = \{\langle f(A+B)x, x\rangle : x \in H, \|x\| = 1\}\\
&\quad \subset \{\langle f(A)+f(B)x, x\rangle : x \in H, \|x\| = 1\}\\
&\quad \subset \{\langle f(A)x, x\rangle : x \in H, \|x\| = 1\} + \{\langle f(B)x, x\rangle : x \in H, \|x\| = 1\}\\
&\quad = W(f(A)) + W(f(B)).
\end{aligned}
$$

□

Similar result for the numerical radius.

THEOREM 3.5. *Let f be a convex decreasing function on $(0,\infty)$ and let $A, B \in \mathbb{B}(H)$ be self-adjoint. Then*

$$w(f(A+B)) \le w(f(A)) + w(f(B)). \tag{3.1}$$

Proof. For every unit vector $x \in H$, we have

$$
\begin{aligned}
\left|\left\langle f\left(\frac{A+B}{2}\right)x, x\right\rangle\right| &\le \left|\left\langle (f(A)+f(B))x, x\right\rangle\right|\\
&\le |\langle f(A)x, x\rangle| + \le |\langle f(B)x, x\rangle|.
\end{aligned}
$$

So,

$$
\begin{aligned}
&w(f(A+B))\\
&= \sup\{|\langle f(A+B)x, x\rangle| : x \in H, \|x\| = 1\}\\
&= \sup\{|\langle f(A)x, x\rangle| + |\langle f(B)x, x\rangle| : x \in H, \|x\| = 1\}\\
&\le \sup\{|\langle f(A)x, x\rangle| : x \in H, \|x\| = 1\} + \sup\{|\langle f(B)x, x\rangle| : x \in H, \|x\| = 1\}\\
&= w(f(A)) + w(f(B)).
\end{aligned}
$$

□

COROLLARY 3.4. *Let f be non-negative convex decreasing function. Then,*

$$\|f(A+B)\| \le \|f(A)\| + \|f(B)\|.$$

THEOREM 3.6. *Let f, g be non-negative continuous functions on $(0,\infty)$ which satisfy inequality (3.1). Then, the functions $f+g$ and $f \circ g$ satisfy also inequality (3.1).*

Proof. In order to prove that $f+g$ satisfy inequality (3.1), we use the fact that f and g are increasing. Then, for every unit vector $x \in H$. We have

$$\begin{aligned}|\langle (f+g)(A+B)x,x\rangle| &= |\langle (f(A+B)+g(A+B))x,x\rangle| \\ &\le |\langle (f(A)+f(B)+g(A)+g(B))x,x\rangle| \\ &\le |\langle (f(A)+g(A))x,x\rangle| + |\langle (f(B)+g(B)x,x\rangle| \\ &= |\langle (f+g)(A)x,x\rangle| + |\langle (f+g)(B)x,x\rangle|.\end{aligned}$$

So that

$$\begin{aligned}&w(f+g)(A+B) \\ &\quad = \sup\{|\langle (f+g)(A+B)x,x\rangle| : x\in H, \|x\|=1\} \\ &\quad \le \sup\{|\langle (f+g)(A)x,x\rangle| + |\langle (f+g)(B)x,x\rangle| : x\in H, \|x\|=1\} \\ &\quad \le \sup\{|\langle (f+g)(A)x,x\rangle| : x\in H, \|x\|=1\} \\ &\quad\quad + \sup\{|\langle (f+g)(B)x,x\rangle| : x\in H, \|x\|=1\} \\ &\quad = w((f+g)(A)) + w((f+g)(B)).\end{aligned}$$

To prove the result of $f\circ g$ note that, for each unit vector $x \in H$.

$$\begin{aligned}|\langle f\circ g(A+B)x,x\rangle| &\le |\langle f(g(A)+g(B))x,x\rangle| \\ &\le |\langle f(g(A))+f(g(B))x,x\rangle| \\ &\le |\langle f\circ g(A)x,x\rangle| + |\langle f\circ g(B)x,x\rangle|.\end{aligned}$$

