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BETA-ALMOST RICCI SOLITONS ON ALMOST COKÄHLER MANIFOLDS

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Abstract. In the present paper is to classify Beta-almost (β-almost) Ricci solitons and β-almost gradient Ricci solitons on almost CoKähler manifolds with ξ belongs to (k, μ)-nullity distribution. In this paper, we prove that such manifolds with V is contact vector field and $Q\phi = \phi Q$ is η-Einstein and it is steady when the potential vector field is pointwise collinear to the reeb vectoer field. Moreover, we prove that a (k, μ)-almost CoKähler manifolds admitting β-almost gradient Ricci solitons is isometric to a sphere.

1. Introduction

In 1982, R. S. Hamilton [15] introduced the notion of Ricci flow to find a canonical metric on a smooth manifold. The Ricci flow is an evolution equation for metrics on a Riemannian manifold defined as follows:

$$\frac{\partial}{\partial t} g = -2S, \tag{1.1}$$

where S denotes the Ricci tensor. Ricci solitons are special solutions of the Ricci flow equation (1.1) of the form $g = \sigma(t)\psi_t^* g$ with the initial condition $g(0) = g$, where ψ_t are diffeomorphisms of M and $\sigma(t)$ is the scaling function. A Ricci soliton is a generalization of an Einstein metric. We recall the notion of Ricci soliton according to [5]. On the manifold

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M, a Ricci soliton is a triple (g, V, λ) with g, a Riemannian metric, V a vector field, called the potential vector field and λ a real scalar such that

$$\pounds_V g + 2S + 2\lambda g = 0, \tag{1.2}$$

where $\pounds$ is the Lie derivative. Metrics satisfying (1.2) are interesting and useful in physics and are often referred as quasi-Einstein ([7], [8]). Compact Ricci solitons are the fixed points of the Ricci flow $\frac{\partial}{\partial t} g = -2S$ projected from the space of metrics onto its quotient modulo diffeomorphisms and scalings, and often arise blow-up limits for the Ricci flow on compact manifolds. Theoretical physicists have also been looking into the equation of Ricci soliton in relation with string theory. The initial contribution in this direction is due to Friedan [12] who discusses some aspects of it. Recently, the notion of almost Ricci soliton have introduced [21] by Piagoli, Riogoli, Rimoldi and Setti.
The Ricci soliton is said to be shrinking, steady and expanding according as λ is negative, zero and positive respectively. Ricci solitons have been studied by several authors such as ([9], [10], [16], [17], [18], [24], [25]) and many others.
Recently, Wang, Gomes and Xia [23] generalized almost Ricci soliton to k -almost Ricci soliton which is defined as follows:

Definition 1.1. A complete Riemannian manifold (M^{2n+1}, g) is said to be a β-almost Ricci soliton, denoted by $(M^{2n+1}, g, V, \beta, \lambda)$, if there exist a smooth vector field X on M^{2n+1} such that

$$S + \frac{\beta}{2}\pounds_V g + \lambda g = 0, \tag{1.3}$$

where λ and β are smooth functions on M^{2n+1}. λ is called soliton function and V is called the potential vector field.

A β-almost Ricci soliton is said to be shrinking, steady and expanding according as λ is negative, zero and positive respectively. A β-almost Ricci soliton is called β-Ricci soliton if λ is constant. A β-almost Ricci soliton is said to be trivial, that is, Einstein if the flow vector field V is homothetic, that is, $\pounds_V g = cg$, for some constant c. Otherwise, it is called non-trivial. A β-almost Ricci soliton is said to be β-almost gradient Ricci soliton if the potential vector field V is the gradient of a smooth function f on M^{2n+1}, that is, $V = Df$, where D is the gradient operator of g on M^{2n+1}. For convanience, we denote $(M^{2n+1}, g, Df, \beta, \lambda)$ as a β-almost gradient Ricci soliton with potential function f.