Then

$$\begin{aligned}&w(f\circ g)(A+B) \\ &\quad = \sup\{|\langle f\circ g(A+B)x,x\rangle| : x\in H, \|x\|=1\} \\ &\quad \le \sup\{|\langle (f\circ g)(A)x,x\rangle| + |\langle (f\circ g)(B)x,x\rangle| : x\in H, \|x\|=1\} \\ &\quad \le \sup\{|\langle (f\circ g)(A)x,x\rangle| : x\in H, \|x\|=1\} \\ &\quad + \sup\{|\langle (f\circ g)(B)x,x\rangle| : x\in H, \|x\|=1\} \\ &\quad = w((f\circ g)(A)) + w((f\circ g)(B)).\end{aligned}$$

□

THEOREM 3.7. *Let $A, B \in \mathbb{B}(H)$ be self-adjoint satisfy $A-B \ge 0$ and let f be a convex and differentiable function whose derivative f' is continuous function. Then*

$$w(f(A)-f(B)) \le w(f'(A))\left[2w(A)+4w(B)\right].$$

If B commute with A. Then

$$w(f(A)-f(B)) \le w(f'(A))\left[2\|A\|+4\|B\|\right].$$

Proof. First, we have for any differentiable convex function

$$f(A) - f(B) \leq f'(A)(A - B). \tag{3.2}$$

For every unit vector $x \in H$. We have

$$\begin{aligned} w(f(A) - f(B)) &\leq w(f'(A)(A - B)) \\ &\qquad \text{(by Eq.(3.2))} \\ &\leq w(f'(A)A) + w(f'(A)B)) \\ &\leq 2w(f'(A))w(A) + 4w(f'(A))w(B) \\ &= w(f'A))[2w(A) + 4w(B)]. \end{aligned}$$

We notice here that f' commute with A every way, for that we found the previous inequality.
If B commute with A, so B double commute with A because A and B are self-adjoint operators and B commute with $f'(A)$. Then, during the previous inequality, we have

$$\begin{aligned} w(f(A) - f(B)) &\leq w(f'(A))\,[2w(A) + 4w(B)] \\ &\leq w(f'(A))\,[2\|A\| + 4\|B\|]\,. \end{aligned}$$

□

Corollary 3.5. *Since $f'(A)$ commute with A and B, so $f'(A)$ commute with $A - B$. Then, we obtain this result*

$$w(f(A) - f(B)) \leq w(f'(A))\,[\|A\| + \|B\|]\,.$$

Corollary 3.6. *Let $A, B \in \mathbb{B}(H)$ be self-adjoint. Then*

$$w(\exp A + \exp B) \leq \exp A(\|A\| + \|B\|).$$

Remark 3.4. If L is an affine function and f is a convex function, then $L \circ f$ reserving all the properties of numerical range and numerical radius of the convex function f.

Our next results based on the inequality of Mond and Pečarić.

Theorem 3.8. *Let $A,\ B \in \mathbb{B}(H)$ be self-adjoints and $\alpha \in [0, 1]$. If f is a convex function on I, then*

(i)

$$f(W(\alpha A + (1 - \alpha)B)) \subset \alpha W(f(A)) + (\alpha - 1)W(f(B)).$$

(ii) If $X \in \mathbb{B}(H)$ be isometry satisfies $XX^* = 1$. Then

$$f(W(X^*AX)) \subseteq W(f(A)).$$

Proof. (i) For every unit vector $x \in H$, we have

$$\begin{aligned} f\Big(\langle \alpha A+(1-\alpha)Bx,x\rangle\Big) &\leq \langle f(\alpha A+(1-\alpha)B)x,x\rangle \qquad \text{(by Lemma 2.2)} \\ &\leq \alpha\langle f(A)x,x\rangle+(1-\alpha)\langle f(B)x,x\rangle \qquad \text{(by Theorem 3.1(1))}. \end{aligned} \tag{3.3}$$

So

$$\begin{aligned} f(W(A+(1-\alpha)B) &\subset W(f(\alpha A+(1-\alpha)B)) \\ &\subset \alpha W(f(A))+(\alpha-1)W(f(B)). \end{aligned}$$

(ii) For every unit vector $x \in H$, we have

$$\begin{aligned} f\Big(\langle X^*AXx,x\rangle\Big) &\leq \langle f(X^*AX)x,x\rangle \qquad \text{(by Lemma 2.2)} \\ &\leq \langle f(A)x,x\rangle \qquad \text{(by Theorem 3.1(2))}. \end{aligned} \tag{3.4}$$

So

$$\begin{aligned} f(W(X^*AX)) &\subset W(f(X^*AX)) \\ &\subset W(f(A)). \end{aligned}$$

□

Similar result for the numerical radius.

THEOREM 3.9. *Let $A,\ B \in \mathbb{B}(H)$ be self-adjoints and $\alpha \in [0,1]$. If f is a convex function on I, then*

(i)

$$f(w(\alpha A+(1-\alpha)B)) \leq \alpha w(f(A))+(\alpha-1)w(f(B)).$$

(ii) *If $X \in \mathbb{B}(H)$ be isometry satisfies $XX^*=1$. Then*

$$f(w(X^*AX)) \leq w(f(A)).$$

Proof. (i) For every unit vector $x \in H$ and Eq.(3.3), we have

$$\begin{aligned} f(w(A+(1-\alpha)B)) &= \sup\{f(|\langle(\alpha A+(1-\alpha)B)x,x\rangle|) : x\in H,\ \|x\|=1\} \\ &\leq \sup\{|\langle f(\alpha A+(1-\alpha)B)x,x\rangle| : x\in H,\ \|x\|=1\} \\ &= w(f(\alpha A+(1-\alpha)B)) \\ &\leq \alpha w(f(A))+(\alpha-1)w(f(B)). \end{aligned}$$

(ii) for every unit vector $x \in H$ and the Eq.(3.4), we have

$$\begin{aligned} f(w(X^*AX) &= \sup\{f(|\langle X^*AXx, x\rangle|) : x \in H,\ \|x\| = 1\} \\ &\leq \sup\{|\langle f(X^*AX)x, x\rangle| : x \in H,\ \|x\| = 1\} \\ &= w(f(X^*AX)) \\ &\leq w(f(A)). \end{aligned}$$

□

The result of Theorem 3.9, also hold for the norm of operators.

Corollary 3.7. *Let $A,\ B \in \mathbb{B}(H)$ be self-adjoints and $\alpha \in [0, 1]$. If f is a convex function on I, then*

(i)

$$f(\|\alpha A + (1-\alpha)B\|) \leq \alpha\|f(A)\| + (\alpha - 1)\|f(B)\|.$$

(ii) *If $X \in \mathbb{B}(H)$ be isometry satisfies $XX^* = 1$. Then*

$$f(\|X^*AX\|) \leq \|f(A)\|.$$

If A is positive defined and we use the inequalities in Theorem 3.9, for the exponential function that one obtains the following inequalities.

Corollary 3.8.

(i)

$$\exp(w(\alpha A + (1-\alpha)B)) \leq \alpha w(\exp(A)) + (\alpha - 1)w(\exp(B)).$$

For each $A,\ B$ be positives and self-adjoints operators and $\alpha \in [0, 1]$.

(ii)

$$\exp(w(X^*AX)) \leq w(\exp(A)).$$

For every X be positive and isometry operators satisfies $XX^* = 1$.

Also, we obtain another result if f is a concave function for example the logarithm function.

Corollary 3.9. *Let $A, B \in \mathbb{B}(H)$ be positives and self-adjoints and $\alpha \in [0, 1]$, then*

(i)

$$\log(w(\alpha A + (1-\alpha)B)) \geq \alpha w(\log(A)) + (\alpha - 1)w(\log(B)).$$

For each $A,\ B$ be positives and self-adjoints operators .