In particular, a Ricci soliton is the 1-almost Ricci soliton with constant soliton λ and an almost Ricci soliton is nothing but the 1-almost Ricci soliton. Recently, Ghosh and Patra studied [14] the k-almost Ricci solitons on contact geometry. In [1], Barros and Ribeiro proved that a compact almost Ricci soliton with constant scalar curvature is isometric to an Euclidean sphere. In this connection, a theorem has also been proved by Wang, Gomes and Xia in [23] for k-almost Ricci soiton which is given as follows:

THEOREM 1.1. [23] *Let $(M^n, g, V, \beta, \lambda), n \geq 3$ be a non-trivial β-almost Ricci soliton with constant scalar curvature r. If M^n is compact, then it is isometric to a standard sphere $S^n(c)$ of radius $c = \sqrt{\frac{2n(2n+1)}{r}}$.*

The above Theorem will be used in later to prove our results.

In the present paper, after introduction, we study almost CoKähler manifolds. In section 3, we characterize β-almost Ricci solitons on almost CoKähler manifolds and prove several important results. In the last section, we consider β-almost gradient Ricci solitons on almost CoKähler manifolds.

2. Almost CoKähler manifolds

In the present section, we give some well known definitions and basic formulae on Almost CoKaehler manifolds which will be very useful in the next sections. An almost contact structure on a $(2n+1)$-dimensional smooth manifold M^{2n+1} is a triplet (ϕ, ξ, η), where ϕ is a $(1,1)$-type tensor field, ξ is a global vector field and η is a 1-form satisfying ([2], [3])

$$\phi^2 X = -X + \eta(X)\xi, \qquad \eta(\xi) = 1, \tag{2.1}$$

Here also holds

$$\phi\xi = 0, \qquad \eta \circ \phi = 0. \tag{2.2}$$

If an almost contact manifold admits a Riemannian metric g such that

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.3}$$

for any vector fields X, Y, then the manifold is called an almost contact metric metric manifold. In such a manifold we can define a fundamental 2-form Φ by

$$\Phi(X,Y) = g(X,\phi Y), \tag{2.4}$$

for any vector fields X, Y. An almost contact metric manifold is said to be an almost CoKähler manifold if both η and Φ are closed. That is, $d\eta = 0$ and $d\Phi = 0$. An almost contact metric manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ is said to be normal if the almost complex structure J on $M \times \mathcal{R}$ defined by (pp. 80 of [3])

$$J(X, f\frac{d}{dt}) = (\phi X - f\xi, \eta(X)\frac{d}{dt}),$$

where f is a real valued function defined on $M \times \mathcal{R}$, is integrable. Moreover, if an almost contact manifold $(M^{2n+1}, \phi, \xi, \eta)$ is normal, then it is said to be a CoKähler manifold. In addition an almost contact metric manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ is CoKähler if and only if $\nabla\phi = 0$, or equivalently, $\nabla\Phi = 0$.
Let $M^{2n+1}(\phi, \xi, \eta, g)$ be an almost CoKähler manifold. Let us consider two operators h and l which are defined by $h = \frac{1}{2}\pounds_\xi \phi$ and $l = R(., \xi)\xi$, where R denotes the curvature tensor and $\pounds$ is the Lie differentiation. These operators are symmetric of type $(1,1)$ and satisfies ([6], [11] [19]) the following

$$h\xi = h'\xi = 0, \qquad \mathrm{Tr}\, h = \mathrm{Tr}\, h' = 0, \qquad h\phi = -\phi h, \tag{2.5}$$

where $h' = h \cdot \phi$. Also in an almost CoKähler manifold, we have ([6], [11] [19])

$$\nabla_X \xi = h'X = h\phi X, \tag{2.6}$$

$$\phi l \phi - l = 2h^2, \tag{2.7}$$

for any vector fields X.

A (k,μ)-contact metric manifold is a generalization of Sasakian and K-contact manifold. In [4] Blair, Koufogiorgos and Papantoniou introduced and studied the notion of (k,μ)-nullity distribution on contact metric manifolds $M^{2n+1}(\phi, \xi, \eta, g)$. A contact metric manifold M^{2n+1} whose curvature tensor satisfies

$$R(X,Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY],$$

for all vector fields X, Y on M^{2n+1}, where $h = \frac{1}{2}\pounds_\xi \phi$ ($\pounds$ denotes the Lie derivative of ϕ along ξ and $k, \mu \in \mathbb{R}$ is known as (k, μ)-contact manifold and ξ is said to belongs to the (k, μ)-nullity distribution. Several authors have studied ([20], [22]) the (k, μ)-contact metric manifold and obtain some interesting results. When k, μ are smooth functions, it is said to be the generalized (k, μ)-nullity distribution. Thus we have the following:

Definition 2.1. An almost CoKähler manifold $M^{2n+1}(\phi, \xi, \eta, g)$ is said to be a (k, μ)-almost CoKähler manifold if ξ satisfies the equation

$$R(X,Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY], \tag{2.8}$$

for all vector fields $X, Y \in \chi(M^{2n+1})$ and k, μ are real constants.
In a consequence of (2.8), we have $l = -k\phi^2 + \mu h$. In view of this, from (2.7) we deduce

$$h^2 = k\phi^2 \tag{2.9}$$

and also we obtain

$$S(X,\xi) = 2nk\eta(Y), \tag{2.10}$$

$$Q\xi = 2nk\xi. \tag{2.11}$$

Definition 2.2. A vector field V on a contact manifold is said to be a contact vector field if it preserve the contact form η, that is

$$\pounds_V \eta = \psi\eta, \tag{2.12}$$

for some smooth function ψ on M. When $\psi = 0$ on M, the vector field V is called a strict contact vector field.

Now we state some well known Lemmas:

Lemma 2.1. *(Poincare Lemma): In Riemannian manifold $d^2 = 0$, where d is the exterior differential operator, that is,*

$$g(\nabla_X \mathrm{grad}\zeta, Y) = g(\nabla_Y \mathrm{grad}\zeta, X), \tag{2.13}$$

for any two vector fields X, Y and for any smooth function ζ.

Lemma 2.2. *([13]) If a vector field X leaves the structure tensor ϕ of the contact metric manifold M invariant, then there exists a constant c such that $\pounds_X g = c(g + \eta \otimes \eta)$.*

3. β-almost Ricci solitons on (k,μ)-Almost CoKähler manifolds

This section is devoted to study β-almost Ricci solitons on Almost CoKähler manifolds.

$$(\pounds_V d\eta)(X,Y) = (\pounds_V g)(X,\phi Y) + g(X,(\pounds_V \phi)Y), \tag{3.1}$$

for all vector fields X, Y on M. Multiplying both sides of (3.1) and then using (1.3), we get

$$\beta(\pounds_V d\eta)(X,Y) = -2g(QX,\phi Y) - 2\lambda g(X,\phi Y) + \beta g(X,(\pounds_V\phi)Y), \tag{3.2}$$

for all vector fields X, Y on M. As V is a contact vector field, from (2.12) we have

$$\pounds_V d\eta = d\pounds_V \eta = (d\psi)\wedge\eta + \psi(d\eta), \tag{3.3}$$

from which it follows that

$$(\pounds_V d\eta)(X,Y) = \frac{1}{2}\{d\psi(X)\eta(Y) - d\psi(Y)\eta(X)\} + \psi g(X,\phi Y). \tag{3.4}$$

In view of (3.2) and (3.4) we obtain

$$\begin{aligned} & \beta d\psi(X)\eta(Y) - \beta d\psi(Y)\eta(X) + 2\beta\psi g(X,\phi Y) \\ = \;& -4g(QX,\phi Y) - 4\lambda g(X,\phi Y) + 2\beta g(X,(\pounds_V\phi)Y), \end{aligned} \tag{3.5}$$

and hence we get

$$\begin{aligned} 2\beta(\pounds_V\phi)Y \;=\;& 4Q\phi Y + 2(\psi\beta + 2\lambda)\phi Y \\ & +\beta\{\eta(Y)D\psi - (Y\psi)\xi\}. \end{aligned} \tag{3.6}$$

Putting $Y = \xi$ in (3.6) we have

$$2(\pounds_V\phi)\xi = D\psi - (\xi\psi)\xi, \tag{3.7}$$

where we use β is positive. Tracing the equation (1.3) gives

$$\beta \mathrm{divV} = -\mathrm{r} - (2\mathrm{n}+1)\lambda. \tag{3.8}$$

Let Ω be the volume form of M, that is,

$$\Omega = \eta\wedge(d\eta)^n \neq 0. \tag{3.9}$$

Taking Lie derivative of the foregoing equation along the vector field V and using the formula $\pounds_V\Omega = (\mathrm{divV})\Omega$ and (3.3) yields

$$(\mathrm{divV})\Omega = (\mathrm{n}+1)\psi\Omega, \tag{3.10}$$

and hence
$$\mathrm{divV} = (\mathrm{n}+1)\psi. \tag{3.11}$$
Using (3.11) in (3.8) we infer
$$r = -(n+1)\psi\beta - (2n+1)\lambda. \tag{3.12}$$
The equation (1.3) can also be exhibited as
$$S(X,Y) + \frac{\beta}{2}(\pounds_V g)(X,Y) + \lambda g(X,Y) = 0. \tag{3.13}$$
Substituting ξ for X and Y in (3.13) and applying (2.10) we obtain
$$\beta g(\pounds_V \xi, \xi) = 2nk + \lambda. \tag{3.14}$$
Replacing Y by ξ in (3.13) and then using (2.10) and (2.12) we deduce
$$\beta \pounds_V \xi = (\beta\psi + 4nk + 2\lambda)\xi. \tag{3.15}$$
Applying (3.15) in (3.14) we have
$$\psi\beta = -\lambda - 2nk. \tag{3.16}$$
Making use of (3.16) in (3.15) we get
$$\beta \pounds_V \xi = (2nk + \lambda)\xi. \tag{3.17}$$
With the help of the first term of (2.2) and (3.17) we obtain
$$\beta(\pounds_V \phi)\xi = \beta \pounds_V \phi\xi - \phi(\beta \pounds_V \xi) = 0, \tag{3.18}$$
and hence from (3.7) it follows that
$$D\psi = (\xi\psi)\xi. \tag{3.19}$$
Taking inner product of (3.19) with respect to Y we get
$$d\psi(Y) = (\xi\psi)\eta(Y), \tag{3.20}$$
from which it follows that
$$d\psi = (\xi\psi)\eta. \tag{3.21}$$
Taking exterior derivative of (3.21) and using (2.13) we infer
$$d^2\psi = d(\xi\psi) \wedge \eta + (\xi\psi)d\eta, \tag{3.22}$$
which implies
$$d(\xi\psi) \wedge \eta + (\xi\psi)d\eta = 0. \tag{3.23}$$
Taking wedge product with η on (3.22) we have
$$(\xi\psi)\eta \wedge d\eta = 0. \tag{3.24}$$

As $\eta \wedge (d\eta)^n$ is the volume element, then $\eta \wedge d\eta \neq 0$ and hence from (3.24) we get

$$\xi\psi = 0, \tag{3.25}$$

and hence from (3.21) it follows that

$$d\psi = 0, \tag{3.26}$$

and hence ψ becomes constant. Integrating (3.11) and then by divergence theorem we infer

$$\psi = 0 \tag{3.27}$$

and accordingly from (3.16) we deduce

$$\lambda = -2nk. \tag{3.28}$$

Thus we can state the following:

Theorem 3.1. *Let $M^{2n+1}(\phi, \xi, \eta, g)$ be a (k, μ)-almost CoKähler manifold with V as a contact vector field. If g is a β-almost Ricci soliton with V as the potential vector field, then it is shrinking, steady or expanding according as k is positive, zero or negative.*

Using (3.27) and (3.28) in (3.12) we get

$$r = 2nk(2n+1). \tag{3.29}$$

By the virtue of the equation (3.29) and the **Theorem 1.1** we can conclude the following:

Theorem 3.2. *Let $M^{2n+1}(\phi, \xi, \eta, g)$ be a compact (k, μ)-almost CoKähler manifold having non-zero k and V as a contact vector field. If g is a β-almost Ricci soliton with V as the potential vector field, then M is isometric to a sphere $S^{2n+1}(c)$ of radius $c = \sqrt{\frac{4n+3}{nk}}$.*

In particular, if $k = \frac{4n+3}{n}$, then the manifold isometric to an unit sphere $S^{2n+1}(c)$. Thus we have the following:

Corollary 3.1. *Let $M^{2n+1}(\phi, \xi, \eta, g)$ be a compact (k, μ)-almost CoKähler manifold with V as a contact vector field. If g is a β-almost Ricci soliton with V as the potential vector field, then M is isometric to an unit sphere $S^{2n+1}(c)$.*