(ii)

$$\log(w(X^*AX)) \geq w(\log(A)).$$

For every X be positive and isometry operator satisfy $XX^* = 1$.

The following property hold only if $\dim H < \infty$.

THEOREM 3.10. *Suppose that $A \in \mathbb{B}(H)$ be Hermitian and f be a convex function, then*
(i) *The set $W(f(A)) = \{\mu\}$ if, and only if, $f(A) = \{\mu I\}$.*
(ii) *$W(f(A))$ is closed but not bounded set.*
(iii) *$W(f(A)) \subseteq [0, +\infty)$ if, and only if, A is positive operator.*

Proof. The first condition follows from the fact $W(A) = \{\mu\}$ if, and only if, $A = \mu I$, where $\mu \in \mathbb{R}$.
(ii) Let $\{x_k f(A_k) x_k\ k = 1, 2, 3, \cdots\}$ be a sequence in $W(f(A))$ converging to $\mu_0 \in \mathbb{C}$ where A_k be self-adjoint operator and x_k is bounded, then there is a subsequence x_{jk} converging to x. We can further consider a subsequence $\{A_{jk}\}_{k\geq 1}$ of $\{A_k\}_{k\geq 1}$ converging to A_0. Then

$$\{x_{jk} f(A_{jk}) x_{jk},\ k = 1, 2, \cdots\} \to \{x_0 f(A_0) x_0,\ k = 1, 2, \cdots\}.$$

Thus, $W(f(A))$ is closed.
(iii) The condition follows from the fact $W(A) = [0, +\infty)$ if, and only if, A is a positive operator. □

THEOREM 3.11. *Assume that $A \in \mathbb{B}(H)$ be self-adjoint and $U \in \mathbb{B}(H)$ be unitary. For all convex function f on $\sigma(A)$, we have*

$$W(f(U^*AU)) = W(f(A)),$$

and for all $X \in \mathbb{B}(H)$ and every convex function f on $\sigma(X^*X)$, we have

$$W(X(f(X^*X)) = W(f(XX^*)X).$$

Proof. (i) For every unit vector $x \in H$, we have

$$\begin{aligned} W(f(U^*AU)) &= W(U^*f(A)U) \\ &\qquad (\text{ by Eq.}(2.1)), \\ &= W(f(A)). \end{aligned}$$

(ii) from the Eq.(2.2), can be verified readily. □

THEOREM 3.12. *Assume that $A \in \mathbb{B}(H)$ be self-adjoint and $U \in \mathbb{B}(H)$ be unitary. For all convex function f on $\sigma(A)$, we have*

$$w(f(U^*AU)) = w(f(A)),$$

and for all $X \in \mathbb{B}(H)$ and every convex function f on $\sigma(X^*X)$, we have

$$w(X(f(X^*X)) = w(f(XX^*)X).$$

Proof. (i) For all unit vector $x \in H$, we have

$$|\langle f(U^*AU)x, x\rangle| = |\langle U^*f(A)Ux, x\rangle| \quad \text{(by Eq.(2.1))}.$$

So,

$$\begin{aligned} w(f(U^*AU)) &= \sup\{|\langle f(U^*AU)x, x\rangle| : x \in H, \|x\| = 1\} \\ &= \sup\{|\langle U^*f(A)Ux, x\rangle| : x \in H, \|x\| = 1\} \\ &= \sup\{|\langle f(A)x, x\rangle| : x \in H, \|x\| = 1\} \\ &= w(f(A)). \end{aligned}$$

(ii) For all unit vector $x \in H$, we have

$$|\langle Xf(X^*X)x, x\rangle| = |\langle f(XX^*)Xx, x\rangle| \quad \text{by Eq.(2.2)}.$$

Then,

$$\begin{aligned} w(Xf(X^*X)) &= \sup\{|\langle Xf(X^*X)x, x\rangle| : x \in H, \|x\| = 1\} \\ &= \sup\{|\langle f(XX^*)Xx, x\rangle| : x \in H, \|x\| = 1\} \\ &= w(f(XX^*)X). \end{aligned}$$