Using the fact that ψ is constant and (3.16) in (3.6) and (2.12) respectively we obtain

$$\beta(\pounds_V\phi)Y = 2Q\phi Y + (\lambda - 2nk)\phi Y \tag{3.30}$$

and

$$\beta\pounds_V\eta = -(\lambda + 2nk)\eta. \tag{3.31}$$

Afterwards we deduce

$$(\pounds_V\phi)Y = \pounds_V\phi Y - \phi(\pounds_V Y). \tag{3.32}$$

Substituting $Y = \phi Y$ in the preceeding equation and using the first term of (2.1) we infer

$$\begin{aligned}(\pounds_V\phi)\phi Y &= \pounds_V\phi^2 Y - \phi(\pounds_V\phi Y)\\ &= -\pounds_V Y + \{\pounds_V\eta(Y)\}\xi + \eta(Y)\pounds_V\xi - \phi(\pounds_V\phi Y),\end{aligned} \tag{3.33}$$

from which it follows that

$$\beta(\pounds_V\phi)\phi Y = -\beta\pounds_V Y + \beta\{\pounds_V\eta(Y)\}\xi + \beta\eta(Y)\pounds_V\xi - \beta\phi(\pounds_V\phi Y). \tag{3.34}$$

Operating ϕ on the both sides of (3.32) and the using the first term of (2.1) and then multiplying by β we have

$$\beta\phi(\pounds_V\phi)Y = \beta\phi(\pounds_V\phi Y) + \beta\pounds_V Y - \beta\eta(\pounds_V Y)\xi. \tag{3.35}$$

Adding (3.34) and (3.35) and using (3.17), (3.31) yields

$$\beta\phi(\pounds_V\phi)Y + \beta(\pounds_V\phi)\phi Y = 0. \tag{3.36}$$

Let us assume that $Q\phi = \phi Q$. Then, by the virtue of (3.30), from (3.36) we obtain

$$QY = \frac{2nk - \lambda}{2}Y + \frac{2nk + \lambda}{2}\eta(Y)\xi, \tag{3.37}$$

which shows that the manifold is η-Einstein. Hence we can state the following:

THEOREM 3.3. *Let $M^{2n+1}(\phi, \xi, \eta, g)$ be a (k, μ)-almost CoKähler manifold with V as a contact vector field and $Q\phi = \phi Q$. If g is a β-almost Ricci soliton with V as the potential vector field, then M is η-Einstein.*

Taking covariant differentiation of (3.37) along an arbitrary vector field X we get

$$\begin{aligned}(\nabla_X Q)Y &= -\frac{X\lambda}{2}Y + \frac{X\lambda}{2}\eta(Y)\xi \\ &\quad + \frac{2nk+\lambda}{2}\{g(Y, h'X)\xi + \eta(Y)h'X\}. \qquad (3.38)\end{aligned}$$

Taking inner product of (3.38) with Z we have

$$\begin{aligned}g((\nabla_X Q)Y, Z) &= -\frac{X\lambda}{2}g(Y,Z) + \frac{X\lambda}{2}\eta(Y)\eta(Z) \qquad (3.39)\\ &\quad + \frac{2nk+\lambda}{2}\{g(Y, h'X)\eta(Z) + g(h'X, Z)\eta(Y)\}.\end{aligned}$$

Contracting X and Z in (3.39) and using (2.5) we have

$$Yr = -Y\lambda + (\xi\lambda)\eta(Y). \qquad (3.40)$$

Contracting Y and Z in (3.39) and using (2.5) we get

$$Xr = -n(X\lambda) \qquad (3.41)$$

In view of (3.40) and (3.41) we obtain

$$(1-n)X\lambda = \frac{\xi\lambda}{2}\eta(X). \qquad (3.42)$$

Substituting $X = \xi$ in the last equation entails

$$\xi\lambda = 0. \qquad (3.43)$$

By the use of (3.43), the equation (3.42) reduces to

$$X\lambda = 0, \qquad (3.44)$$

from which it follows that λ is constant and hence we have the following:

THEOREM 3.4. *Let $M^{2n+1}(\phi, \xi, \eta, g), n > 1$, be a (k, μ)-almost CoKähler manifold with V as a contact vector field and $Q\phi = \phi Q$. Then the β-almost Ricci soliton becomes β-Ricci soliton.*