□

REMARK 3.5. The first assertion in Theorem 3.1, and the second assertion in Theorem 3.2, are true for every real function whe A is normal operator and U unitary operator.
If X is isometry operator in the second equality in previous theorem, then

$$w((f(X^*X) = w(f(XX^*)).$$

COROLLARY 3.10. *Let $A \in \mathbb{B}(H)$ be self-adjoint and $U \in \mathbb{B}(H)$ unitary. for all convex function f on $\sigma(A)$, we have*

$$\|f(U^*AU)\| = \|f(A)\|,$$

and for all $X \in \mathbb{B}(H)$ and every convex function f on $\sigma(X^*X)$, we have

$$\|X(f(X^*X))\| = \|f(XX^*)X\|.$$

3.2. Inclusions and Inequalities for sum of n operators. Now, we will generalize some results to the sum of n operators.

THEOREM 3.13. *Let $A_1, \cdots, A_n \in \mathbb{B}(H)$ be self-adjoint and let $\alpha_1, \cdots, \alpha_n$ be positive real numbers such that $\sum_{i=1}^n \alpha_i = 1$. Then*

1. $W(f(\sum_{i=1}^n \alpha_i A_i)) \subset \sum_{i=1}^n \alpha_i W(f(A_i))$ *for every non-negative convex function on* $(0, \infty)$.
2. $W(\sum_{i=1}^n \alpha_i f(A_i)) \subset W(f(\sum_{i=1}^n \alpha_i A_i))$ *for every non-negative concave function on* $(0, \infty)$.

Proof. Let f be convex function on $(0, \infty)$. Then

$$\begin{aligned}
W(f(\sum_{i=1}^n \alpha_i A_i)) &= \{\langle f(\sum_{i=1}^n \alpha_i A_i)x_i, x_i\rangle : x_i \in H,\ \sum_{i=1}^n \|x_i\| = 1\} \\
&\subset \{\langle \sum_{i=1}^n \alpha_i f(A_i)x, x\rangle : x_i \in H,\ \sum_{i=1}^n \|x_i\| = 1\}\} \\
&\subset \sum_{i=1}^n \alpha_i \{\langle f(A_i)x_i, x_i\rangle\ x_i \in H,\ \sum_{i=1}^n \|x_i\| = 1\}\} \\
&= \sum_{i=1}^n \alpha_i W(f(A_i)).
\end{aligned}$$

The proof of the second assertion similar to the first only for the concave function. □

The same result for the numerical radius.

THEOREM 3.14. *Let $A_1, \cdots, A_n \in \mathbb{B}(H)$ be self-adjoint and let $\alpha_1, \cdots, \alpha_n$ be positive real numbers such that $\sum_{i=1}^n \alpha_i = 1$. Then*

1. $w(\sum_{i=1}^n \alpha_i f(A_i)) \leq w(f(\sum_{i=1}^n \alpha_i A_i))$*for every non-negative concave function on* $(0, \infty)$.
2. $w(f(\sum_{i=1}^n \alpha_i A_i)) \leq \sum_{i=1}^n \alpha_i w(f(A_i))$ *for every non-negative convex function on* $(0, \infty)$.

Proof. Let f be non-negative concave function on $(0,\infty)$. Then

$$\begin{aligned} w(\sum_{i=1}^{n}\alpha_i f(A_i)) &= \sup\{|\langle\sum_{i=1}^{n}\alpha_i f(A_i)x_i, x_i\rangle| : x_i \in H,\ \sum_{i=1}^{n}\|x_i\| = 1\} \\ &\le \sup\{|\langle f(\sum_{i=1}^{n}\alpha_i A_i)x_i, x_i\rangle| : x_i \in H,\ \sum_{i=1}^{n}\|x_i\| = 1\} \\ &= w(f(\sum_{i=1}^{n}\alpha_i A_i)). \end{aligned}$$

The proof of the second assertion is similar to the first only for the non-negative convex function. □

Remark 3.6. Theorem 3.14 may not be true if f is not non-negative function. To see this one make $f(t) = -\log t$ where $t \in (0,+\infty)$ for the convex function and $f(t) = -t^r$, where $t \in (0,+\infty)$ and $r \ge 1$ for the concave function.