Making use of (3.37) in the equation (3.30) yields

$$(\pounds_V \phi)Y = 0, \qquad (3.45)$$

as β is positive.
In view if the (3.45) and the Lemma 2.2 we observe that

$$(\pounds_V g)(X, Y) = c\{g(X, Y) + \eta(X)\eta(Y)\}, \qquad (3.46)$$

from which we can conclude that the β-Ricci soliton is non-trivial. Thus we are in a position to state that

THEOREM 3.5. *Let $M^{2n+1}(\phi, \xi, \eta, g)$ be a (k, μ)-almost CoKähler manifold with V as a contact vector field and $Q\phi = \phi Q$. Then the β-Ricci soliton is non-trivial.*

Now we shall consider a special type of β-almost Ricci soliton where the potential vector field V is pointwise collinear with the reeb vector field ξ. Then we have

$$V = \alpha\xi, \tag{3.47}$$

where α is a non-zero smooth function on M in the connection that V is non-zero.
Taking covariant derivative of (3.47) with respect to any vector field X and using (2.6) we infer

$$\nabla_X V = (X\alpha)\xi + \alpha h'X. \tag{3.48}$$

The eqution (1.3) can also be represented as

$$2S(X, Y) + \beta g(\nabla_X V, Y) + \beta g(X, \nabla_Y V) + 2\lambda g(X, Y) = 0. \tag{3.49}$$

Applying (3.48) in (3.49) we have

$$\begin{aligned} &2S(X, Y) + \beta\{(X\alpha)\eta(Y) + \alpha g(h'X, Y)\} \\ &+\beta\{(Y\alpha)\eta(X) + \alpha g(X, h'Y)\} + 2\lambda g(X, Y) = 0. \end{aligned} \tag{3.50}$$

Putting $Y = \xi$ in (3.50) and using (2.10) we get

$$\beta(D\alpha) + \beta(\xi\alpha)\xi + (2nk + 2\lambda)\xi = 0. \tag{3.51}$$

Putting $X = Y = \xi$ in (3.50) we have

$$\beta(\xi\alpha) = -\lambda - 2nk. \tag{3.52}$$

Making use of (3.52) in (3.51) yields

$$\beta D\alpha + \lambda\xi = 0, \tag{3.53}$$

from which it follows that

$$\beta(X\alpha) = -\lambda\eta(X). \tag{3.54}$$

Putting $X = \xi$ in (3.54) we find

$$\beta(\xi\alpha) = -\lambda. \tag{3.55}$$

Using (3.55) in (3.52) we have

$$k = 0. \tag{3.56}$$

By the virtue of the **Theorem 3.1** and (3.56) we observe that β-almost Ricci soliton becomes steady. Thus we can state the following:

THEOREM 3.6. *Let $M^{2n+1}(\phi, \xi, \eta, g)$ be a (k, μ)-almost CoKähler manifold with V as a contact vector field such that V is pointwise collinear with the reeb vector field ξ. If g is a β-almost Ricci soliton with V as the potential vector field, then it is steady.*

4. β-almost gradient Ricci soliton on (k, μ)-almost CoKähler

In this section we characterize β-almost gradient Ricci soliton on (k, μ)-almost CoKähler manifolds. Then the potential vector field V can be expressed as

$$V = Df, \tag{4.1}$$

where f is a smooth function on M^{2n+1}.
By the help of (4.1) the equation (3.49) reduces to the equation

$$S(X, Y) + \frac{\beta}{2}\{g(\nabla_X Df, Y) + g(X, \nabla_Y Df)\} + \lambda g(X, Y) = 0. \tag{4.2}$$

Making use of (2.13) in (4.2) we have

$$\beta \nabla_X Df = -QX - \lambda X. \tag{4.3}$$

Taking covariant derivative of (4.3) with respect to any vector field Y we obtain

$$\begin{aligned} \beta \nabla_Y \nabla_X Df &= \frac{1}{\beta}(Y\beta)\{QX + \lambda X\} - (\nabla_Y Q)X \\ &\quad -Q(\nabla_Y X) - (Y\lambda)X - \lambda(\nabla_Y X), \end{aligned} \tag{4.4}$$

as β is positive. In view of (4.3) and (4.4), we can represent the curvature tensor as follows:

$$\begin{aligned} \beta R(X, Y)Df &= \frac{1}{\beta}(X\beta)\{QY + \lambda Y\} - \frac{1}{\beta}(Y\beta)\{QX + \lambda X\} \\ &\quad -\{(\nabla_X Q)Y - (\nabla_Y Q)X\} - \{(X\lambda)Y - (Y\lambda)X\}. \end{aligned} \tag{4.5}$$