Corollary 3.11. *Let $A_1,\cdots,A_n \in \mathbb{B}(H)$, $B_1,\cdots,B_n \in \mathbb{B}(H)$ be self-adjoint and let $\alpha_1,\cdots,\alpha_n$ be positive real numbers such that $\sum_{i=1}^{n}\alpha_i = 1$. Then*

$$w(\sum_{i=1}^{n}\alpha_i A_i + B_i)^r \le \sum_{i=1}^{n}\alpha_i^r 2^{r-1}(w(A_i^r) + w(B_i^r))$$

for all $r \ge 1$.

Corollary 3.12. *Let $A_1,\cdots,A_n \in \mathbb{B}(H)$ be self-adjoint and let $\alpha_1,\cdots,\alpha_n$ be positive real numbers such that $\sum_{i=1}^{n}\alpha_i = 1$. Then*

1. *$\|\sum_{i=1}^{n}\alpha_i f(A_i)\| \le \|f(\sum_{i=1}^{n}\alpha_i A_i)\|$ for every non-negative concave function on $(0,\infty)$.*
2. *$\|f(\sum_{i=1}^{n}\alpha_i A_i)\| \le \sum_{i=1}^{n}\alpha_i\|f(A_i)\|$ for every non-negative convex function on $(0,\infty)$.*

Our final result in this part is a consequence of Furuta-Mićić-Pečarić-Seo inequality.

Theorem 3.15. *Let $A_i \in \mathbb{B}(H)$ be self-adjoints for each $i \in \{1,\cdots,n\}$ and $\sum_{i=1}^{n}\alpha_i = 1$. If f is a convex function on I, then*

1.

$$f(W(\sum_{i=1}^{n}\alpha_i A_i)) \subset \sum_{i=1}^{n}\alpha_i W(f(A_i)).$$

2. *If* $X_i \in \mathbb{B}(H)$ *be isometry satisfies* $\sum_{i=1}^{n} X_i X_i^* = 1$. *Then*

$$f(W(\sum_{i=1}^{n} X_i A_i X_i^*)) \subseteq \sum_{i=1}^{n} W(f(A_i)).$$

Proof. For the first assertion, we used the lemma **??** the we obtain our result.
The second assertion is a consequence of the third assertion in Lemma 2.5 and the result of Theorem 3.1. □

Also we can verified readily this property for the numerical radius.

THEOREM 3.16. *Let* $A_i \in \mathbb{B}(H)$ *be self-adjoints for each* $i \in \{1, \cdots, n\}$ *and* $\sum_{i=1}^{n} \alpha_i = 1$. *If* f *is a convex function on* I, *then*

1.
$$f(w(\sum_{i=1}^{n} \alpha_i A_i) \leq \sum_{i=1}^{n} \alpha_i w(f(A_i)).$$

2. *If* $X_i \in \mathbb{B}(H)$ *be isometry satisfies* $\sum_{i=1}^{n} X_i X_i^* = 1$. *Then*

$$f(w(\sum_{i=1}^{n} X_i A_i X_i^*)) \leq \sum_{i=1}^{n} w(f(A_i)).$$

We can extract the following from the above result.

COROLLARY 3.13. *Let* $A_i \in \mathbb{B}(H)$ *be self-adjoints for each* $i \in \{1, \cdots, n\}$ *and* $\sum_{i=1}^{n} \alpha_i = 1$. *If* f *is a convex function on* I, *then*

1.
$$f(\| \sum_{i=1}^{n} \alpha_i A_i \|) \leq \sum_{i=1}^{n} \alpha_i^n \| f(A_i) \|.$$

2. *If* $X_i \in \mathbb{B}(H)$ *be isometry satisfies* $\sum_{i=1}^{n} X_i X_i^* = 1$. *Then*

$$f(\| \sum_{i=1}^{n} X_i A_i X_i^* \|) \leq \sum_{i=1}^{n} \| f(A_i) \|.$$

Furthermore, if $f = x^p$ where $p > 1$ we have.

COROLLARY 3.14.

1.
$$\| \sum_{i=1}^{n} \alpha_i A_i \|^p \leq \sum_{i=1}^{n} \alpha_i^n \| A_i^p \|.$$

For each $A_i \in \mathbb{B}(H)$ *be positive, self-adjoints and for each* $i \in \{1, \cdots, n\}$ *and* $\sum_{i=1}^{n} \alpha_i = 1$.

2.

$$\|\sum_{i=1}^{n} X_i A_i X_i^*\|^p \leq \sum_{i=1}^{n} \|A_i^p\|.$$

For every X_i be isometry operarors satisfies $\sum_{i=1}^{n} X_i X_i^* = 1$.

And, if $0 < p < 1$, we have

Corollary 3.15.

1.

$$\|\sum_{i=1}^{n} \alpha_i A_i\|^p \geq \sum_{i=1}^{n} \alpha_i^n \|A_i^p\|.$$

For each $A_i \in \mathbb{B}(H)$ be positive, self-adjoints and for each $i \in \{1, \cdots, n\}$ and $\sum_{i=1}^{n} \alpha_i = 1$.

2.

$$\|\sum_{i=1}^{n} X_i A_i X_i^*\|^p \geq \sum_{i=1}^{n} \|A_i^p\|.$$

For every X_i be isometry operarors satisfies $\sum_{i=1}^{n} X_i X_i^* = 1$.

References

[1] Fadi Alrimawi, Omar Hirzallah, and Fuad Kittaneh, *Norm inequalities involving convex and concave functions of operators*, Linear and Multilinear Algebra (2018), 1–16.

[2] Richard Bouldin, *The numerical range of a product, ii*, Journal of Mathematical Analysis and Applications **33** (1971), 212–219.

[3] Sever S Dragomir, *Hermite Hadamards type inequalities for convex functions of selfadjoint operators in Hilbert spaces*, Linear Algebra and its applications **436** (2012), 1503–1515.

[4] Sever S Dragomir, *Inequalities for the numerical radius of linear operators in Hilbert spaces*, Springer, 2013.

[5] CK Fong and JAR Holbrook, *Unitarily invariant operator norms*, Canad. J. Math. **35** (1983), 274–299.

[6] Karl E Gustafson and Duggirala KM Rao, *Numerical range*, Springer, 1997.

[7] Paul Richard Halmos, *A Hilbert space problem book*, Springer Science and Business Media, 1982.

[8] Frank Hansen and Gert K Pedersen, *Jensen's inequality for operators and lowner's theorem*, Mathematische Annalen **258** (1982), 229–241.

[9] Fumio Hiai, *Matrix analysis: matrix monotone functions, matrix means, and majorization*, Interdisciplinary Information Sciences **16** (2010), 139–248.

[10] John AR Holbrook, *Multiplicative properties of the numerical radius in operator theory*, J. reine angew. Math. **237** (1969), 166–174.

[11] Fuad Kittaneh, *Notes on some inequalities for hilbert space operators*, Publications of the Research Institute for Mathematical Sciences **24** (1988), 283–293.

[12] Fuad Kittaneh, *Numerical radius inequalities for hilbert space operators*, Studia Mathematica **168** (2005), 73–80.

[13] B Mond, JE Pe, et al, *Convex inequalities in hilbert space*, In Houston J. Math., Citeseer, 1993.

[14] J Pemcarisc, T Furuta, J Miścisc Hot, and Y Seo, *Mondpencariéc method in operator inequalities*, Inequalities for Bounded Selfadjoint Operators on a Hilbert Space, Element, Zagreb, 2005.

[15] RP Phelps, *Convex functions, monotone operators, and differentiability*, second ed. Lecture Notes in Mathematics, 1964.

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