Taking inner product of (4.5) with ξ and using (2.10) we obtain

$$\begin{aligned}\beta g(R(X,Y)Df,\xi) &= \frac{1}{\beta}\{2nk\eta(Y)+\lambda\eta(Y)\}-\frac{1}{\beta}\{2nk\eta(X)+\lambda\eta(X)\}\\ &\quad -\{g((\nabla_X Q)\xi,Y)-g((\nabla_Y Q)\xi,X)\}\\ &\quad -\{(X\lambda)\eta(Y)-(Y\lambda)\eta(X)\}. \end{aligned} \tag{4.6}$$

Taking covariant derivative of (2.11) with respect to X we have

$$(\nabla_X Q)\xi = 2nkh'X - Qh'X. \tag{4.7}$$

Applying (4.7) in (4.6) we get

$$\begin{aligned}\beta g(R(X,Y)Df,\xi) &= \frac{2nk+\lambda}{\beta}\{(X\beta)\eta(Y)-(Y\beta)\eta(X)\}\\ &\quad +\{S(h'X,Y)-S(X,h'Y)\\ &\quad -\{(X\lambda)\eta(Y)-(Y\lambda)\eta(X)\}. \end{aligned} \tag{4.8}$$

Also $g(R(X,Y)Df,\xi) = -g(R(X,Y)\xi,Df)$. Then using (2.8) we find

$$\begin{aligned}g(R(X,Y)Df,\xi) &= -k\{(Xf)\eta(Y)-(Yf)\eta(X)\}\\ &\quad -\mu\{\eta(Y)g(Df,hX)-\eta(X)g(Df,hY)\} \end{aligned} \tag{4.9}$$

Using (4.9) in the equation (4.8) we have

$$\begin{aligned} &-k\beta\{(Xf)\eta(Y)-(Yf)\eta(X)\}-\mu\beta\{\eta(Y)g(Df,hX)-\eta(X)g(Df,hY)\}\\ = &\ \frac{2nk+\lambda}{\beta}\{(X\beta)\eta(Y)-(Y\beta)\eta(X)\}+\{S(h'X,Y)-S(X,h'Y)\}\\ &-\{(X\lambda)\eta(Y)-(Y\lambda)\eta(X)\}. \end{aligned} \tag{4.10}$$

Replacing X by hX and Y by h^2Y in (4.10) yields

$$Q\phi X - \phi QX = 0, \tag{4.11}$$

as we have taken k as a non-zero real number. Let $\{e_i, \phi e_i, \xi\}$, $i = 1,2,3,...,n$, be an orthonormal $\phi-$basis of M such that $Qe_i = \sigma_i e_i$. Then we have $Q\phi e_i = \sigma_i \phi e_i$. Substituting e_i for X in the last equation we get

$$Q\phi e_i = \sigma_i \phi e_i. \tag{4.12}$$

Making use of ϕ-basis and (2.11) we obtain

$$
\begin{aligned}
r &= g(Q\xi,\xi) + \sum_{i=1}^{n}[g(Qe_i,e_i) + g(Q\phi e_i,\phi e_i)] \\
&= 2nk + 2\sum_{i=1}^{n}\sigma_i. \qquad (4.13)
\end{aligned}
$$

As σ_i are the eigen values, $\sum_{i=1}^{n}\sigma_i$ is constant and hence r is constant. Hence, following the **Theorem 1.1** we can conclude:

THEOREM 4.1. *Let $M^{2n+1}(\phi,\xi,\eta,g)$ be a compact (k,μ)-almost CoKähler manifold with V as a contact vector field. If g is a β-almost gradient Ricci soliton with V as the potential vector field, then M is isometric to a sphere $S^{2n+1}(c)$ of radius $c = \sqrt{\frac{2(n+1)(4n+3)}{r}}$, where $r = 2nk+2\sum_{i=1}^{n}\sigma_i$.*

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